Modern Optics
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Lecture – 06
Maxwell's equations and electromagnetic waves (Contd.)

So, in this section, we will discuss Electromagnetic Wave Propagation in Absorbing Media. We have seen the propagation characteristics of electromagnetic waves in dielectric medium isotropic dielectric, and now in an observing medium.

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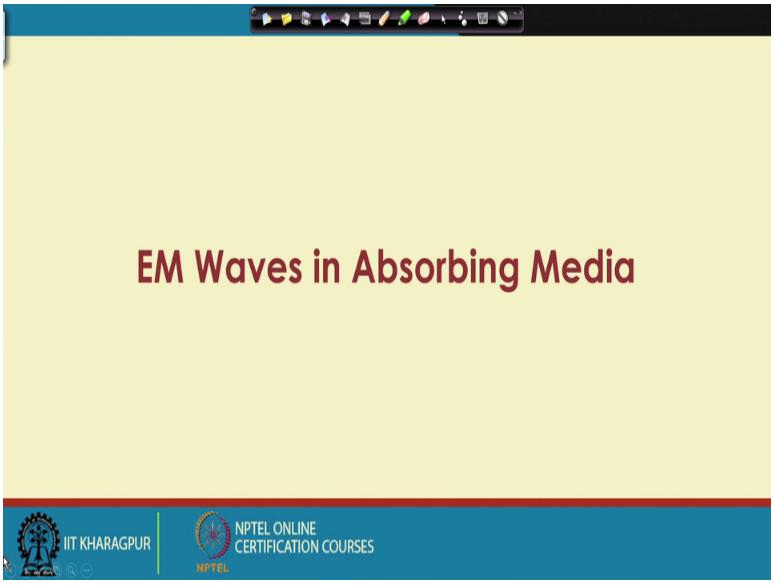

Mostly absorbing media; primarily where conductors conducting medium.

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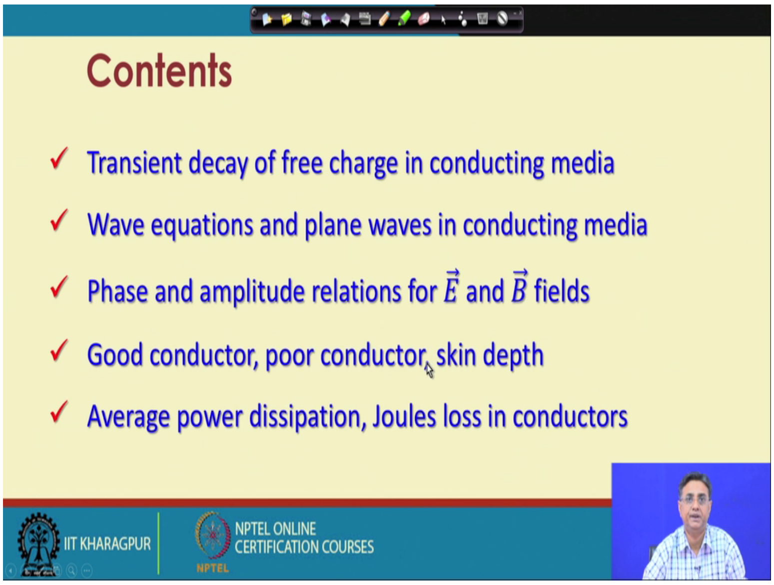

So, today in this discussion, the following topics will be discussed that the transient decay of free charge in a conducting medium, then we will address the wave equations for the conducting medium and the resulting plane waves. Phase, we will look at the phase and amplitude of the electric field and the magnetic field of the plane waves. Then will categorize the good conductors and poor conductors. We will see that the electromagnetic waves cannot propagate in conducting medium as a result, there will be only a small skin depth penetration of the electromagnetic waves will characterize that and look at the quantity expression, how it depends on the propagation core and the frequency of the electromagnetic waves. Then, we will see the average power dissipation joules loss in conductor.

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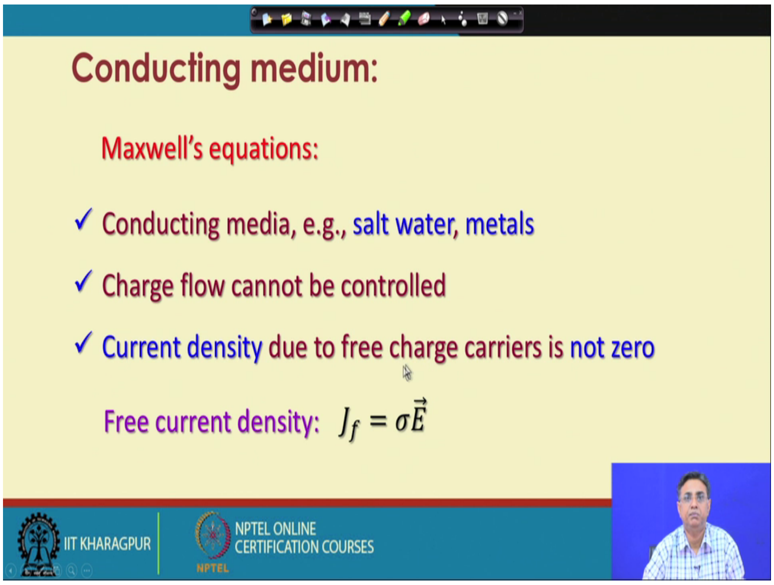

So, a conducting medium will look at the Maxwell's equation; for example, conducting medium salt, water, metals. In such a medium the charge flow cannot be controlled and the current density due to free charge carriers is not 0. Free charge, free current density J f is given by the Ohm's law sigma into E a, that should be a vector here.

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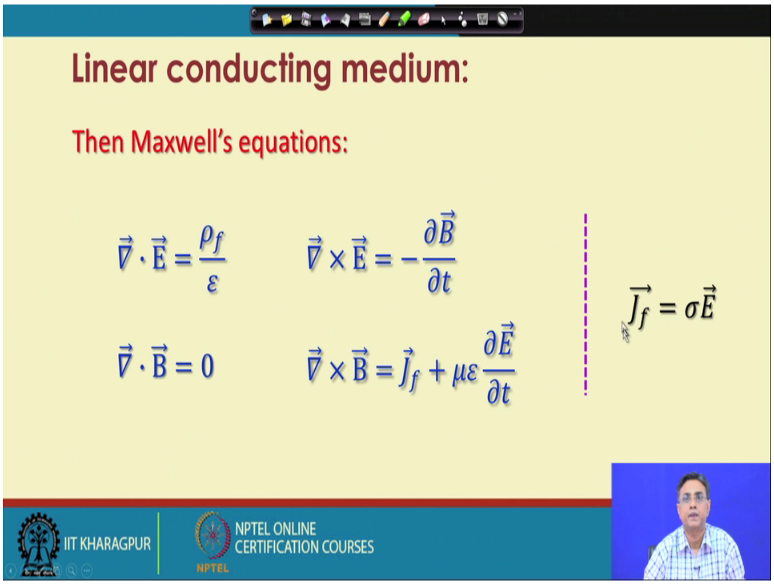

So, now will look at the Maxwell's equation; in this case, del dot E equal to free charge rho f by epsilon and the curl equation for the electric field del cross E equal to minus del B del t, then del dot B equal to 0 and del cross B equal to J f; the current density due to free charges and mu into epsilon del E del t. So, this J f is equal to sigma into epsilon which will substitute in place of J f and will reorganize this Maxwell's equation to form the wave equation.

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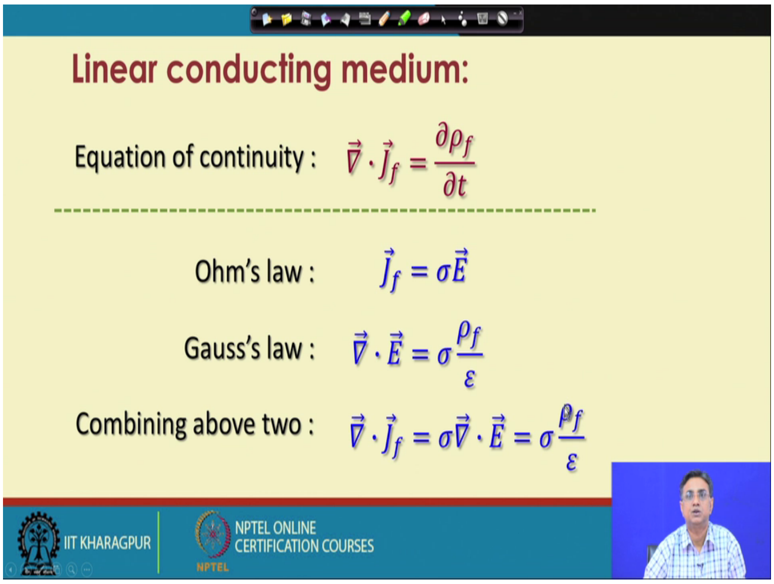

Also, we have this equation of continuity; the del dot J equal to del rho f del t is the free charge, rate of change of the free charge there should be a minus sign and then this J f is the charge density; the current density free charge due to the free charges. Now Ohm's law states that J f equal to sigma into E Gauss's law gives you the del dot E equal to sigma rho f by epsilon.

Now, if I combine these two equations, then we can write that del dot J is equal to del dot J is equal to sigma del dot E using this equation in this, and then if I use del dot E equal to sigma rho f by epsilon, then I can write del dot J is equal to sigma rho f by epsilon. So, I have been able to write an expression which relates this J f and rho f through sigma and epsilon.

Now, because the left hand side of this del dot J is equal to del rho f del t and the right hand side; this left hand side is equal to this which is equal to this. So, I can write del rho f del t is equal to sigma rho f by epsilon naught.

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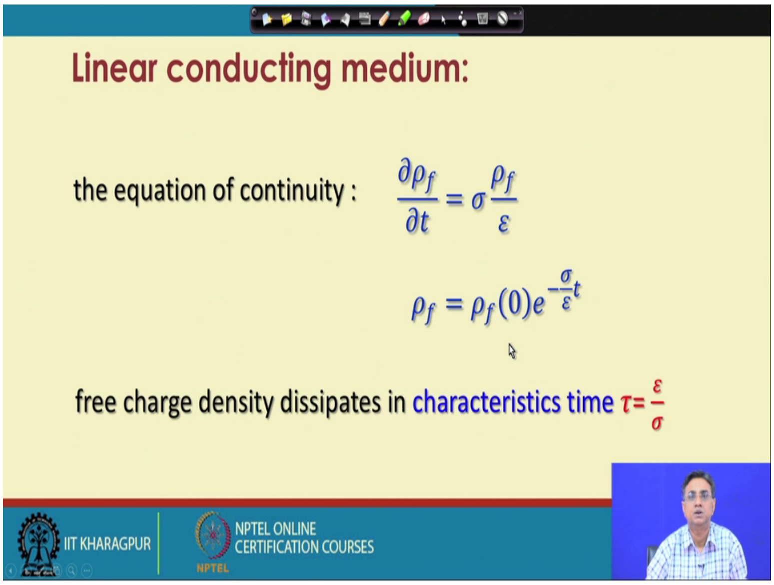

So, equating this to this, we can write this the equation of continuity in this from the del rho f del t is equal to sigma rho f by epsilon. And if you solve this equation, this gives you the solution rho f equal to rho f 0; that is a time t equal to 0, the free charge, times E to the power of minus sigma by epsilon times t. So, it shows that the free charge decays exponentially and the decaying depends on this factor sigma by epsilon naught that is the conductivity and the sigma. So, free charge density dissipates in the characteristic time tau sigma by epsilon by sigma that is the time over which the charge decays to 1 upon E times of the original.

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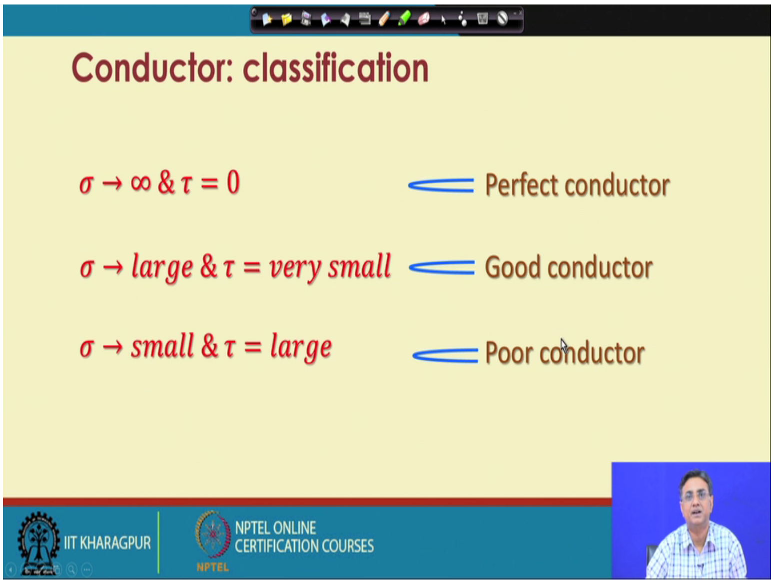

So, looking at this consequence, we can classify the conductors, if sigma tends to infinity and tau this characteristic time, the dissipation time is equal to 0 that is instantaneous, then it is a perfect conductor. And if sigma is large reasonably large and tau is small, then this conductor is a good conductor. But the reverse will happen, if sigma equal to small and tau is very large; that is the characteristic time for charge dissipation is quite large in that case the conductor will be classified as a poor conductor.

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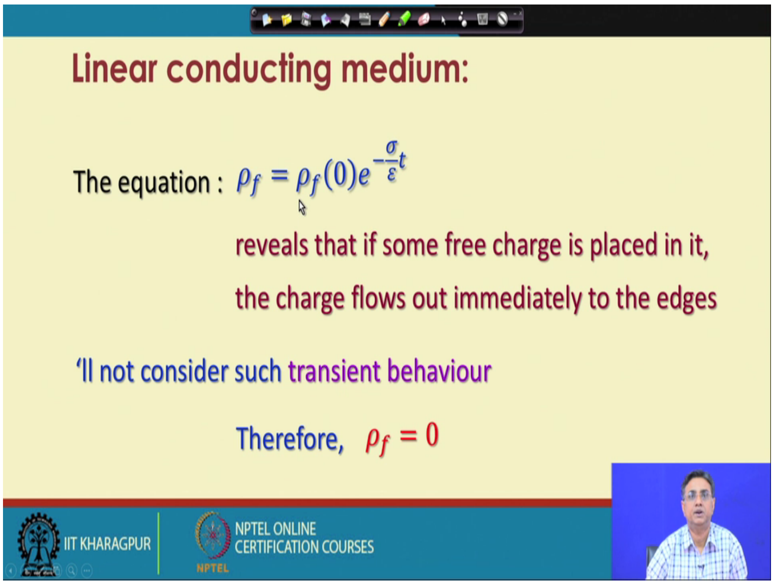

So, this equation that rho f equal to rho f 0 e to the power of minus sigma by t into t sigma by epsilon into t, it reveals that if some free charge is placed inside the conductor, the charge flows out immediately to the edges; that is in no time, the charge dissipates. So, we will not consider such transient behavior while looking at the electromagnetic wave propagation in a conductor. Therefore, we will put this free charge rho f is equal to 0 in the equation arising out of the Maxwell's equation.

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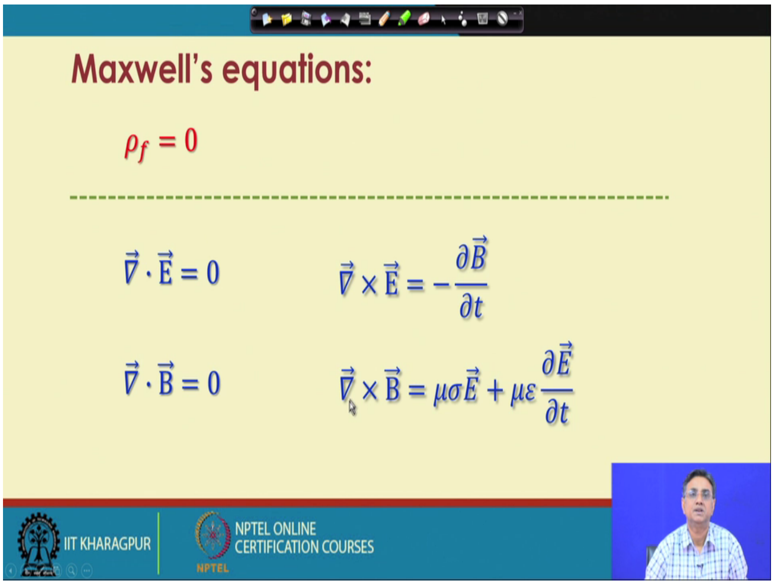

So, del dot E equal to 0 del dot B equal to 0 and del cross E equal to as it is and del cross B will now be represented as mu sigma E which is for J; that is J equal to sigma into a and mu epsilon del E del t. So, this is the new form of the Maxwell's equation when we have use this rho f equal to 0.

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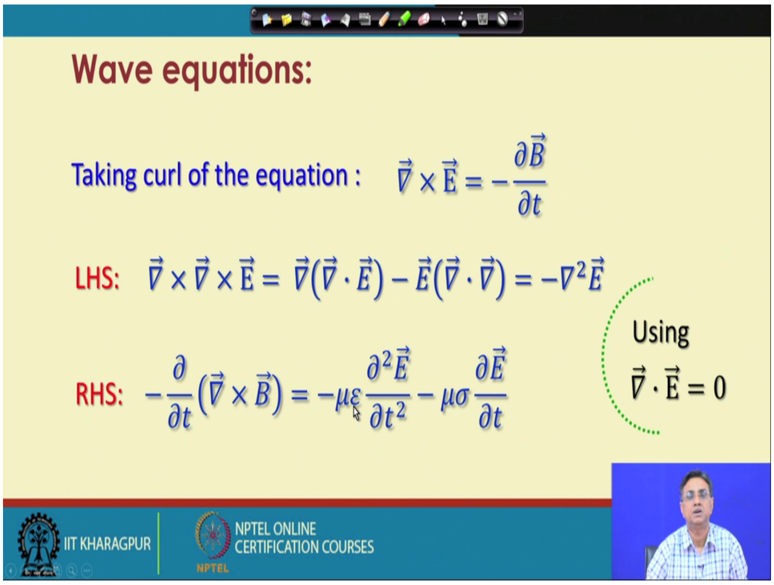

Now, if we take the curl of the equation del cross E equal to minus del B del t that is the third Maxwell's equation, taking the curl on either side so the left hand side gives you del cross del cross E which is equal to del of del dot E and E of del dot. Del, which is a very well known identity, and this identity in this case will result in only minus del square E, because del dot E is equal to 0 which has come from the consequence of this equation.

So, as a result, we can write this left hand side del cross del cross E is equal to minus del square E and the right hand side that is minus del B del t, you should take the curl of this so you have to write minus del del t of del cross B which is equal to minus mu epsilon del square E del t square minus mu epsilon del E del t. This has come from this equation. I take curl of del B del t which will be equal to mu epsilon del E del t plus mu epsilon del to E del t square. Exactly that is what is written here. So, I have been able to calculate the del cross del cross E and del cross del B del t.

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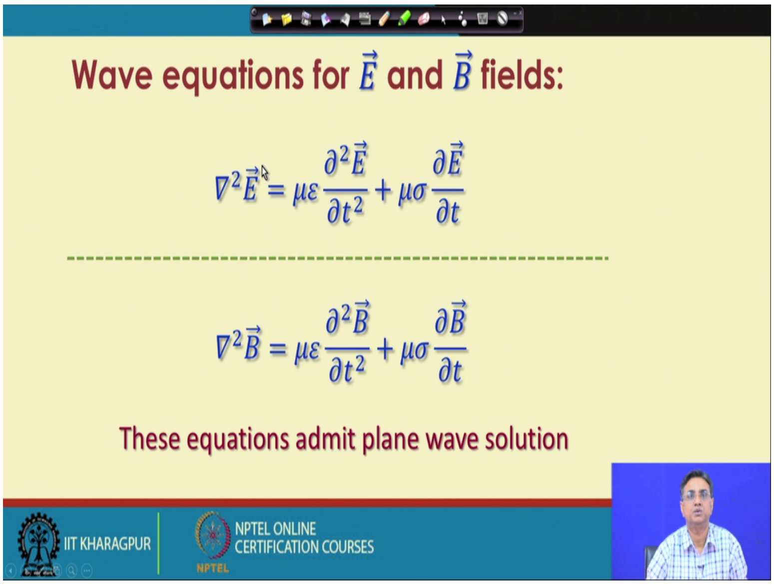

So, equating these two, we can write this equation del square E equal to mu epsilon del square E del t square plus mu epsilon mu sigma del E del t, this is for the electric field. And if we proceed by taking the curl of the forth equation; that is del cross del cross B del cross del cross B, then for del cross E if I write this and also for del cross E if I write this then we can arrive at the this equation del square B equal to mu epsilon, which is a similar equation except that E is now replaced by the B vector. These equations, if we look at these two equations these equations admit plane wave solution.

So we will try, we will look for the plane wave solution and we will try and look and on substitution of this plane wave solution into these equation, we will see what is the nature of the propagation constant, what is the amplitude of the electric field and the magnetic field and is there any phase relation between the electric field, and the magnetic field.

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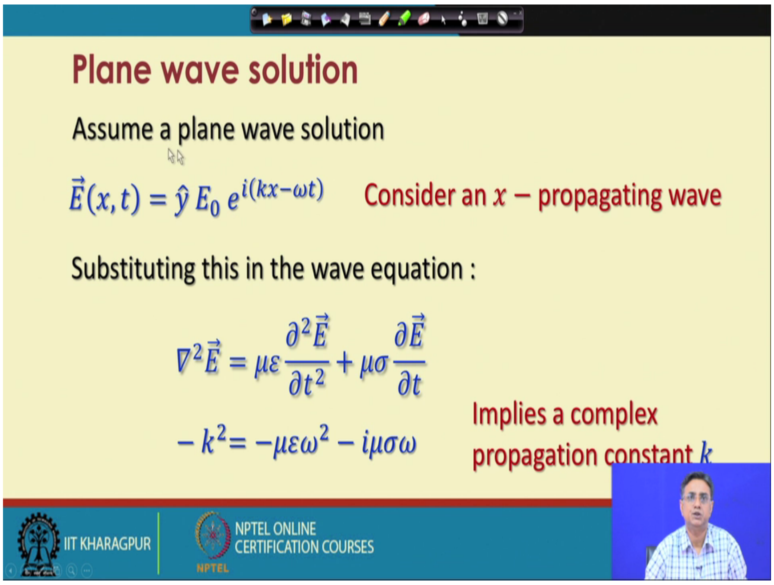

Accordingly, we assume a plane wave solution consider and x propagating electromagnetic wave. So, in that case we can write this electric field it is propagation along x as a function of x, but it is polarized in the y direction. So, we can write this equation y cap E naught E to the power of i k x minus omega t. Substituting this equation into the wave equation that is del square E equal to mu epsilon, del square E del t square plus mu epsilon del E del t. We get that for an operator del square E operating on this, this equation plane wave equation we get i k and i k twice so that gives you minus k square. And operating twice that is taking the time derivative twice will take out i; minus i omega twice, so that gives you minus omega square multiplied by mu epsilon, and del E del t will give you i into omega times mu into omega. So, this equation when substituted with this solution, then we get that minus k square equal to minus mu epsilon omega square minus i mu sigma omega all of them are having a minus sign.

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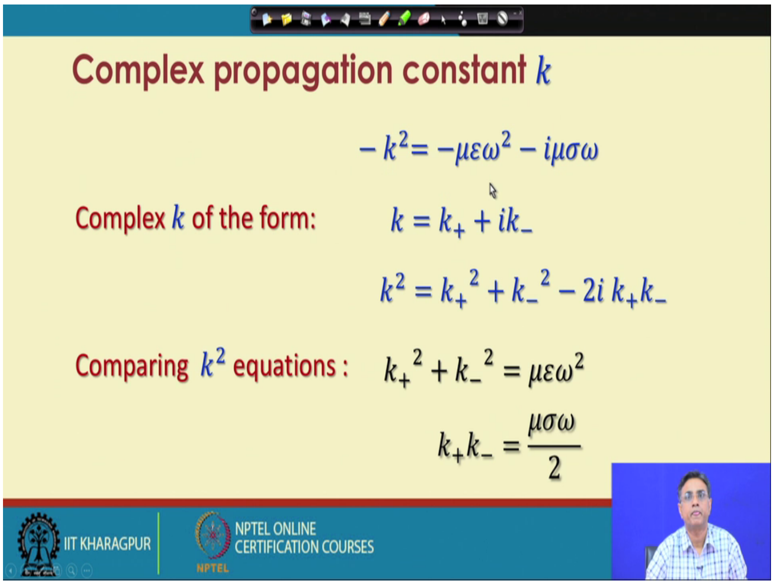

Now, it tells you that the square of the propagation constant is also imaginary. That means, the complex k of the form you can consider that k is k plus plus i k minus and if you take the mod of this k square; k square will be equal to k plus square plus k minus square minus twice i k plus k minus.

So, you have been able to write the imaginary part and the real part for k square and now we can compare these two equation k square equal to this also k square equal to this. And on comparing we get that k square plus plus k minus square is equal to mu sigma omega square, whereas k plus k minus the product of these 2 quantities; the real and imaginary part of the propagation constant will give you mu sigma omega by 2.

So, we have been able to see the complex propagation constant whose real part is mu epsilon omega square and the imaginary part is this. Of course, this is the real and imaginary part of the k square quantity.

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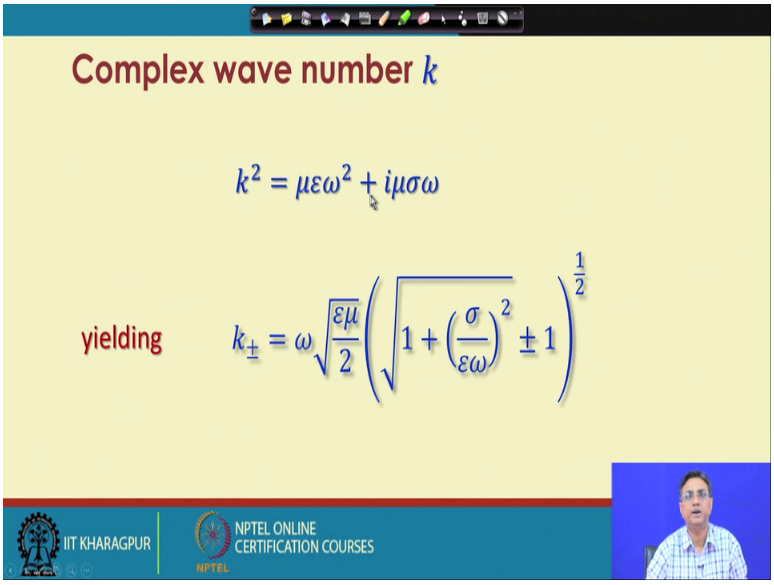

So, k square equal to we can write that mu epsilon omega square plus i mu sigma omega. So, this on solving because I have these equations k square plus k square minus then k plus k minus. So, I can use this we can 2 and then you minus into this, then we can add, you can frame k plus minus k minus square.

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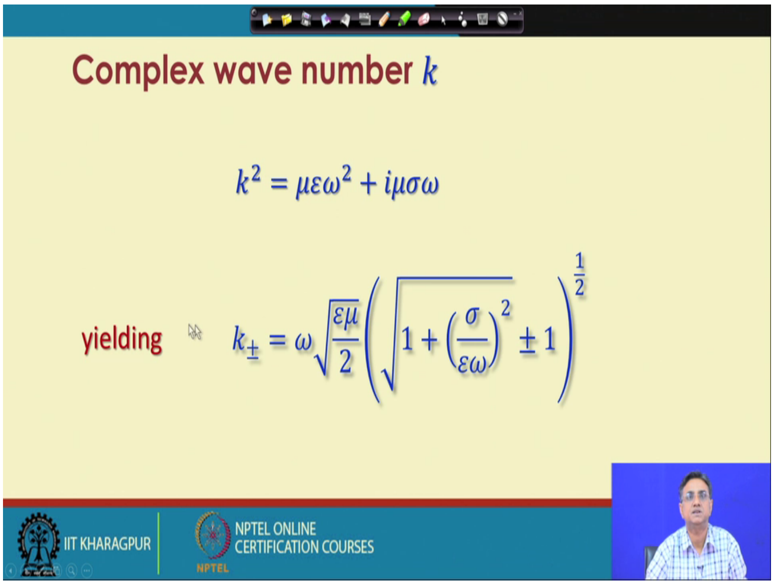

Which will yield the solution of this k plus minus; the k plus and k minus put together will give you the solution of this equation is a quadratic equation whose solution will be this. So, once it is plus 1; next for the plus value of k and minus 1 for the minus value of k.

So the plane wave solution; having known that the k the propagation constant has 2 values k plus and k minus and can be written as k equal to k plus plus i k minus. We can write the plane wave solution..

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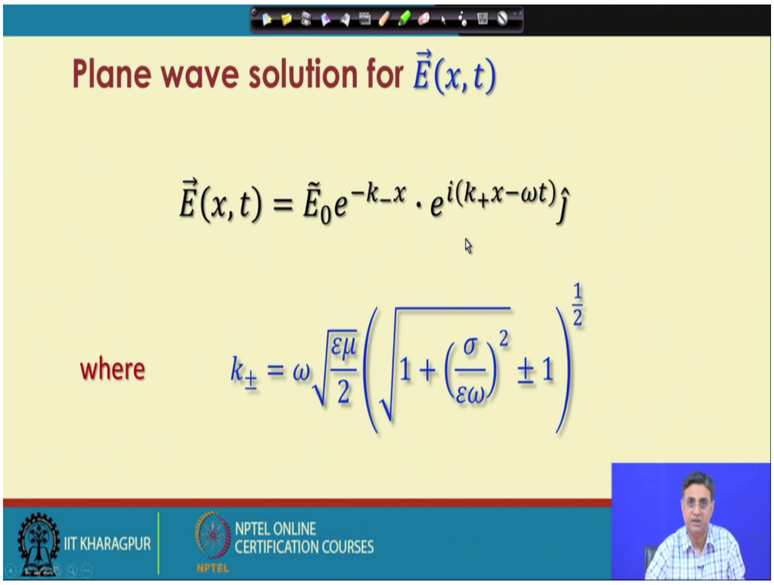

we can rewrite in fact, the plane wave solution as: E 0 E to the power of minus k minus x into E to the power of i k plus x minus omega t. In fact, you can see that this part accounts for the phase k plus is the real part of the propagation constant, k minus is the imaginary part and this imaginary part has given you the attenuation of the amplitude E 0 E to the power of minus k minus x.

So, it tells you the amplitude of the electromagnetic wave which is traveling along the x direction in a conductor decays as E to the power of minus k minus x. So, this k minus k plus and k minus these 2 quantities are given by this which are the functions of omega epsilon mu and sigma. So, all these 4 quantities are going to decide the k plus value and the k minus value that is how what will be the phase instantaneous phase and what will be the loss in the amplitude of the electromagnetic wave in the conducting media.

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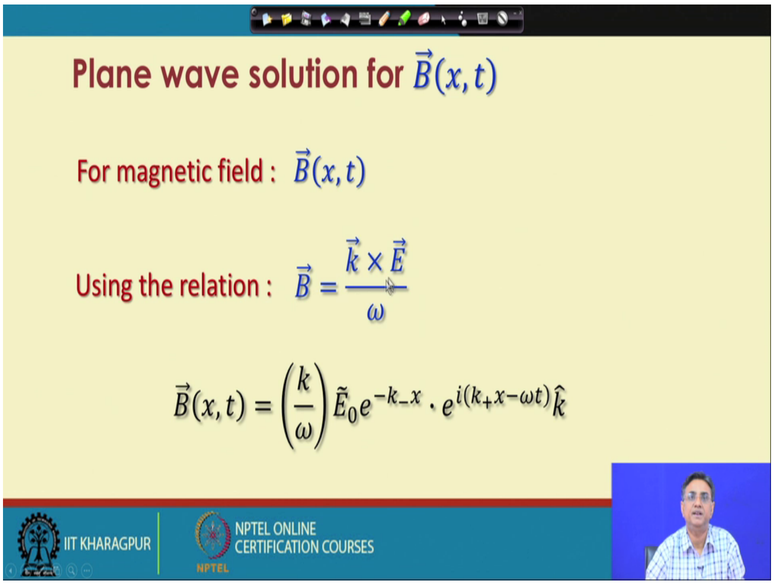

For the magnetic field we know that B equal to this can be connected to the electric field through this equation. Now, since k is complex B and E, they are connected by a complex quantity; that means the phase related with this k will also be attached to the phase of this electric field and in general electric and magnetic field will have a phase difference. And writing this B field in term using this electric field value known; we can write in this equation. So, k by omega which is a consequence of this equation; so, I am trying to write this electric if the magnetic field in terms of the amplitude of the electric field, the attenuation factor of the electric field and the phase of the electric field, but this is a complex amplitude.

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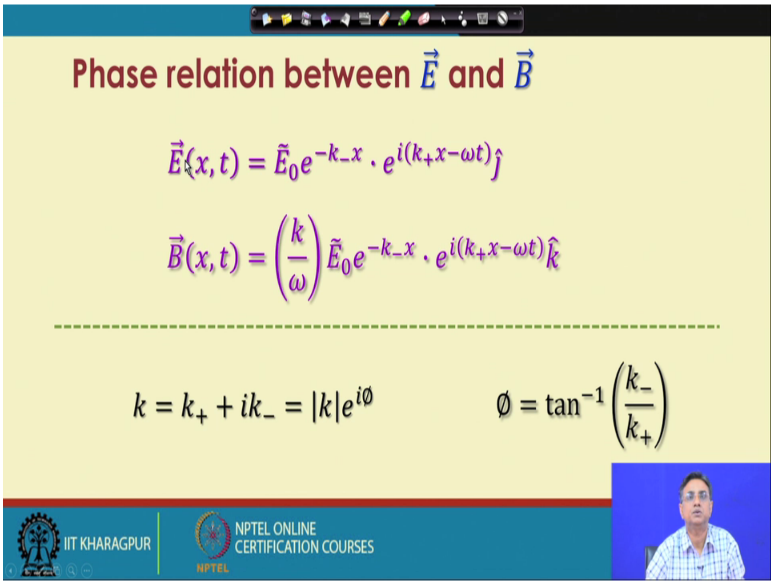

So, let us look at this. So, we could write the electric field as this and magnetic field in the same way k by omega and the electric field quantity. But you can see that they are oriented in a mutually orthogonal direction this is along the y direction, this electric field direction is y that is, and y polarized the electromagnetic wave, and the magnetic field is along the z direction.

Now, this k plus and k minus these 2 quantities which constitute the complex propagation constant can be written as k mod into E to the power of i phi. Where, phi is the phase of the phase factor at associated with this propagation constant, and this phi the value of the phi will be equal to tan inverse k minus by k plus.

So, this is the phase that is associated with the propagation constant, and we will see how this phase is going to be related to the phase of the electric field and the magnetic field. Now we could write k equal to k E to the k mod E to the power of I phi with phi equal to this and this is the relation which connects electric field and the magnetic field k. Therefore, k decides the phase of B with respect to E.

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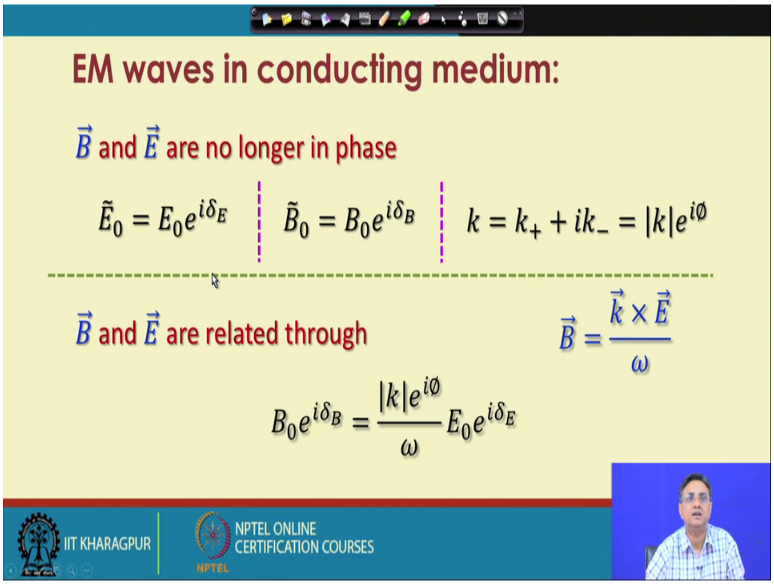

So, let us see that we can write this B and E in this form as we have seen B and E their no more in phase. So, we attach different phases delta E for the electric field and delta B for the magnetic field, this. And we also remember recall that k has a phase of E to the power of phi. So, you have three quantities electric field has a phase of delta E, magnetic field has a phase of delta B, and the propagation constant has a phase of phi.

So, B and E are then related through this equation, because B equal to k E by omega. So, because k and E they are at right angles, so k into E by omega if you look at the magnitude, therefore B is equal to k. Then E by omega into E to the power of i phi is the phase for this k and E to the power of I delta is the phase due to the electric field. So, this is the way the electric field and the magnetic field they are related.

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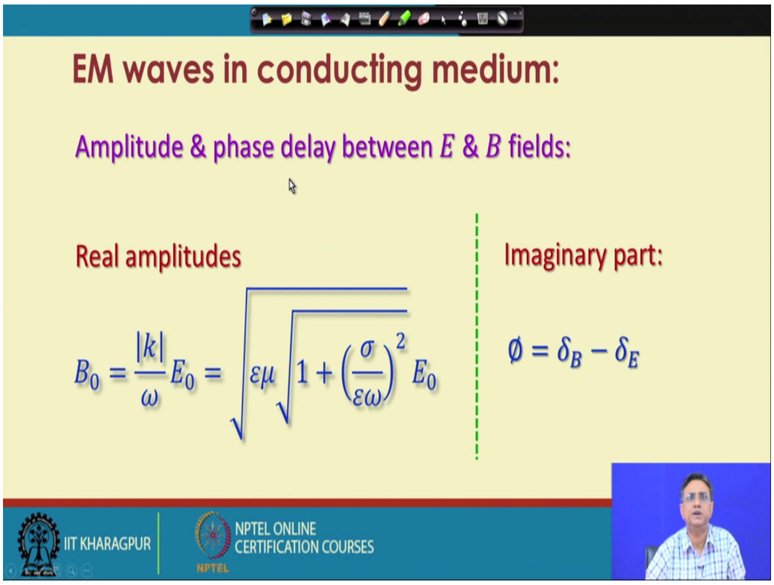

So, the amplitude and phase delay between E and B fields, you have this real part of the amplitude of the electric field, and the magnetic field they are related in this way where the value of k and omega we have to consider. So, this is the connection that this gives you the value of k by omega, and the imaginary part that gives you the phase difference between the magnetic field and the electric field.

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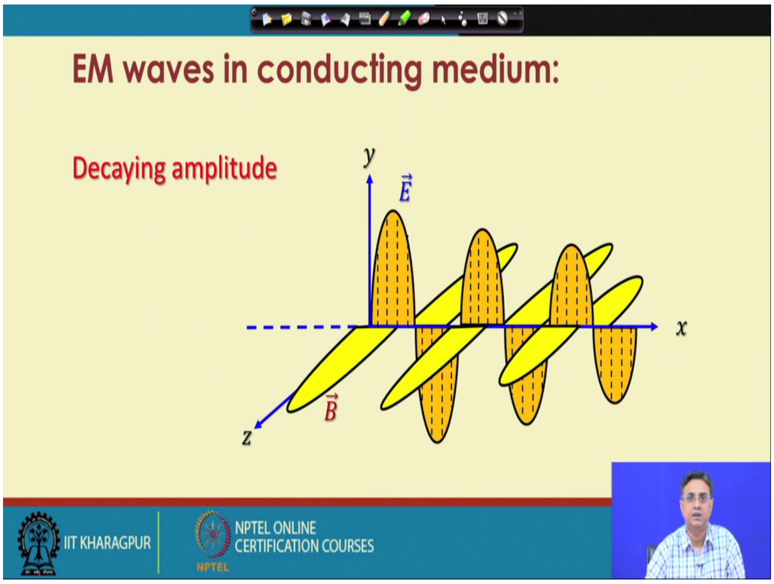

So, through this figure we tried to depict how the electromagnetic wave is propagating in a conducting medium. You can see that the field amplitude decays as it moves along the x direction and also the electric field and the magnetic field they have a phase difference as it propagates along the conducting medium.

So, these are the electric field directions and these are the representation for the magnetic field. So, it shows that the electric field and the magnetic field the decay as it propagates through a conducting medium. This, that means, that an electromagnetic wave cannot propagate cannot penetrate a conductor, it only penetrates wont up to a certain depth which is referred to as the skin depth and will try to quantify the skin depth of electromagnetic waves in a conductor.

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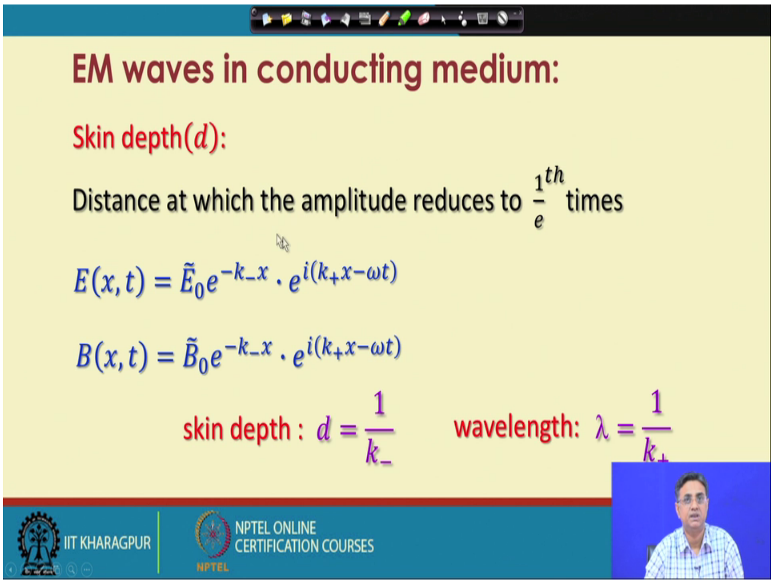

So, in a conducting medium the skin depth is the distance at which the amplitude reduces to 1 upon E times the original amplitude. So, within the before original ampli by saying original amplitude, I mean that before it heats the conducting medium or within a within a point at a point within the conducting medium the field if it is E 0, then after the skin depth the field will be E 0 by e. So, the electric field in a conducting medium the plane wave solution is E 0 complex amplitude, and this is the decay factor, and this is the phase factor, and in the similar fashion the magnetic field has a complex amplitude and a decay factor, and a phase factor.

So, the skin depth is to account for this quantity that is by the distance over which this E 0 the amplitude falls to one upon falls to E 0 by E one upon E times E 0 that is I have to put this value to be equal to 1. So, k x into d will be equal to 1 that gives you d equal to 1 by minus k. So, this is the skin depth. This k plus and k minus they have different significances k minus gives you the skin depth, whereas, k plus accounts for the wavelength of the electromagnetic wave. So, twice pi by lambda equal to k plus we can write in this form also by putting twice pi equal to 1.

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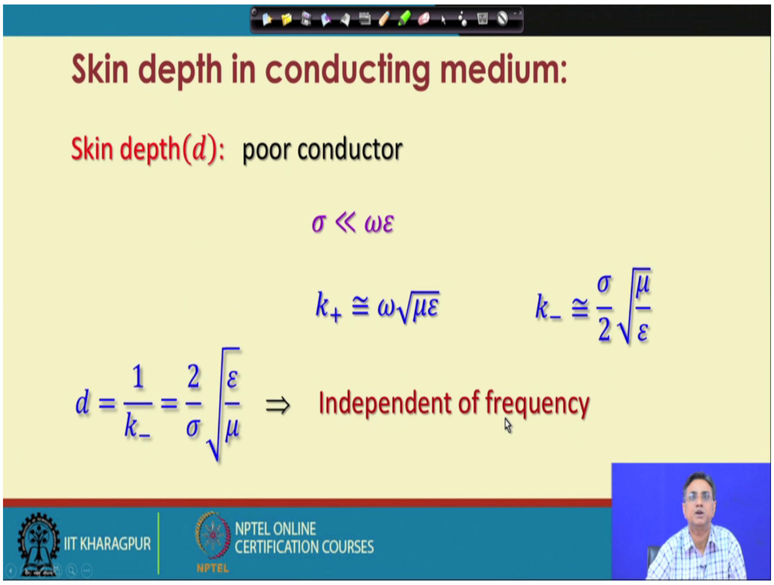

Now, for a poor conductor: for a poor conductor the skin depth can be written as to account for the skin depth for a poor conductor we have sigma which is much much less than omega epsilon so k plus becomes approximately equal to omega under root mu into epsilon k minus will be approximately equal to sigma by 2 under root mu by epsilon. You can see that the skin depth d which is 1 upon k minus is equal to 1 upon k minus if you right this will become the reciprocal of this quantity, that is 2 by sigma epsilon by mu this is this does not involve the frequency of the electromagnetic wave that is it is independent of the frequency. For a poor conductor the skin depth does not depend on the frequency of the electromagnetic wave that is trying to travel through the medium.

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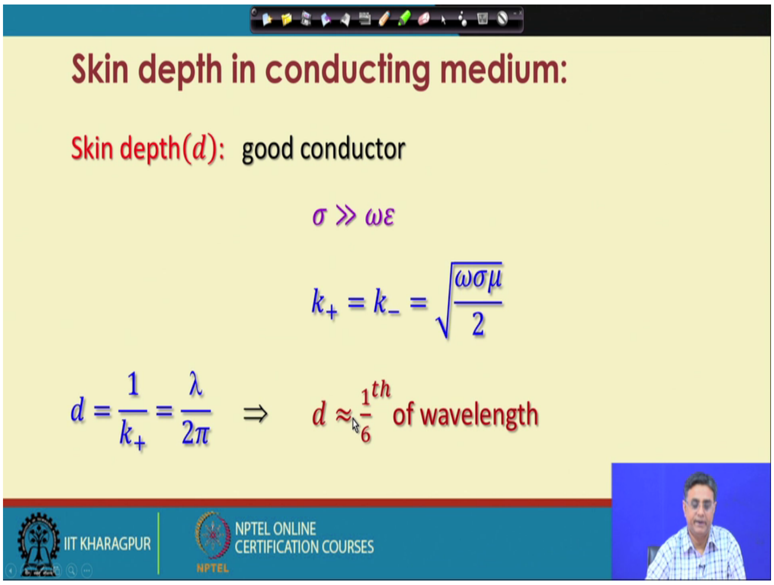

In the case of a good conductor, we have sigma which is much much greater than omega epsilon. This is the way for define this good conductor that is conductivity is very high. Then this k plus will be approximately equal to k minus, because you can ignore that one in the solution of k plus and k minus, then both of them are approximately same.

So, you can write this k plus or k minus is equal to this; therefore, the skin depth d which is the reciprocal of k plus will be equal to lambda by twice pi and d equal to one sixth of the wavelength. So, in agood conductor the electromagnetic waves only travel through one-sixth of the of the wavelength distance. The energy density of an electromagnetic wave is given by this we have seen that.

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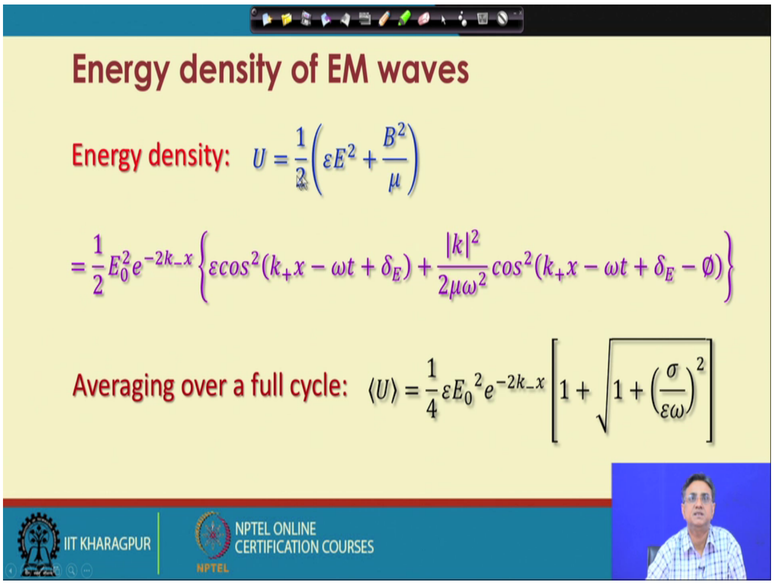

So, if you substitute in the sin cosine solution of the electric field and magnetic field. We can rewrite this equation in this form and then you can calculate the average energy density associated with the electromagnetic wave is given by this.

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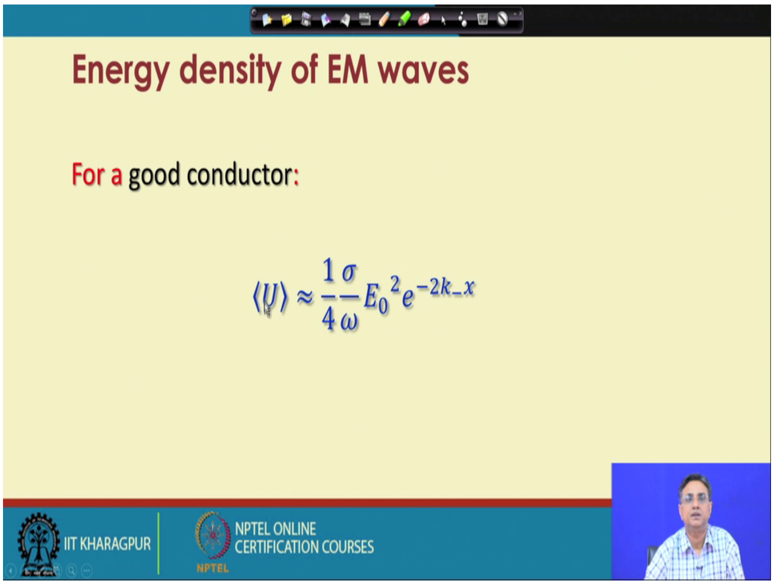

So, for a good conductor, you can calculate the average energy density associated with the electromagnetic wave is equal to this.

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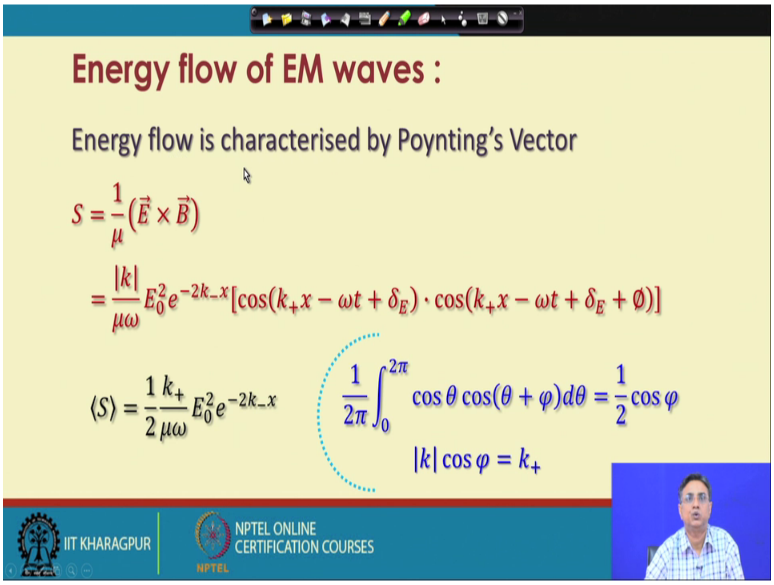

And the energy flow is characterized by the pointing vector that is s equal to 1 upon mu E cross B and if we evaluate this expression where for a cycle the integration is equal to half into cosine phi. In this case, it is not equal to half this is, because there is a phase difference between the electric and magnetic field which is like the power factor. So, it appears in the expression of k plus as k into cosine phi. So, we can calculate the pointing vector in this case as well.

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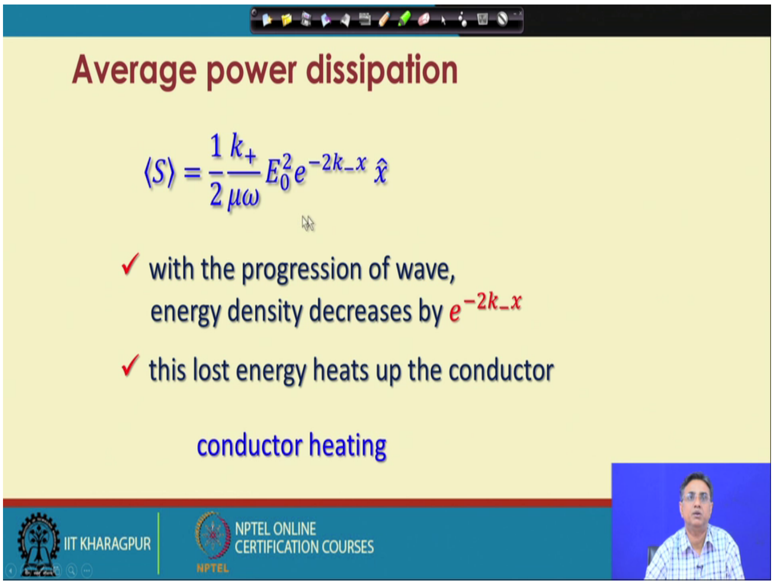

So, the average power; we can calculate in this form with the progression of the wave the energy density decreases with a factor of this and this lost energy heats of the conductor which you call the conductor heating.

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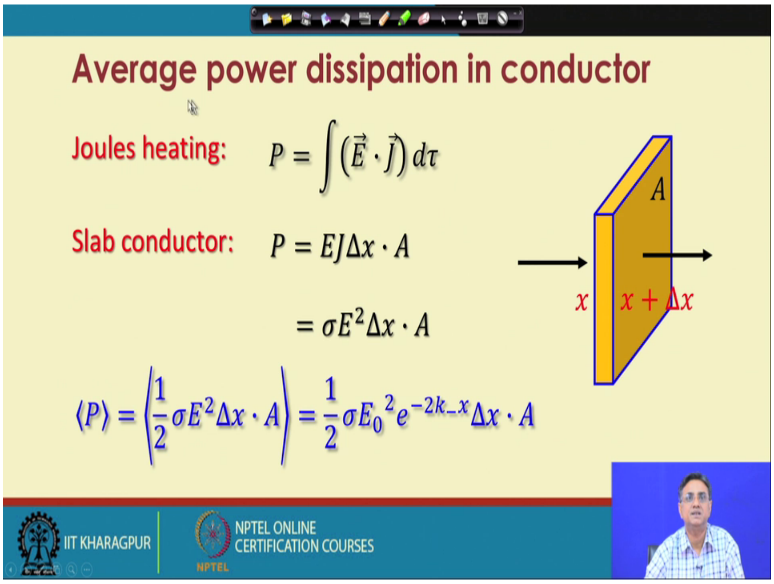

Next will consider the average power dissipation in a conductor; let us consider a slap in the slab; this power we can write in this form that E into J delta x into A, J is equal to sigma E square. So, we can rewrite this equation. Now if you calculate the power; average power, then we can write this equation.

Now, let us consider the average power that is flowing in to the slab, we have calculate the expression for average power whether it is going in or going out will be decided by this factor.

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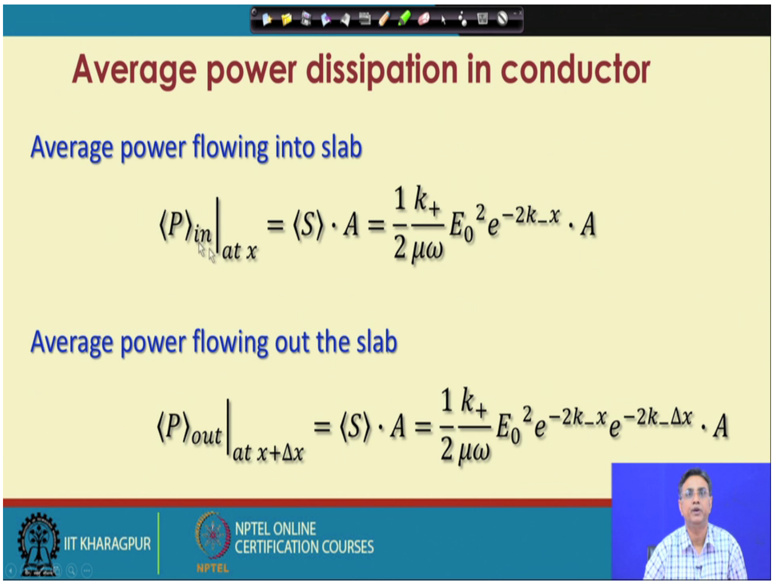

So, power flowing in will be given by this into A whereas, the power flowing out from the slab this k minus k into x will be replaced by twice minus k into x plus delta x. So, this factor will account for the thickness of the slab.

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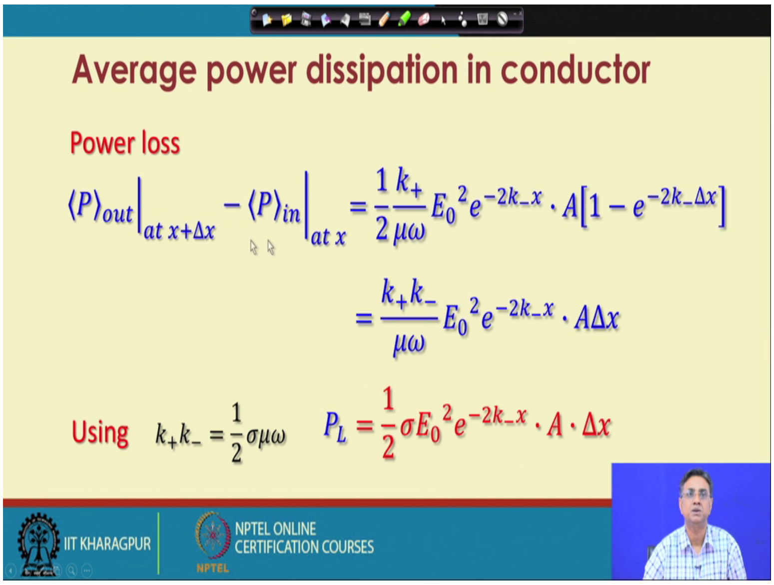

So, the difference between the power that is going in and going out will give you the power that is lost within the conductor. If we evaluate this equation, you can rewrite this equation in this form using k plus k minus equal to this; which we have seen in the solution for k, so you can calculate the power loss is equal to this.

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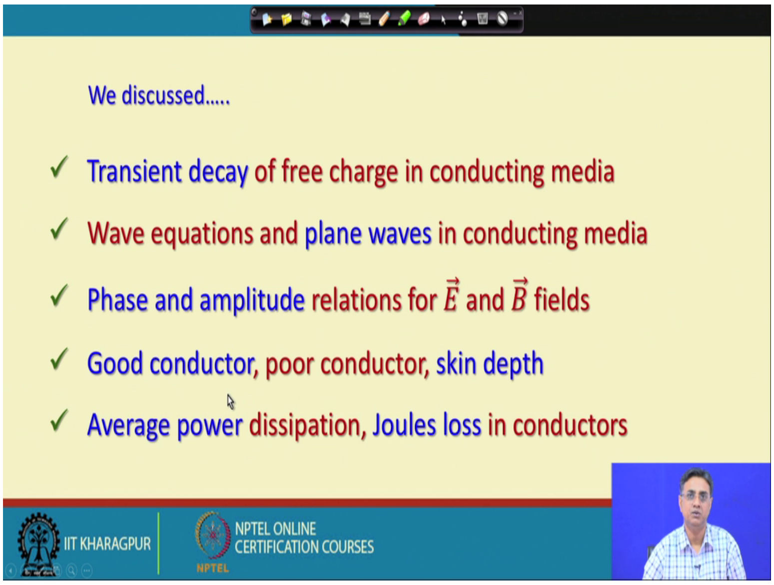

So, this is how we have discussed the transient decay of the free charge in conducting medium wave equations and plane wave solutions in a conducting medium, phase and amplitude relations for the electric field and the magnetic fields.

We have also classified the good conductor and poor conductor. And we looked at the skin depth for good conductor and poor conductor. For a good conductor, it is only the one-sixth of the wavelength of the electromagnetic wave, whereas for a poor conductor, it does not depend on the frequency. And we also have calculated the average power dissipation through a conducting medium which actually accounts for the joules loss in the conductor.

Thank you.