Mathematical Methods in Physics-I
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Lecture – 20
A: Linearly Dependent-Independent Function (Cont.)
B: Inner Product Of Functions

Ok. So, welcome back student, in our last class we stopped something wronskian the concept was straightforward.

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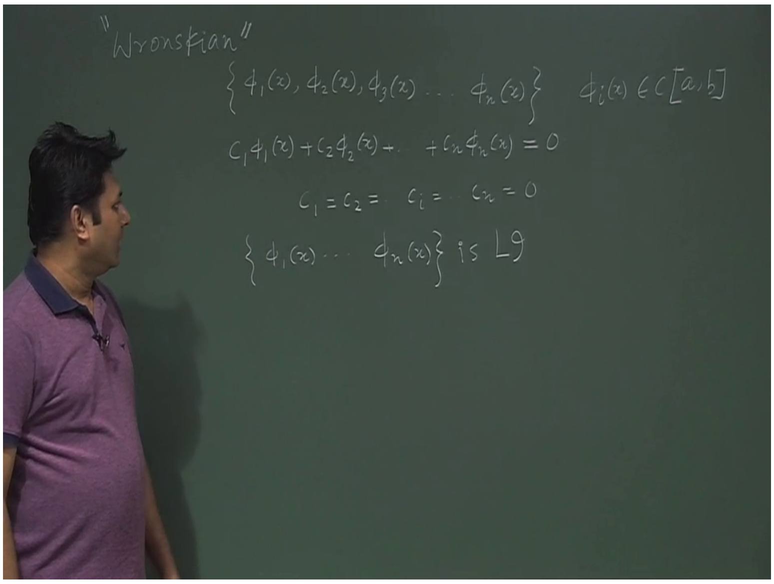

We wanted to find out how to, how we get a set of linearly independent function, how we know that a given set of linearly independent a given set of function is linearly independent then wronskian is a method. What was the wronskian, say let me remind you once again say this is a set of functions phi 1, phi 2, phi 3 set of functions, in general it belongs to the continuous function in the closed interval ab.

Now, the question is if I make a linear combination of this function and say the linear combination, I make the linear combination in such a way that this is equal to 0 then if the only condition for that is ci is equal to ci, c2 is equal to ci is equal to cn is equal to 0. If this is the only condition then we can safely say that the functions the set of functions is linearly independent li. Now this is a last day we described all these things elaborately so no need to discuss it further also we define something called wronskian and the wronskian is if the set of function phi 1002C phi 2 is given the wronskian is nothing, but the determinant of this quantity.

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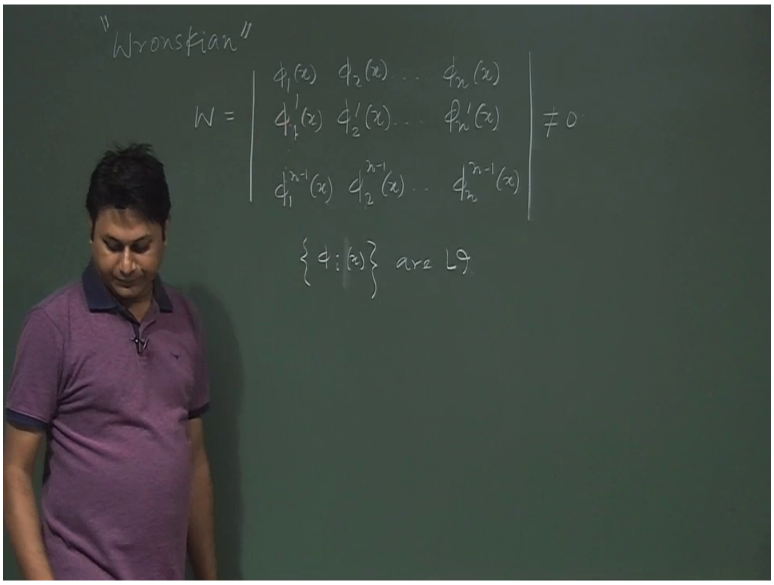

Then enter this quantity phi 1, phi 2, phi 3, phi n is the set of function that is already given to me.

Then phi 1, phi 2, prime phi 3 prime and phi n prime is a first order derivative of the given set of function and I can make a matrix like form by making all the functions with the derivative up to n minus 1. So, it will be a n by n matrix form, now I try to find a find out what is the determinant of this quantity if this is not equal to 0 then the condition is the set of function phi i better to put a x inside function of x are linearly independent functions by definition. Now, go back to some examples I know that, now I will try to find out some examples, say examples 1.

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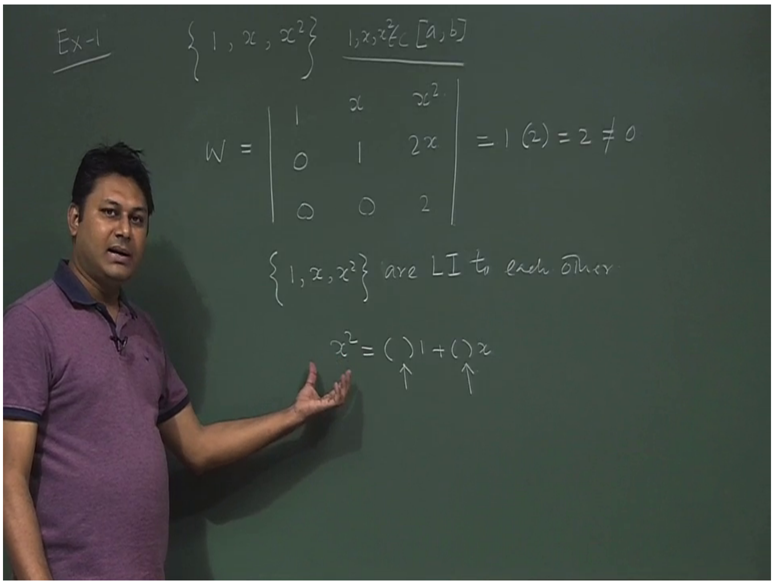

Let us take a set of function 1, x, x square 3 function 1, x, x square both are continuous in the limit, some limit a to b. So, 1, x, x square belongs to this, now I want to find out whether this function this set of functions is linearly independent or not. So, my wronskian I will like to calculate, it will be 1, x, x square derivative of this entire row whatever we have. So, derivative of 1 with respect to s is 0, then derivative of this quantity will be 1 derivative of this is 2 x then I need to do the derivative once again because this is 3 functions that is given; that means, the order is 3, if it is given up to 3 then I need to go up to 2 order of derivative.

So, here again if I do the derivative it will be 0 it will be 0 and this final value will be 2, this is the quantity I have this is my wronskian as simple as that, what will be the value here it will be simply 1 multiplied by 2, which is 2 which is not equal to 0. So, the wronskian of this entire given set of functions is not equal to 0 that essentially means the set of functions 1, x, x square are linearly independent to each other. You should also appreciate the fact that what is the meaning of linearly independent set of function; that means, one function any one of this function cannot be written in terms of other functions.

So, I cannot write for example, x square in terms of this into 1 plus this into x it is not possible that I can write these things if I write this then this and this coefficient if I have some non 0 coefficient here and here then; obviously, this function is linearly related to other functions there is a linear combination. So, these 2 functions these functions can be represented in terms of this and this function with non zero coefficient then these things will be linearly dependent in their things.

So, linearly independent thing will not be there it will be linearly dependent, but here I cannot write this because my treatment or my recipe suggests that whatever the function is given here set of functions are not linearly dependent to each other because I cannot write any of the function in terms of any other 2 functions with non 0 coefficients here.

Let us go to another example this is example 1. So, I will go to another example say 2 example 2.

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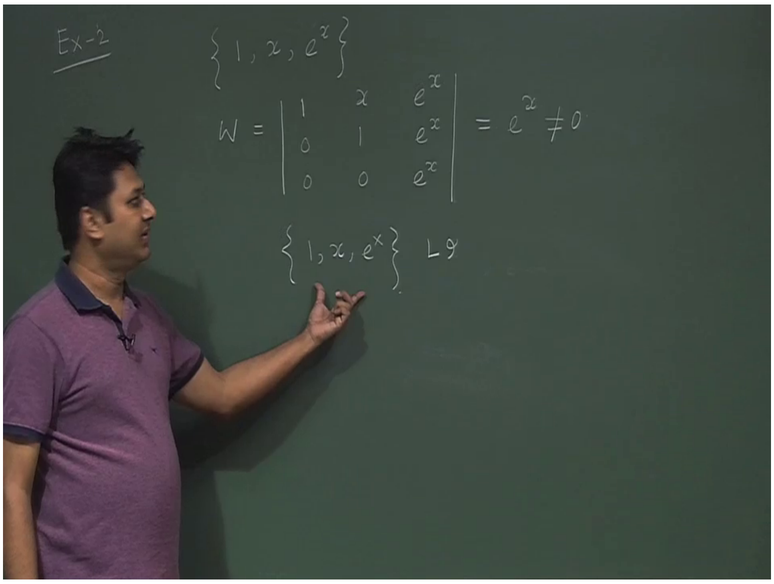

Let us take the function 1, x, e to the power x what about this can I write can I say this is a linearly dependent or independent functions, my wronskian will be 1, x, e to the power x. First row I need to put all the functions as it is in the second row I need to make a derivative of this quantity, e to the power x derivative we remain e to the power x no problem with that, 0 0 it will be e to the power of x.

Now, if I calculate the determinant of this quantity you will find that 1 if I go with this column these 2 are 0. So, I should not bother about calculating this. So, the first 1 is say it will be e to the power x now; obviously, it is not 0 because it is e to the power x that is 1 issue, second thing even if I have some other quantity which is which is a function of x here then I can consider is as not equal to 0 because the determinant has to be 0 uniquely without putting any kind of value in x if it is a function of x. So, it will not be 0 of any value of x, here we have e to the power x in our hand. So, no problem with that e to the power x will never vanish because it is whatever the I mean never vanish for any value of x.

So, we should consider this as a not equal to 0 so; that means, again 1 x e to the power x this set of functions, this set of function is linearly independent again I have a set of functions which are linearly independent. So, now, we let us take another example you need to practice this with different functions, I am taking only 3 functions you can extend these functions.

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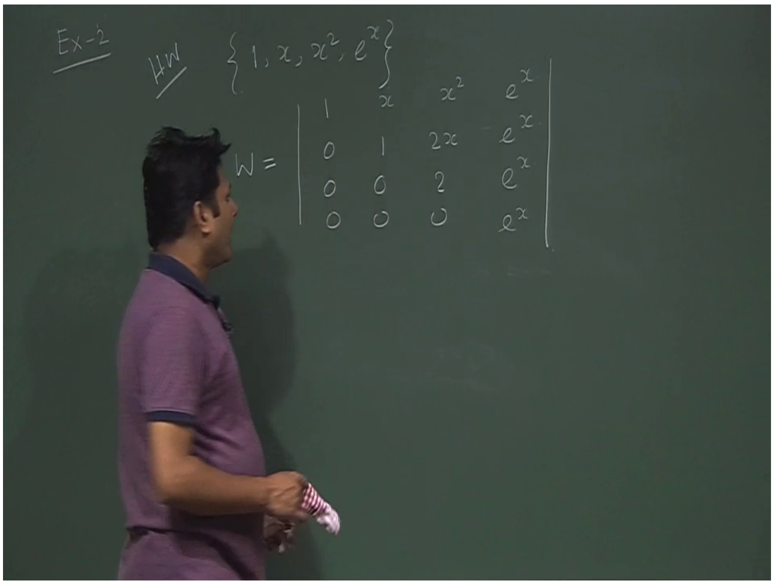

For example you can try out with this bit, let me put it as a homework you can try out with this function 1, x, xa square, e to the power x the procedure will be same let me help you a bit 1, x, x square e to the power x you need to put here then 0, 1, 2 x, e to the power x derivative of 0 0 derivative of this to e to the power x and 0, 0, 0, e to the power x.

So, instead of having 3 by 3 I have a 4 by 4 matrix, now if I take the determinant of this entire quantity I will then I need to calculate the whether it is 0 or not since it is 4 by 4 you need to calculate the determinant meticulously and you will see that again it will be not equal to 0 so; that means, this set of functions is again linearly independent, let us try with these functions also this is another functions.

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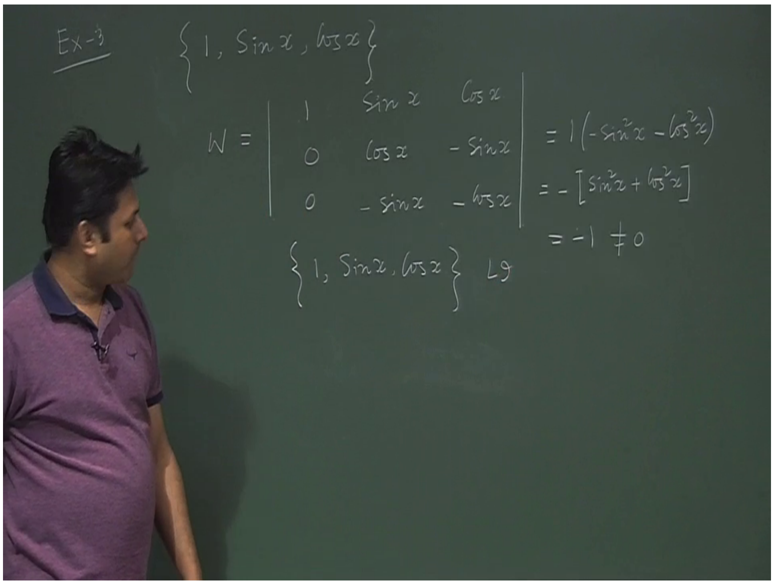

Another set of function 1 sin x cos x this is another set of functions 1 sin x cos x.

So, wronskian can be represented by 1 sin x cos x 0 the derivative of sin x is cos x, the derivative of cos x is minus sin x and 0 cos x is minus sin x, minus sin x is minus cos x, then I have 1 multiplied by minus sin square x minus cos square x or minus sin square x plus cos square x which is minus of 1 or not equal to 0; that means, 1 sin x cos x is a another set of functions which is also linearly independent.

So, 1 sin x cos x they are linearly independent. So, all the set of functions we are taking seems to be linearly independent that does not mean that there is no functions which are linearly dependent. So, let me change this thing to sin square and cos square.

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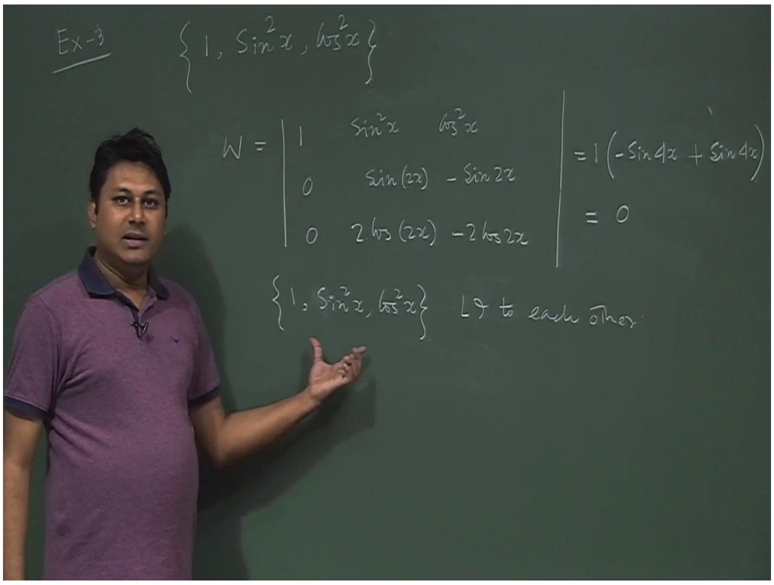

So, I have also another set of functions like this instead of having sin x I put it sin square x instead of cos x I now put cos square x, wronskian again 1 sin square x cos square x 0, sin square is 2 sin x, cos x, cos square is is 2 cos x sin x with a negative sign, this quantity whatever the quantity we have is nothing, but sin of 2 x sin 2 x is 2 sin x cos x.

So, it is sin of 2 x so let me replace this as sin of 2 x, again this quantity is minus of sin 2 x now if Imake a derivative once again it will be 0 this quantity is cos of 2 x with 2 and this quantity is minus of 2 cos of 2 x sign derivative 2 and then the 2 will be there now if I calculate the wronskian then I can have something. So, it will be 1 and then if I take 2. So, it will be the first term will be sin x multiplied by cos 2 x with 2. So, it will be minus of sin 4 x something like that minus the same quantity.

But now I should have a plus sign, this minus will be added with the minus sign there and it will be plus of sin 4 x or it will be 0. So, now, I find that this is a set of functions whose wronskian is 0 when; that means, the set of functions 1 sin square x cos, square x these set of functions we have a set of functions these set of functions is linearly independent to each other. Why this is linearly independent now you can readily understand because I can write any of the function in terms of other for example, I can write 1 is equal to sin square x plus cos square x as simple as that.

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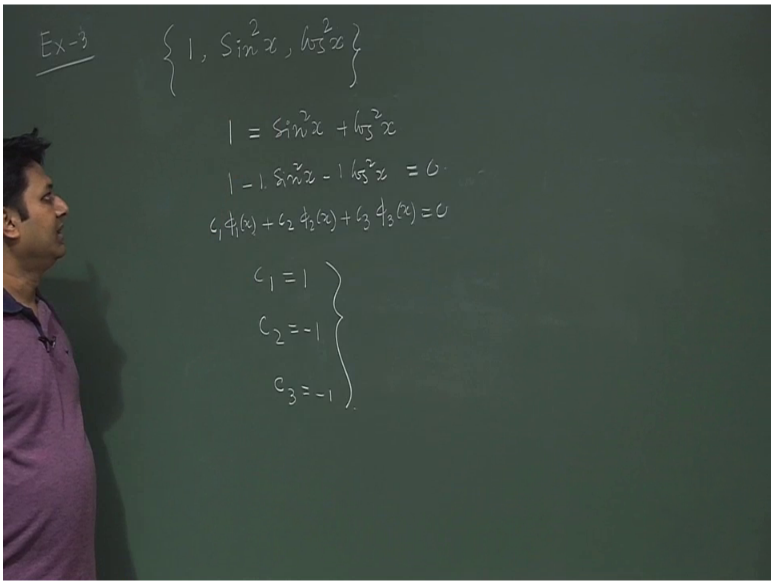

That means I can write if I make a linear combination then 1 minus 1 into sin square x minus 1 into cos square x is equal to 0, I can make a equation with no problem; that means, phi 1 x c 1 plus c 2 phi 2 x plus c 3 phi 3 x is equal to 0 please note that this is 0 even if it is 0 the c 1 c 2 c 3 is not equal to 0, but they have some values non 0 values c 1 is here 1 c 2 is here minus 1 and c 3 is here minus 1.

So; that means, I have a non trivial solution here for this equation and since a function here 1 can be represented in terms of other 2 then we can readily say that these set of functions whatever is given is linearly dependent to each other . So, now, we are try to do something if you, if you note carefully that in the vector space what we are doing just changing, in the vector space whatever the concept we have in terms of vector we just replace this or replicate the same thing in function space; that means, in vector space we have a concept of linearly independent and dependent vectors. Here we mimic the same concept with the same definition we try to find out whether a set of function which behaves like a vector can be linearly independent or not.

So, you can appreciate that then the inner product which is defined there also can be represented here. So, let me remind you the inner product say e 1, e 2 in vector space.

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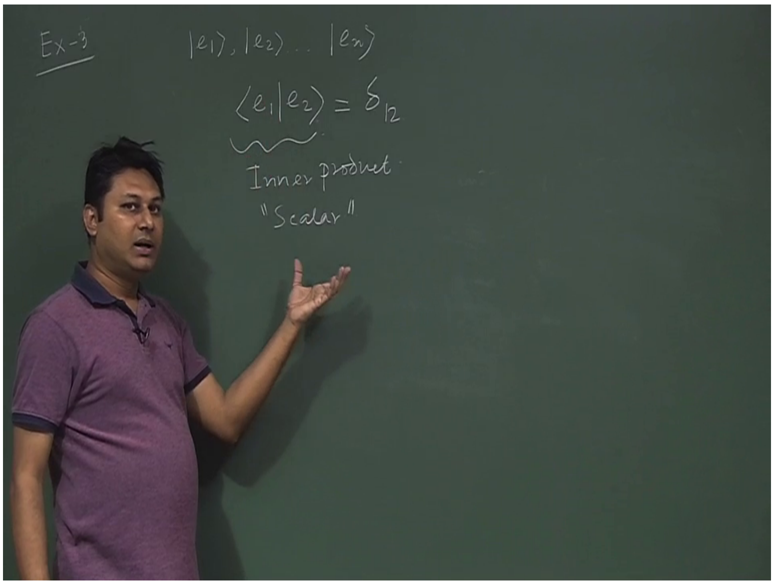

We have this quantity, these are the vectors I put in the vector sign k th sign. So, these are the vectors and these are the basis you remember that these was something like a delta 1 2 if they are normalized and if they are orthogonal, but the important thing is that this quantity was something inner product. That is the first thing you note you I believe you remember how this notation appear and this appear because of the fact that we are taking these things from the dual space and when we are taking these things from the dual space we need to change the dimension and all these things elaborately we have explained these things.

Now, the important thing is that inner product whatever the product I have here this has to be a scalar quantity this is a simple scalar. So, I have a vector quantity, I operate something over that I am getting something scalar. So, here also in function space we can define something, this is in vector space in function space we can define something.

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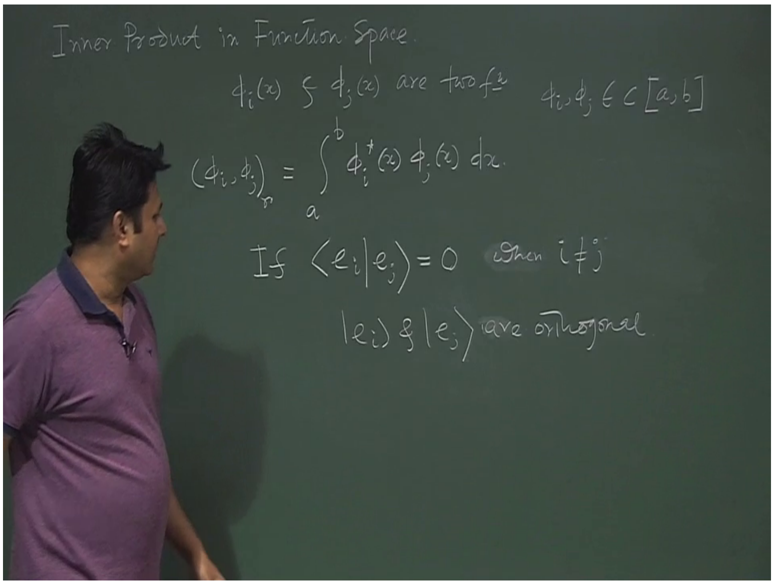

So, inner product, inner product in function space. So, now, I need to define what is the meaning of inner product in function space, say phi i x and phi j x are 2 functions both belongs to phi i and phi j, both belongs to the continuous the set of continuous function in the closed interval ab, phi 1 phi i and phi j both are continuous and then in some interval ab, these 2 functions like these 2 vectors.

So, I can define it is defined in this way phi i phi j in the bracket, this is the definition of the inner product in function space in normally books they give in this way this is the definition, it is quite same as if you try to match you with the vector space and the vector space the definition was something like this here it is with the bracket. It is given a to b definition phi i star x, phi j x in general rx dx, now this is interesting. I have 2 functions in my hand phi i and phi j, I am saying that these 2 functions are the 2 elements in the function space I want to define the inner product of these 2 functions and I am saying that by definition the inner product is this multiplied by this and I will take the integration a to b over dx with a function r multiplied by, this r function is normally called the density function or weight function, density function or the weight function, this is called density function or weight function.

But in many cases we normally take these as 1 to life to make life simple. So, rx is nothing, but 1 if I define this in terms of the density function I need to put a r as suffix here that my inner product is defined in this way with r. Now forget about the r because I am taking r equal to 1 which is always we take then my definition simplified become more simplified and I have this, please note 1 star is here in this case if you remember when we make a dual conjugation then we need to make a star of that.

Here what we are doing the similar kind of things we are making the integration by taking a star, but if phi 1, phi 2 they are real functions then we do not need to put this star here that is 1 issue second thing also you can take you can note that this integration gives you some scalar. So, whenever you have a function this is a variation and when you make a integration from some this limit to some up lower limit to some upper limit it converted to a scalar so; that means, whatever the definition I am giving this definition is obeying the fundamental rule of inner product that inner product is something which can give you as a result a scalar quantity or some numbers.

So, here we will getting something like this. So, by definition we have this is my inner product fine. So, now, I know what is the linearly dependent and independent function, now I know what is the meaning of inner product when I know the meaning of inner product then; obviously, the next thing we need to know that what is the value of this inner product because if in vector space if say ei, ej was 0 when I is not equal to j. That means, ei and ej where orthogonal, are orthogonal rather this is from vector concept we already know that 2 2 vectors are there if I make a inner product of 2 vector if I get it is 0 then they are orthogonal to each other; that means, 2 vector if they are perpendicular to each other if I make a dot product of that we will get 0.

Here also we can define the thing in this way, we can define the thing in this way.

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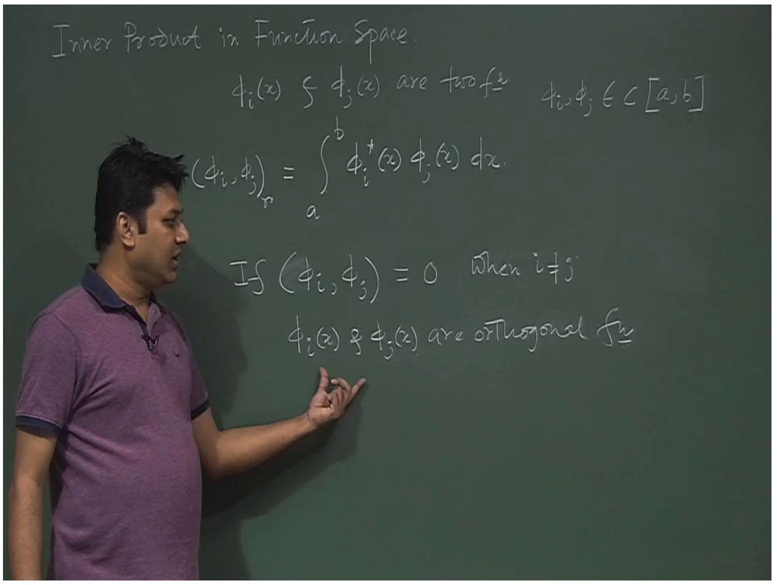

So, if phi I, phi j this quantity is equal to 0 when i is not equal to j then these 2 functions phi ix and phi jx are orthogonal functions, orthogonal functions. Exactly the same definition exactly with the same concept, in that case 2 vectors who are associated with that which is perpendicular to each other. Here we are doing the same thing, but taking the functions and by definition I can put a inner product and this inner product suggests that I have a function I have a number by doing this operation which may be 0.

If this operation is 0 this quantity is 0 with i is not equal to j then these 2 are related to each other and this relation is they are orthogonal to each other, these 2 functions are orthogonal to each other well at this point also I can say something this is the inner product of 2 different functions.

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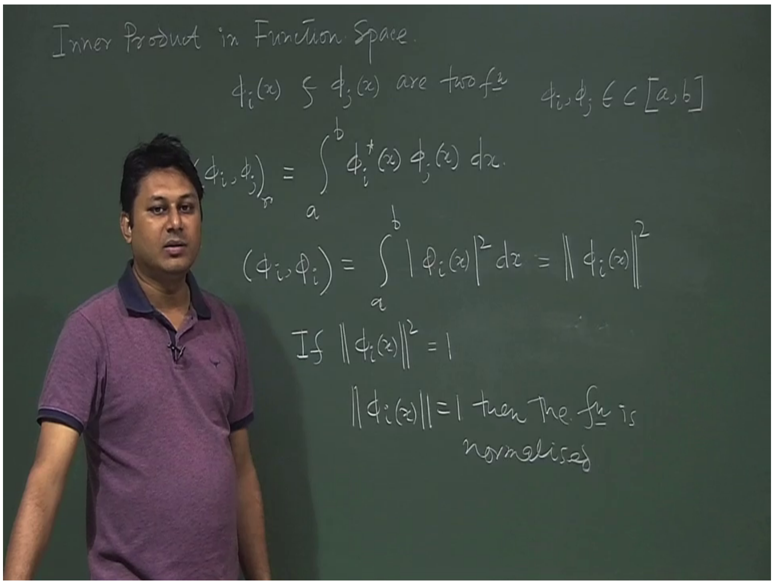

I can do the same thing like a vector and make the inner product of 2 same function phi i and phi i no problem with that absolutely ib, here we have this quantity in my hand this is essentially the norm of the function, norm of the function.

Now, if this quantity norm of this; that means, integration on this quantity is equal to 1 this is rather norm square norm is this quantity. So, root over of these things is basically norm, but anyway if this quantity is equal to 1 this is 1 means this is also 1, the square is the same thing then the function is normalized this function is normalized. So, and this point I like to I like to make you note that these things we always use in quantum mechanics that last day I mentioned about the wave function and all these things we always make the wave function normalize.

So, here the normalization procedure is exactly the same; that means, what we are doing that whatever the function I am taking as a wave function it has a norm it has a norm with unity, here we are doing the same thing at this point I would like to stop. In the next class so far we define a very important concept. So, far we define a very important concept linearly dependent independent function, then we define also the inner product of the function and now I say that if the norm of a function is 1 then it is normalized.

So, now we are going to something very important which is the basis because the concept of basis is same that when 2 vectors are forming a basis they are orthonormal to each other. In the same concept using the same way we can try to find out whether we can make some basis of functions which are orthogonal to each other and also they have unit norm. If we find this kind of set of functions which are which are following this rule then we can use this as a basis; that means, any function now can be represented with that orthogonal set of functions, orthonormal set of functions. So, that part we will do in the next class. So, see you in the next class with all this detailing orthonormal function and all these things.

Thank you.