Classical Field Theory
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Lecture - 16

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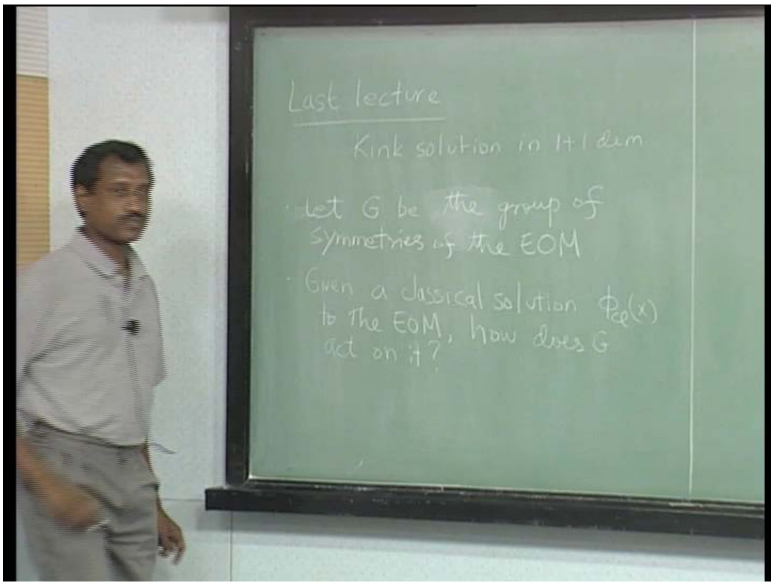

Of the equation of motion or may be the lagrangean, so on and so forth. So, you already know. You should give me some equations so we can work out for the symmetries and very early on, we saw that, I mean this was the motivation for defining a group. So, they have to form a group. So, that is what you guaranteed. But now we can ask different a question. Given a classical solution to the equations of motion, how does g act on it? There are exactly two possibilities.

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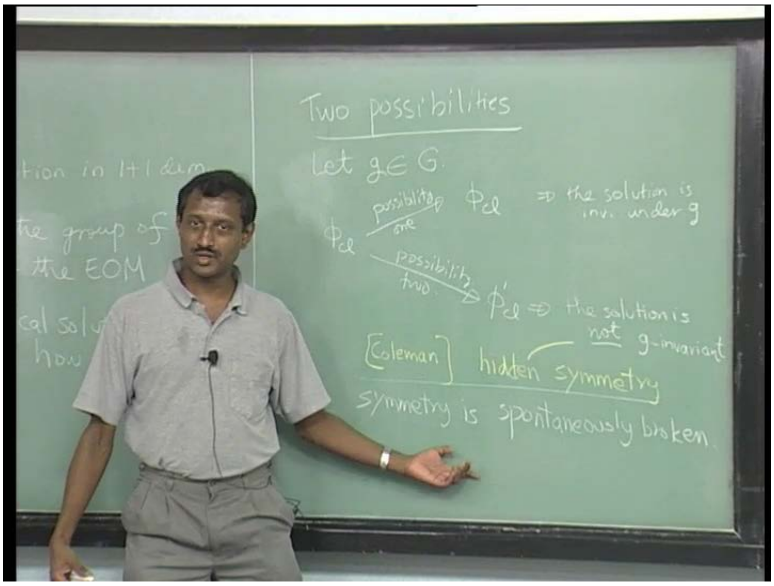

Again, something which we saw explicitly. The first one is, so, let us pick; let us just choose let g be some element of the group. So, you can ask. So, when we say that g is a symmetry of the equation of motion, in other words, we know how g acts on the fields. So, by the way, here even though I am using a scalar field, if there are many many fields, a single scalar field, you can see that, what I mean by that is the full set of fields in the theory.

So, we know how this group acts on an arbitrary configuration. Obviously, you can work out what it acts on a classic equation. So, let us, so, the question is, we take phi classical and act on it with g. So, the question is what happens. So, one possibility is, it is invariant. So, it goes to, so, this arrow indicates the action of this element g. So, it goes to phi classical. So, we say that it is invariant. So, implies that the solution, the classical solution is invariant under g.

So, possibility two is, when it is not. Now, the key point here is that, since it is the symmetry of the equations of motion, if it is not invariant, it has to go to another solution. Whatever is the g action, it will map solutions to solutions. So, then it goes to phi prime of classical. Again by classical, I indicate that it is a solution to the classical equations of motion or the Euler Lagrange equations of motion and implies that the solution is not g invariant. So, that is a name which goes with this sort of thing. We say that. So, the symmetry, but g is indeed a symmetry of the theory.

It is just that the solution is very much similar to the fact that we live in a world where there is no translation invariants. Does not mean that the fundamental interactions etcetera may, I mean, we know, we expect them to be invariant under those things. We put that as one of the axioms or whatever as input. But we live in a world which is where everything is hidden. So, the terminology for this is, it should be a correctly called hidden symmetry. This is what Coleman argued for that it should be called hidden symmetry, because the symmetry is there. It is not like there is anything, just because a solution is not invariant. Does not mean anything about the symmetry of the theory. So, he wants to or he wanted to call it hidden symmetry, but sadly he lost out to the rest of world he argued and so out of respect to him, we should actually remember that it is actually a hidden symmetry. Because, physically that is what is happening, but the rest of the world calls it, the symmetry is broken or to be more precise, symmetry is spontaneously broken.

This is very different from, you can always write, so, when we are looking at systems, we look for systems which have certain symmetries. But look here. Once you say that, you do not want a certain symmetry; you can explicitly break the symmetry. But this is not quite in that category. This is the symmetry, the theory has the symmetry. But just that you are in a configuration or in a, this thing that where the symmetry is not an invariance. So, this is called spontaneous symmetry breaking. So, this looks very abstract, but we can start with examples and I am going to, for a change, start with a stat mech example.

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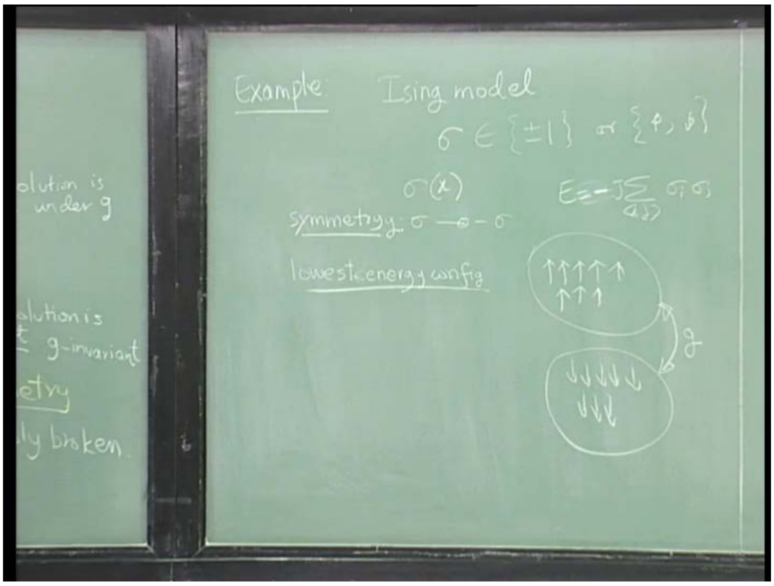

So, we will look at examples and there is a reason for this. It turns out there is a very nice connection between statistical mechanics and quantum mechanics and it is roughly that, you think of time. You replace it, go to imaginary time, you take t to become i beta; beta is inverse temperature and there seems to be some correct connection between these things. So, that is one connection. The other connection is that, if you look at many body theory in condensed mater physics, those are non relativistic systems. But just for a moment, suppose we assume that we are going to look at time independent solutions, just for simplicity. Then, there is not much of a difference between relativistic and non relativistic systems, at least within that sub class.

So, you will find that what we might call Hamiltonian in our field theory could be mapped to some free energy in some satellite system. It is a 1 into 1 map. Beautifully it works. So, and there you try to minimize or extremize the free energy. Here, we can extremize the Hamiltonian or the energy, which as I explained to you, which is the same as extremizing the Lagrangian, because it related to via a Legendre transformation. So, the example is going to be deliberately from statistical mechanics to just to show you that, whatever I am saying has a much larger regime achieve of validity. But just to have a frame work, I always discuss the relativistic field theories. Also it is my expertise.

So, the example is, so, is the icing model. So, in the icing model, at every lattice site, there is a spin, which can take up or down values. So, sigma takes values in plus or minus 1 or more pictorially it can take two values, which we sort of indicate by up or down. So, in a continual description at every, if you are at every lattice site, if you have a spin field, then you just sort of replace it by saying it as a field, which takes, which is a two valued and has some value at every point. So, this theory has a symmetry, which is sigma goes to minus sigma. We know that. All these up spins, we can call them down; we can call down up. So, there is a symmetry. Of course, you can break this symmetry by coupling it to a magnetic field or something. But we are not talking about that. We are assuming this is just a simple nearest neighbor interaction.

So, this is a symmetry. So, now, you can ask what are the, so, let us look at the vacuum or the ground state lowest energy configuration. What would that be? So, lowest energy configuration, so, energy is in lattice nearest neighbor, some j sigma i sigma; j should become minus sign. So, that, it, let us parallel guys become lowest energies configuration is to have all of them lined up. But this is not unique. There exists for exactly the same energy, you can always flip all the signs and you can get something similar to this. So, that is degeneracy. Now you can see. So, let us call this g. We can see that the action of g takes you from one ground state to another ground state.

So, suppose you are in this particular ground state. You can see that, this I mean, this other ground state is related to it by this z 2. But you would say that this z 2 is spontaneous broken. So, this is one example and this is where the symmetry is a discrete symmetry. But coming back to this. Here, nowhere did I have to actually say that; look there is the symmetry is discrete or continuous. It does not matter. So, we, so, what will do now is to look at an example in field theory now. Here, I could even; here I could get a continuous symmetry by replacing these things by Heisenberg spins. So, then these things can take a rotation worth of those things. So, that is just a generalization of that. But I do not want to get on to that. So, Heisenberg ferromagnetic would be a natural generalization, which will we have a rotation symmetry. So, let us look at spin theory.

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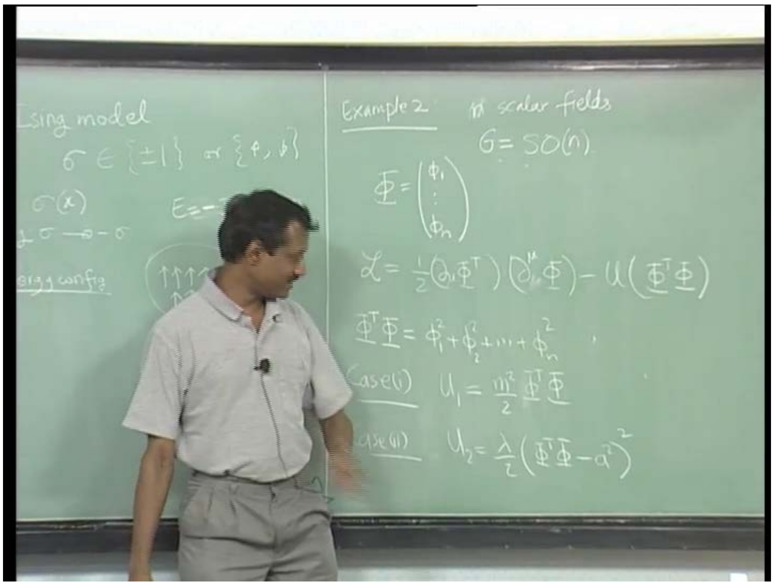

So now, example one is this one, is the following. Just take, let us take n scalar fields. One thing is, I just defined g here to be the full group of symmetries. B, I mean, I will write out possibly some sub group of it because I am going to focus on those guys. But that theory can have a lot more symmetries.

So, let us take G to be S O n. You will see that, in our example, it will be n o n. But does not matter. So, let us just take n scalar fields and I write them as a inter pull and then, choose a Lagrangean, which is S O n invariant. So, it just. So, what is phi transpose phi? Which is, S O n is a set of transformations, which keeps this sort of combination invariant. Even this is invariant, because it is actually d mu of phi 1 whole square d mu of phi 2 whole square plus plus so on and so forth. This symmetry, since it is just a global symmetry, in the sense, it does not depend on coordinates. Does not matter the derivative being there.

Oh Yes, this I am not allowed to do this. Thank you.

So, what we will do now is to consider two different examples of a potential and there is one potential, which we will be coming back again and again. Do not for a moment think that I do not know other potentials. In some sense, you will see that, that particular potential capture is the simplest one, which will capture all the ideas that I need. So, we will, so, first example is the following. So, let us just call it u 1. I just choose it to be and case two. So, this is the one which we already had seen in the early example, where it was only one scalar field. But now, I have n scalar fields.

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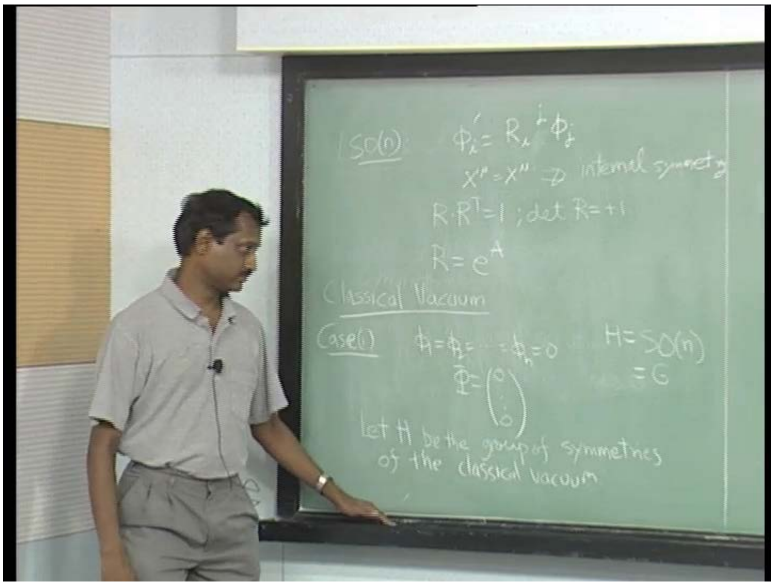

How does the S O n act? S O n acts phi j and it does not act on; so, implies it is an internal symmetry. That is, one which does not act on the space time co-ordinates. So, it is only action field space. This is I index here. Index is the index for n scalar fields and of course, R is an n by n matrix satisfying R R transpose equal to 1. In fact, I do not need to put in the determinant condition. But because that is the continuous part of the group, so I want to, so, I can break it up into a discrete z 2 and this thing, which we know, determinant R equal to plus 1. So, these are the two conditions I impose this. Of course, we all also know that R, once I do this, can be written as e power A. Let A is any anti symmetric matrix n by n matrix.

So, we know all these things. So, now, the question is, let us look at what is a classical vacuum. So, in most situations where you study, when you talk of symmetry is spontaneously broken, typically with respect to the ground state or the lowest energy configuration, rather than an arbitrary configuration. You can, whatever I said can be done. I mean, here I did not make the assumption that this is a classical vacuum, but we will see for many physical things, we would, the first nontrivial solution you look at is the classical solution.

So, the classical vacuum. So, that would be the lowest energy state. So, obviously, that is the one where phi is a constant. So, this term, when you go to the Hamiltonian thing, you just get a flip of signs that you get a sum of squares. You have done all that. So, all we need to do is to forget about this and think of phi as just a number, a constant and minimize your action. So, but you look here. This as a minimum, when it is just a quadratic in, this thing, so, it has a minimum at 0.

So, case one, it is just; now the question to you is, what can you say about the symmetry of the solution. Anybody from this side? It is invariant under the full action of G because if I start out, this is equivalent to saying that this phi equal to all 0s' and any rotation of all 0s', 0 vector is the vector which will remain as 0 vector in this n dimensional fictitious place. So, let us, let me just define. So, let H be the group of symmetries of the classic. This will also form a group. Obviously, again the same argument. But in this case, here H is also equal to S O n. So, the idea of spontaneous symmetry breaking is to say that H is the sub group of G. Strict sub group.

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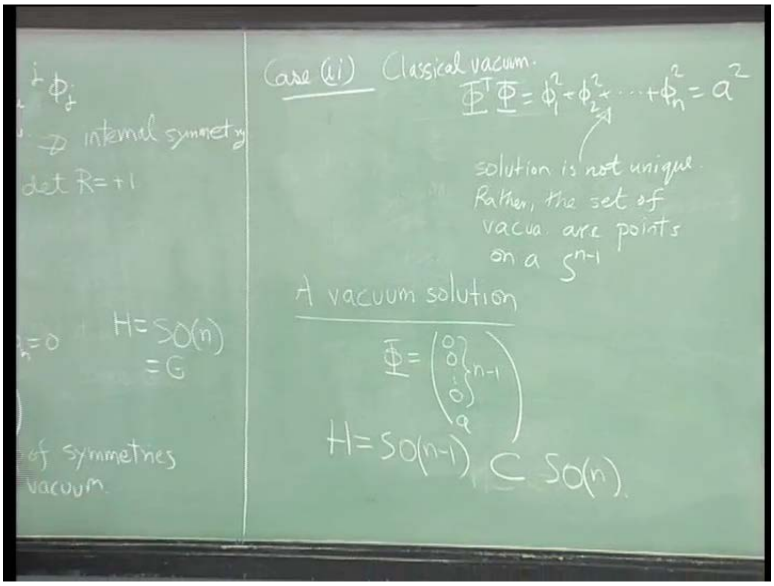

So, this is case one. What about case two? Now, again this is bounded from below. I have adjusted the constants, obviously, to make this have a 0. So, where is the 0? This is at phi transpose phi equal to a square. Classical vacuum.

So now, you can see that, this is a set of points on a n minus 1 dimensional sphere. So, this is a set of vacua. So, there are many vacua now. Many solutions. Here it was, the solution was unique. But here, solution is not unique. Rather, the set of vacua are points on a n minus 1 dimensional sphere. Some of you may not know what this is. But if I put n equal to 3, you will see a normal two-dimensional sphere and if you take n equal to 2, you get a circle and higher dimensions. These are just natural generalizations.

So, if you have trouble picturing this, there are two ways out. Do not picture it or put n equal to 3, if you want to have this vivid picture in your mind. So, both ways are fine. So, let us look at this. So now, the question is, let us pick A vacuum. For instance, A vacuum solution is of this n minus 1 times and put an a. I am just solving that. I am putting phi 1 to phi 1 and phi n minus 1 equal to 0 and phi n, I am choosing it to be plus a. I could have chosen it to be minus a, but I do not care. I just choose one of them. So now, the question is, what is H?

So, we know that, if we take S O n and act on these things, if I take a arbitrary S O n, it will mix it and in general, it is like saying that, so, let us put n equal to 3 because you want to picture things. So, if I choose a vector, this way if I do a rotation, arbitrary rotation, then I will go to some other thing. So, in general, I will generate all the components back. So, it will be moving in some motion on the 2 dimensional sphere, n equal to 3.

Now, the question is, what is the sub group which preserves this? It is S O n minus 1. It is the set. So, let us take, in 3 dimension, if you pick a direction, the sub group of rotations, which preserves it is the rotations, which about that axis. But that is in 3 dimension. So, and that picture is actually lousy. In general, what you want do is, you want to look at rotations, which makes all the 0 components and do not leave, which leave this untouched. That is S O n minus 1, which will agree with the other case. If in 3 dimension, 3 becomes, it becomes S O 2. S O 2 is the one which mixes x and y; leaves z untouched for instance.

So, in general, you have to think of it in that fashion. So, h is very simple. It is S O n minus 1 and clearly, strictly a sub group of S O n. Of course, you could do this, if you think of A as a free parameter. If you take a to 0, suddenly you find an enlargement of the symmetry and then, H becomes, everything becomes 0. Then, it goes back to this case, even though it is not n square power 4. But it does not matter. It is some 4^{th} power.

But nevertheless it still has the same property that the ground state lowest configuration is same as out here.

So now, we have something very nice. We have H. This has checks up to and… So, this is the case, where you have spontaneous symmetry vacuum. So, you can see that there is a nontrivial set of vacua. So, all of them have to have the same energy. All of them have same energy 0. That is why they are degenerate. Now, there is something very interesting, which comes out of situations like this, where, you have a continuous; so, this is not like you have a, so, this is unlike the example of the icing model, where you have a discrete group. This is the continuous group.

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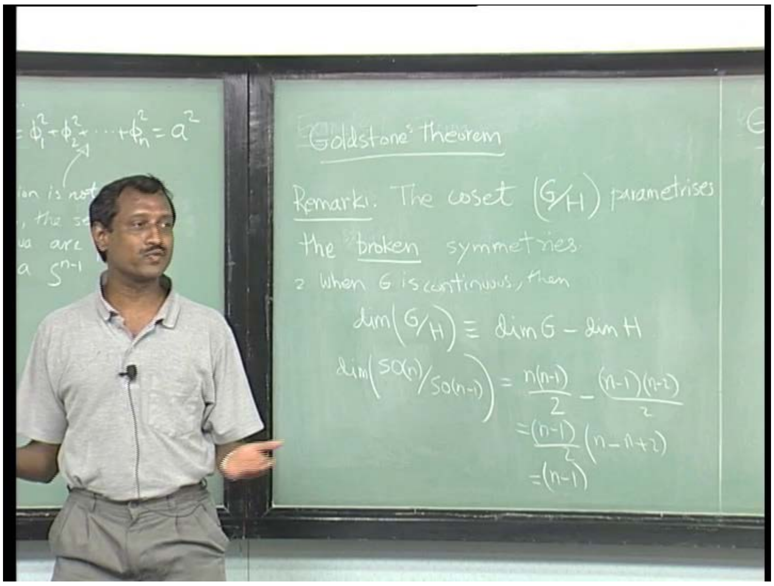

So, there is a very nice result, which is due to Goldstone, called Goldstone's theorem, which is the following. No, before that, let us just discuss before we discuss Goldstone's theorem, just I need one more bit of this thing. So, you can ask, what are the set of transformations now? So, in G, which are not in H. What would they be parameterized by? Do you know any set which parameterizes that?

Exactly.

So, it is a coset. So, it is a set of elements in G, which is not in a H. So, it is just a remark. So, the coset G mod H parameterizes the broken. So now, I am sort of and this is, by the way true for whether G is continuous or not. But just I need one more small remark, which is, let us assume that G remark 1 or remark 2 is, when G is continuous, G is continuous, then I will define something called dimension of the coset. I need to explain something is out here. When you have a continuous group, dimension cannot be the number of elements in that set because it is infinite. So, this is just a number of parameters.

So, coming back to our example here, dimension of S O n will be equal to as many, the number of parameters, that is the number of independent anti symmetric n by n matrices. So, that is n into n minus 1 by 2. So, in particular, dimension of, should be equal to dimension of S O n, which is n into n minus 1 by 2 minus dimension of S O n minus 1. So, I should just subtract by this. So, it will become n minus 1 into n minus 2 divided by 2. I can pull out the common factor. It is n minus 1. It is not unrelated to the fact that there are n minus 1 0 is out here. Roughly it is the set of directions in which, independent directions in which you could go. So now, I am ready to state Goldstone's theorem. I will write it again.

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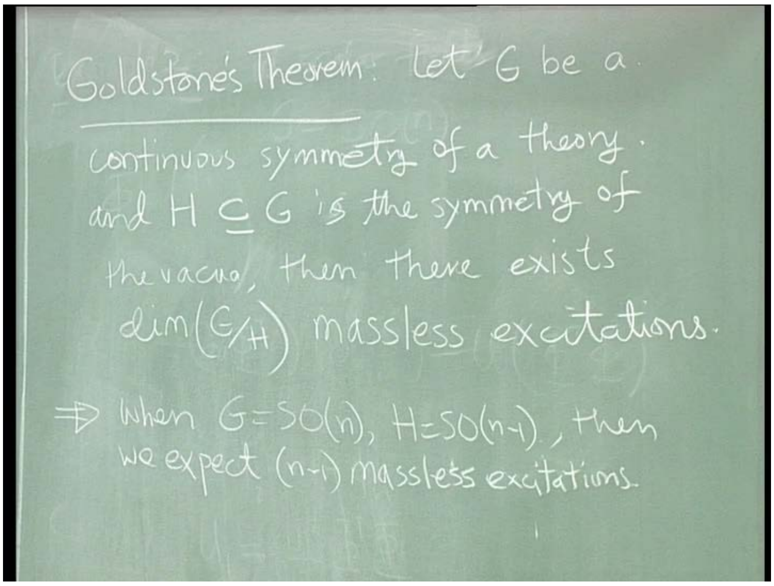

So, let G be a continuous symmetry of a theory and H less than or equal to G and just one comment. If suppose, sometimes it can happen that H is some discrete group. In the sense of counting, dimension of a discrete group what would it be? 0. Just remember that, if the group is broken down. So, we will see one situation, where I can show that, you start out with S O 2. S O 2 broken to S O 1. What is S O 1? Nothing right. But actually you can break it down to some z q, which is a discrete group. In this kind of continuous counting, it is 0. So, this formula still holds, if h is a discrete group. Even though g can be continuous, it can be broken down to some discrete groups. But those do not count. So, it is only the continuous parts of H that matter and H is the thing, is the symmetry of the vacua. Then, there exists dim G mod H mass less. By mass less, it means excitation which do not cost you energy. But that makes sense, right? If I am going to go from here through to something, where I do a slight rotation, where it turns out something else, I can always do that infinitesimally. Energy does not change.

So, that is what one means by this. But in field theory, we mean something slightly more. That is what I want to show you. So, now, I will be more, since I want to be explicit, should I choose, I will choose n equals to 2 for satis. Whatever I say goes through for any higher these things, but let us choose. So, let us, to illustrate this, let, so, according to, so, application of this, it tells you in this example. That implies that, when, so, you can see that, when G equal to H, then your dimension is 0, which is the first case. In this case, there are no mass less excitations. In fact, the only excitations have a mass given by the parameter that m which we added, would be this thing. The mass of this thing. I will explain that a little bit more later. So, when g equals to S O n and H equal to S O n minus 1, then we expect n minus 1 mass less excitations. What we mean by that is that, there is no potential for that. There are excitations which have no potential. Not just quadratic. It is just that it does not matter what translations in that thing is completely, that is what one means.

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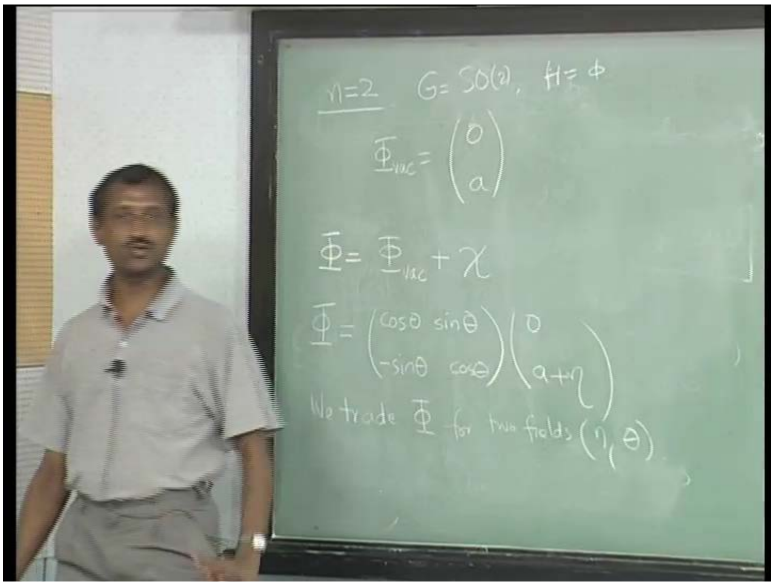

So, let us now choose an example of n equal to 2, so that, I can be, let us choose n equal to 2. So, then G would be, with of case two of course, G is now S O 2 and H is actually, let us just for simplicity, just say it is null. I will discuss. Again, we will come back to this example. There is a possibility of it not being some discrete sub group. But for now I just sort of leave that. I think of it as 0. So, we repeat whatever we did here. So, phi vacuum it is something like this. So, what we will do now is to expand, look for a, so, what you mean by excitations is to ask it is with respect to this thing. So, what you do is, you take a field and you write it as follows. Write it as phi vacuum plus, say some chi, and so, we look at chi and we expand about these things. But I will be more clever. I will choose chi in a slightly different manner, I will choose, I choose it as follows. I will chose phi not like this.

You know quite often in this course, we will do something like this and expand things through all orders in this thing, in phi and you get an effective vac. So, it is like you trade your field phi, I mean for this field chi. So, that is what you are doing. So, I want to write it in a slightly different manner. So, I will write this as phi for our example. So, this is the S O 2 transformation which is broken, which acts on a. So, plus, I will write an eta. So, what I have done ism I trade the field phi for two fields. One is an angle field theta and the other is this eta. So, we trade. So, this is chi, the analog of chi. Here, we trade phi of two fields eta and theta and their excitations about the vacuum. If I switch off eta and theta, both of them will go back to the classical solution. So, I can think of asking, taking them to be small. Take eta and theta to be small. But what will I say, it does not actually require that. So, what we will do now is to plug this into the action and ask what happened.

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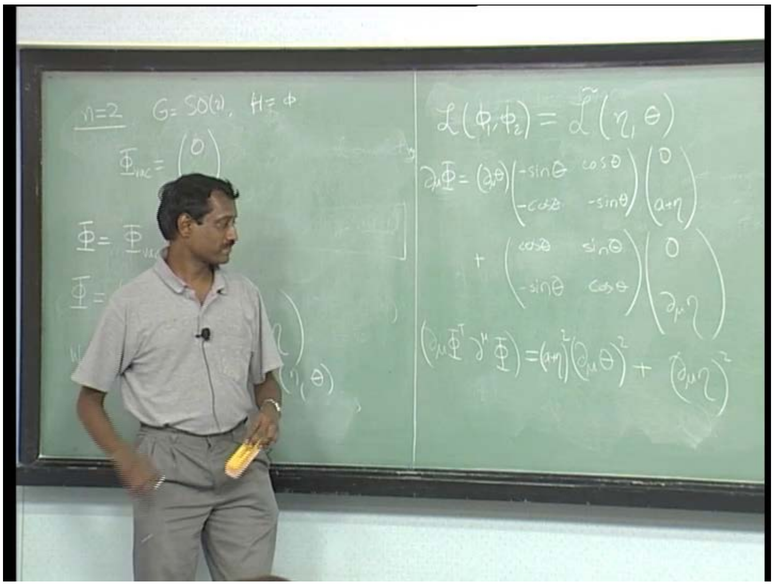

So, we have l, which is a function of phi 1 and phi 2. Now, which you rewrite as L of eta and theta. So first step is to work out what of phi d mu would look like. d mu can act to these guys. So, I get, I will do it a little fast. Hopefully you would not mind. So both, I am not saying that both eta and theta are constants. They are fields. So now, we just got to go ahead and put this into the, we need to put it into the, so, we need to compute d mu phi d mu phi transpose of course, I will let you work this out and for and I will tell me what you get. So, one thing, so, this will look. Here, I will, by this I mean d mu theta d nu theta plus you will see something like this.

You may think that I have cross terms. There are not and dimensionally speaking, if you think of this as some kind of length, you would see that this would come with some. There might be cross terms. You check those things. I would not. I would not write those things.

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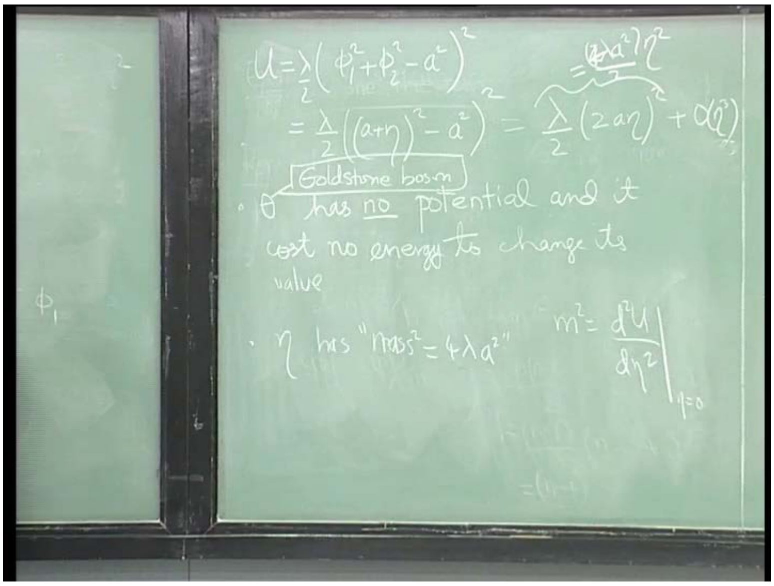

What about the potential? Potential was, but by definition it is independent. What does it, phi 1 square plus phi 2 square? We can see that the angle will completely disappear. It depends only on A plus eta whole square. So now, let us look at this thing. This may have a, I mean this looks like, I mean, yeah, what am I trying to say? If this thing has a kinetic energy term for theta, but there is no potential energy for theta. It is completely flat. So, there is, so, this is your mass less excitation. So, you can see that theta has no potential and it costs no energy obviously, to change its value.

So, what about eta? eta of course, sees this full potential. You can expand it and you if you look at, so, this will have a term which will look like lambda by 2 eta. No, the leading term, eta, now wait, let me just do this, into 2 a eta whole square eta power 4 or higher powers plus eta cube. Let me leave it at this. This is what you get. The a square cancels. So, the leading term is this sort of a thing. So, you get something which looks like this term is equal to lambda 2 lambda a square eta square, which I will write as 4 lambda a square upon 2 and eta has mass square equal to 4 lambda a square. So, this object is precisely what we will call the mass less excitation. So now, I need to make a remark from quantum field theory.

Suppose, you have in quantum field theory, you have the analog of a garden field, which is just you have phi being n square upon 2 phi square. For a scalar field, when you quantize the theory, you find that the quanta of the theory R have mass n. So, for the excitations in the quantum field theory of this classical field theory, which we look at, so, that would correspond to the mass. So, in that sense, the co efficient of the eta square is, for higher order things what we usually do is to call them eta actions and we treat them separately. But what this will tell you is that, this eta field has excitation, which with some mass square. So, this mass will be square route of this object. But this theta has blow up these things. So, it is a mass less excitation or mass less particle.

So, in quantum field theory, this field, theta field becomes and since it is a scalar field, so, this is called a Goldstone Boson. So, theta is called a Goldstone. So, the content of Goldstone theorem is to say that there is dim g minus dim h goldstone bosons. There are certain funny symmetries called Fermionic symmetries, for which you will find that this kind of the analogous excitations are Fermionic. So, you can get goldstone fermions. But so, in theories with super symmetry, for instance that happens. But we will not be discussing that in this elementary course. So, this is the very important, I mean we have done it for this simple case. But what you will see is, it is exactly similar to whatever happens here, happens in the more general case, which I looked at and even more general examples.

So, for instance, if you take spin systems like the Heisenberg anti Ferromagnets, so, there again, if you pick a ground state configuration where all the spins are lined up; obviously, there exists the symmetry is broken, the rotation symmetry and then, you will expect goldstone bosons. Those have a name. They are called Magnons. So various, I mean, there is a sense in which even phonons, so, what is happening is that, if you are in a lattice, translation symmetry is broken. But there is subtlety because when I mean, in understanding these systems, where the G acts on space time, when it is not an internal symmetry things, when things, when it is an internal symmetry, things are very clean. There are subtleties in the statement. But nevertheless I think I will make a sort of a twice a wrong statement to give you a feel for things, is that you break translation in variants in a crystal, so, what that implies is that you expect is mass less excitations. So, one for each direction and so, if you are in a 3 dimensional crystal, so, we expect three mass less excitations. What would you call them? They are called phonons.

Similarly, if you have some, so, the vibrational modes or the phonons, when you quantize, second quantize the many body theory, there again is the same thing. So, this is a very very powerful theorem. It just tells you how many mass less excitations that you have in the theory. Another point is that, the symmetry as I have discussed, is a global symmetry. It is constant in, all the parameters were constant at space and time. Yes, yes, yes. So, actually yes. There is, so here, I was very clever in parameterizing. If I done it in this; I could do it in this fashion.

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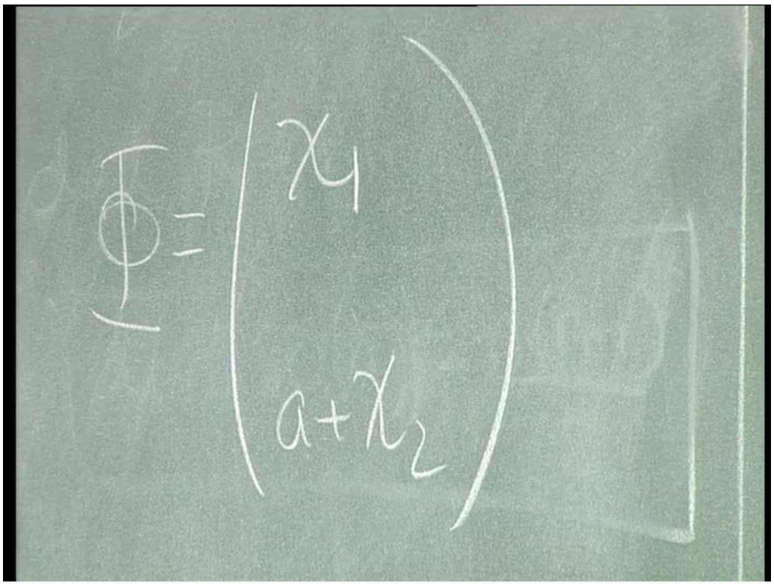

So, let us for a moment do it and then, let us write that phi would be, let us call the doublet as chi 1 and chi 2. So, then I would write chi 1 and then I would write a plus chi 2. Then, in all these cases, you would not, what would not be obvious is the following. You can expand it to a second order and you will indeed see that there is a mass less guy. But you would not be sure that it will be true to all orders or whatever. But actually if you go back and think about its origin, it has to be true to all orders. But for that, you have to have this clever parameterization. I am not saying you cannot see it in this. The reason I choose, for instance, I choose l equal to 2 is because I can quickly write the explicit matrix for that.

But, in fact, let me show you the parameterization for S O n in general. How we would do that? It is exactly like this. Except, here we wrote the rotation matrix, which corresponds to the part, which is not invariant. So, there I have to just write out elements of the coset and that is not very hard. It would not be unique, but I can still write. So, let me just show you that. For n greater than 2.

Yeah, doublet of fields, we can write down the potential. We can draw the potential.

Yes, actually let me, maybe I should, instead of writing something more general, let me draw this picture for you. It is called the Mexican hat. It is called the Mexican hat potential.

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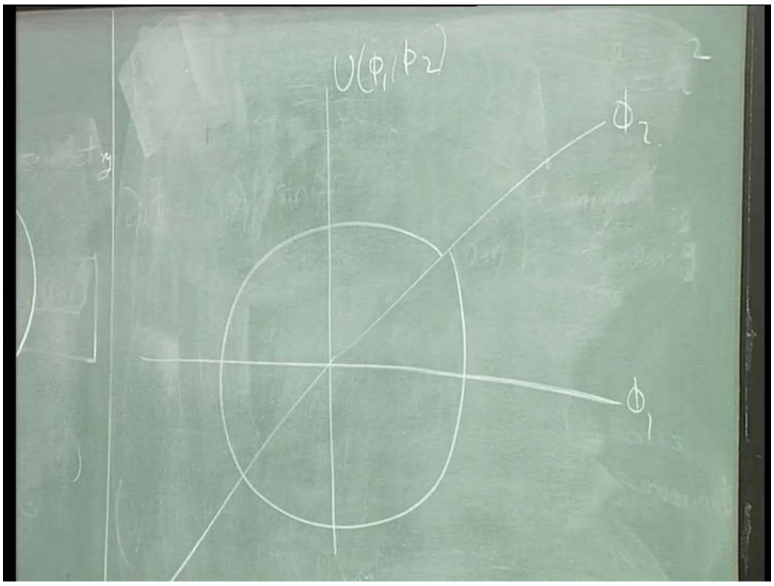

So, what we have is, yeah, so, this is a good idea. So, what we want to plot is u 1 u 2 u, which is the function of phi 1 and phi 2 in this space. So, let me first draw a circle of radius A. Maybe I should draw the, what I will do is, I will first draw the hat and then, I will then, I will draw the co-ordinate or whatever, because I am not so good at drawing. So, how does it go? So, the Mexican hat looks like this.

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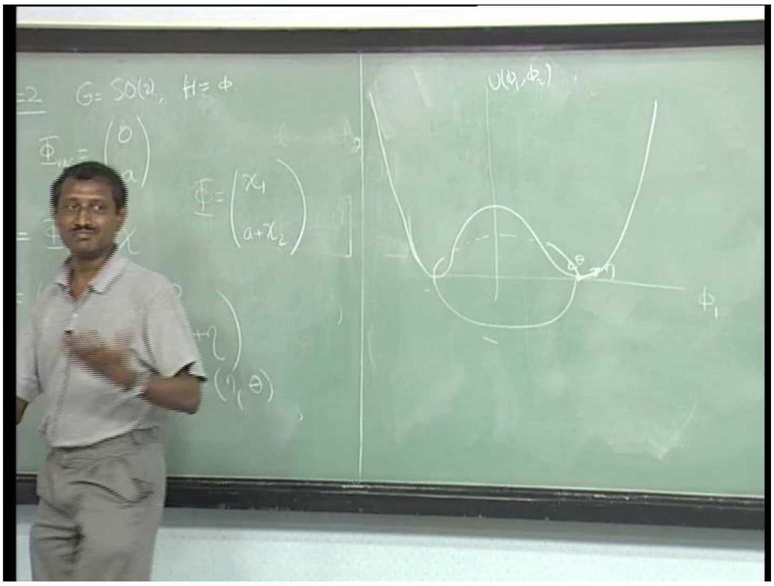

So, actually I will do it in this fashion. I should have thought of this. I first plotted within the phi 1 phi 2 space. Say, double bell potential at 0 is this thing and to get the phi 2 thing, I just take about this and revolve it. So, you can see that, have you ever seen the Mexican sombrero or whatever? This is exactly, I mean, it looks exactly like that. So, a choice of vacuum, for instance, would be to choose say this one. So now, I can draw this surface of revolution like this. So, what you can see here is that, if you pick this particular point, the eta excitation is going up here. It sees this. So, this is the direction of the eta excitation, while this direction is the theta excitation. So, it is going along the flat potential.

So, this is exactly the picture, which you should remember. But there is, so, this is the flat direction in which there is no; so, think of a marble rolling in this thing and the marble chooses your vacuum. If you just tap it, it can go to the next point. There is no this thing, but if you try to, you need to give a much harder kick to make it go up the hill and that harder kick is measured, if you wish, it is the second derivative about this point which is exactly this.

So, the mass square can be written if you wish. It can be written as d square mu by d eta square at or rather, let us call it at eta square d square u by d eta square at eta equal to 0 or I can rewrite in terms of the original radial variable called phi 1 square plus phi 2 square as r square or something, d square u by d r square at r equal to a square. So, this is, so, what you can see here is that, this is basically that solve that is going on out here. But I mean, the power of this statement here is that, once you have a group structure in this thing, it is a continuous group or it is a lee group, then you can actually write out explicitly what it is. So, I think I will stop here.