Physics of Turbulence
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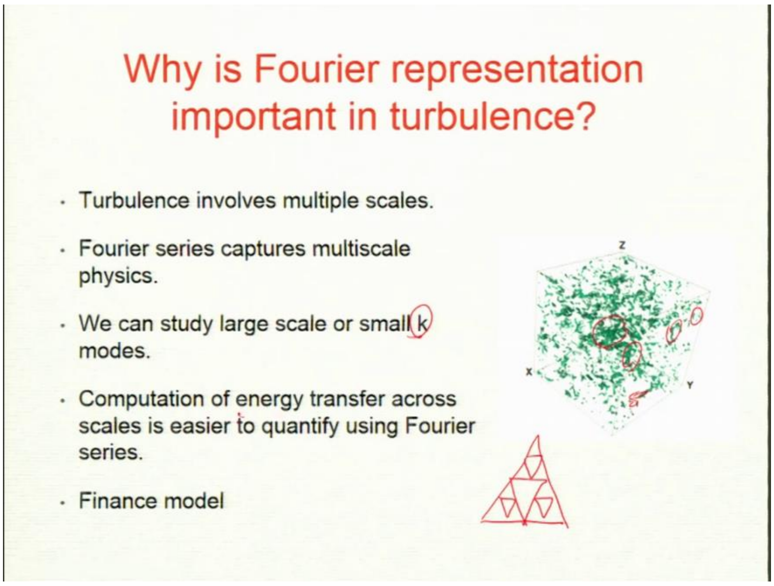
Lecture - 06
Fourier Space Representation Definition

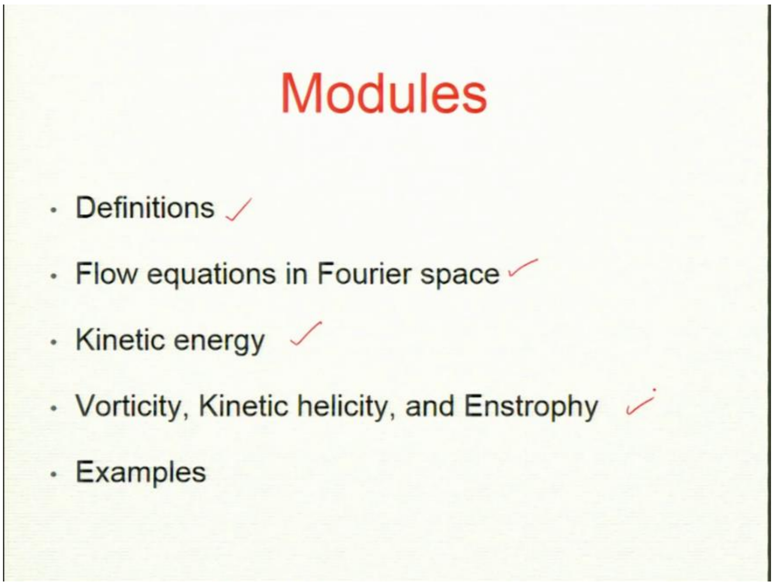
So, last couple of lectures were on flow equations in real space; we were discussing velocity field equation for the velocity field energy in real space. Now, we are going to Fourier Space. I will motivate by some logic or some ideas and also over time will become clear you know why Fourier space is very important for description of multiscale problems; problems which has many scales. In fact, I have been giving example turbulence being one of them.



Fourier space important in turbulence. Now, like we have bigger structure here, the small structure, in fact, this is somewhat similar to what we have in the sky the structures in the universe we have very big structures super galaxy galaxies, then we have stars, then planets and within planet they are different scales. So, if use in fact, linear scale that is not a good idea or in fact, you cannot if we describe in terms of linear series it becomes quite problematic to represent it. So, Fourier tells you that you can de describe a different scale. So, first Fourier mode corresponds to large scale, then you go down to second Fourier mode with half the scale, third Fourier mode is one-third, so this is very convenient ok. Now, we will see its importance over time we really realize that without this you cannot really describe many aspects of turbulence. Turbulence is a multiscale problem; Fourier space captures the multiscale physics very well. I will keep emphasizing it over time. Small wave number corresponds to large scale; k is 1 by l.

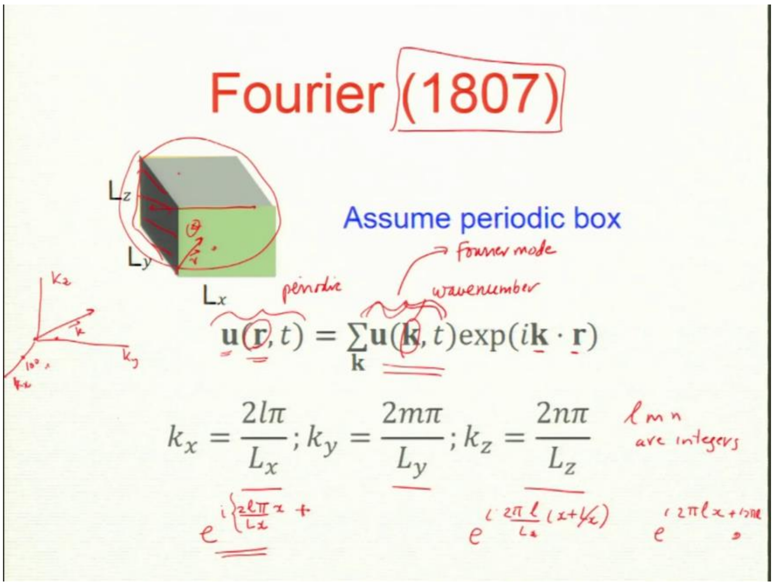
As I said when you supply energy in a bucket of water at large scale by stirring it then it goes to small scale and the scale the energy at every scale in fact, similar to this problem if there are energy at different scales right, now energy flows from one scale to other scale and the example like money flow. We have central government which has budget allocated for the whole nation, then it goes to state level, district level, village level and this is not in a real space; these are a collective phenomenon; you cannot really think of a real space. So, this is scale by scale transfer or in fact, we can think of these factors know this example which is given.

So, here it is important to use Fourier kind of representation; scale different scale has different physics and we will use Fourier transforms to especially study this energy transfer that is what we will do.



So I will again divide this Fourier description in five parts. So, we will review definitions. There are variations in this definition. So, I stick to my definition for this course and please stick to the signs.

A flow equation Fourier space then we will do, then kinetic energy is very similar to what we did in reall space but I am just going to do all of what was done in real space in Fourier space and then we will do vorticity, kinetic helicity and enstrophy then I will illustrate in fact, the same examples which I did in real space and we see some benefits right away so but not all of it we will all of it will take time. So, first start with definitions.



So, Fourier; so in fact, I just want to say that the first time Fourier gave his theory was in 1807. So, it is 200 years back and he solved heat equation using Fourier transfer. Fourier does not well convergence of Fourier series it is very difficult to prove, the proof was given many years later.

So, it is like engineering kind of problem; so he gave a series and he said well we can use that series to solve this problem, but the proof was given much later. He wrote a book and named that book, he put all that idea. a Fourier series the idea is that we have box. So, this is a 3D box we can also do for 1D and 2D but I will illustrate for 3D, but if you can do for 3D then always do for 2D right or 1D.

So, this box has to be periodic we can also generalize it to box with non periodic boundary conditions or vanishing boundary condition, but periodic is easiest to illustrate. I will stick for this five modules to the assumption that box is periodic. Periodic means if I go across by distance L_x other corner, I should have a same value of the function and this periodicity can be for any function.

So, here I am going to use velocity field, pressure field, energy.

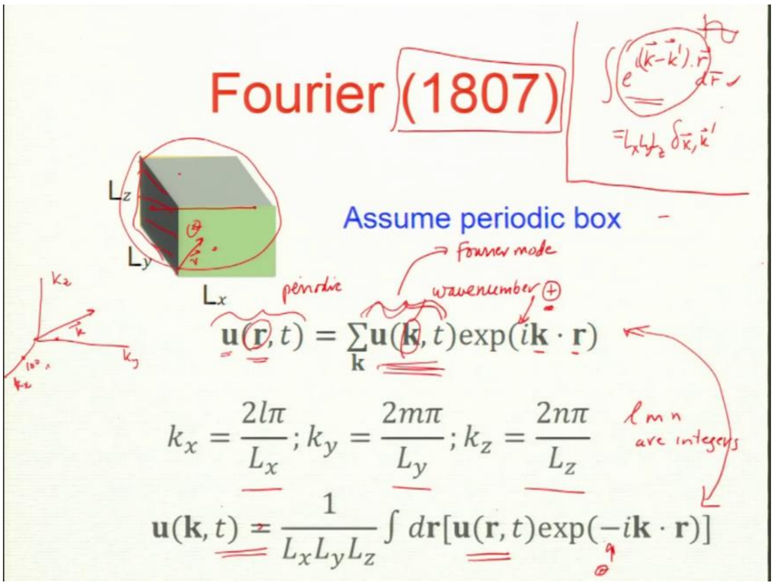
Assume this is periodic then the definition is here. So, $\boldsymbol{u}(\boldsymbol{r}, t)$ is vector field. I have gave for vector field but you can also write scalar field straight forward. So, basically there are three components; for every component I can write like that. So, these are real space, $\boldsymbol{r}$ is a real, this is in fact, my general notation $\boldsymbol{r}$ means real, t is time. So, I just keep time in it, so we our; function is Fourier time, this is wave number and $i\boldsymbol{k}\cdot\boldsymbol{r}$.

So, this is a vector $\boldsymbol{r}$, but now there is a corresponding space. Now, Fourier space is not this real space description this one now I can also have a Fourier space description. So, this axis will be k_x, k_y, k_z and this is a different space ok. My k is discrete.

I have equi-space points in this Fourier space and so I can draw a vector $\mathbf{k}$. So, it has k_x component, k_y component and k_z component and they are given by these numbers ok. Now, you can see that $\exp(i\boldsymbol{k}\cdot\boldsymbol{r})$; $\mathbf{k}$ let us put along x direction. So, I will I do not want to write all this $2\pi x/L_x$. Now, similarly you write for y $2m\pi/L_y$.

So, you can just put $\exp(i2\pi l/L_x)$. So, the new thing which is coming is $\exp(i2\pi l)$, l, m, n are integers. They can take both values negative and positive, for Fourier periodic box it is both take both positive and negative value. So, this function is periodic, similarly the y part will also be periodic same with z. So, these basis functions are periodic and that is why we need to use a periodic box.

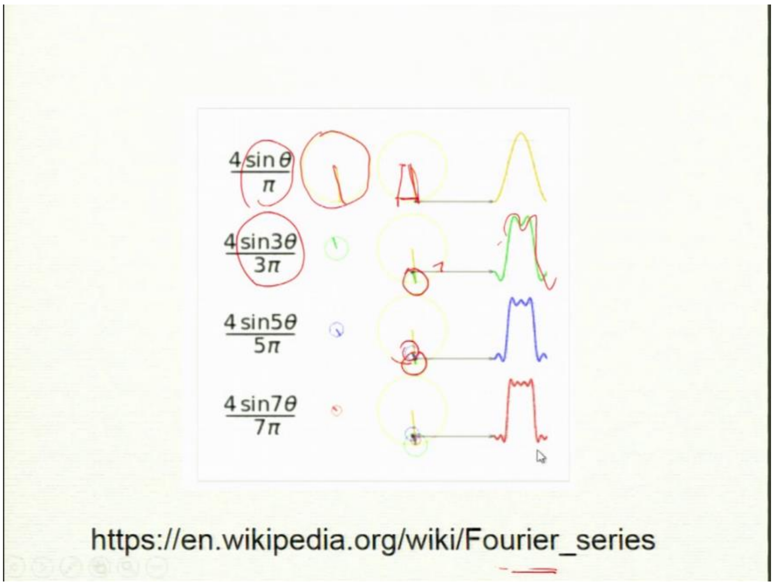
Any periodic function can be represented as a series. Amplitudes in Fourier space are called Fourier mode. This is what Fourier assume for 2D plate. We solved for 2D plate heat equation but we will solve for 3D flows or 2D flows also.



Now, you can invert, so given $\boldsymbol{u}(\boldsymbol{k}, \mathrm{t})$ you can get $\boldsymbol{u}(\mathrm{r}, \mathrm{t})$. This is called from inverse transformation. So, I can compute $\boldsymbol{u}(\boldsymbol{k}, \mathrm{t})$ given $\boldsymbol{u}(\boldsymbol{r}, \mathrm{t})$.

So, periodic function if I sum over the volume it should give you 0. So, one simple way to when you; you can do it for all three but a periodic function $\sin x$ over whole period is 0 integral.

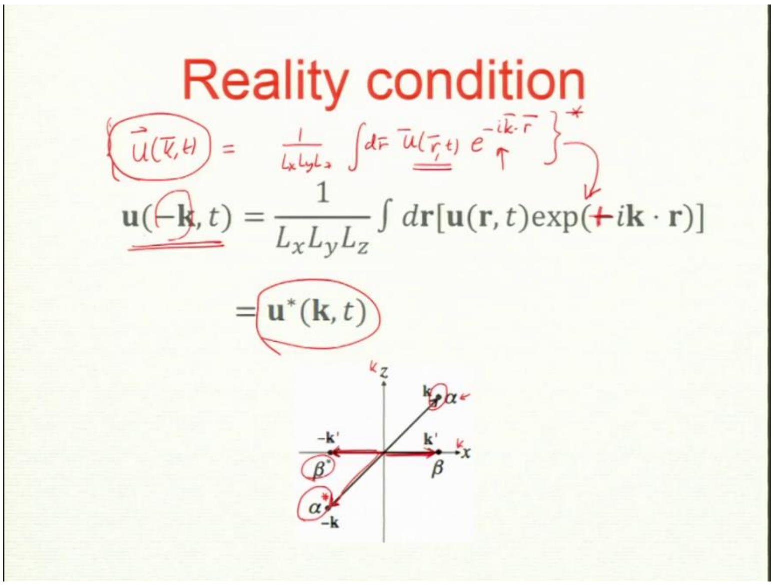
$\boldsymbol{dr}$ is volume. So,



This is a nice thing I saw in Wikipedia. This is a circle and there is a there is a rod which is going around the circle with constant velocity angular velocity. If you plot the y component, it looks nice sin curve.

I want to get $\sin\theta + \sin 3\,\theta$. You make another circle of radius one-third. So, now, this is loop within loop this is called epi-circles. This first guy is going with constant velocity and other guy is also going around like that and there is of course, there has to be relation between the two and then this. You can construct similarly.

So, these are 3 epicycles, you can make 4 epicycles. Look at Wikipedia a Fourier series and you will find this example. In fact, it is animation.



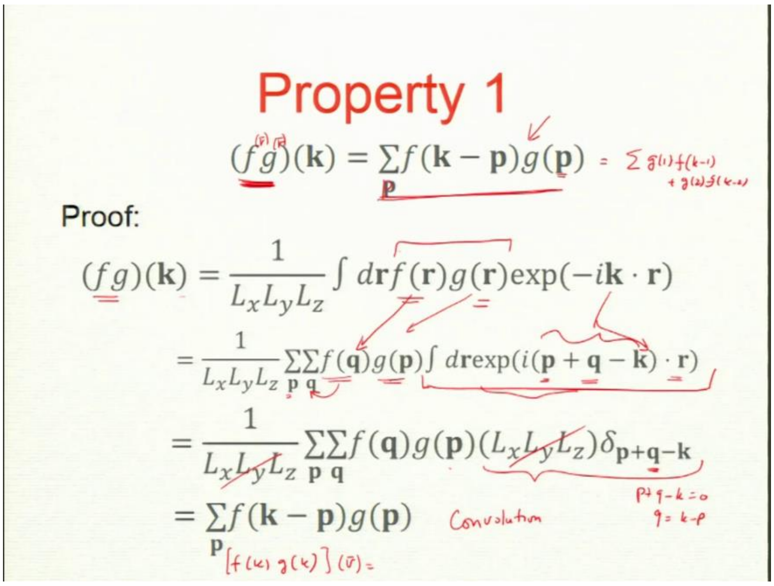
According to the Fourier series, any periodic function can be represented as a Fourier series.

Now, velocity field is real; that means, there is some condition for Fourier modes. Reality condition: $\boldsymbol{u}(-\boldsymbol{k},t) = \boldsymbol{u}^*(\boldsymbol{k},t)$. See above figure.

If I take a complex conjugate of this what I just wrote.

This is very important, and we need this all the time. Advantage of this relation is that we do not need to save all the modes in computer; we need only half of the modes.

Now, I will illustrate this using a figure here. α is a complex number.



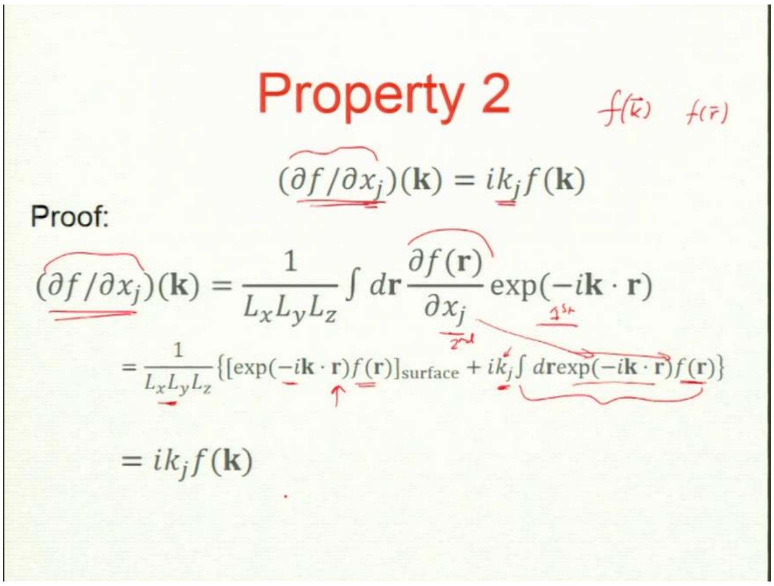
Fourier series has some very interesting properties. See the above figure for Property 1.

This is called convolution. If I do the Fourier transform of a product in the Fourier space, I will have sum.

This is could be expensive. I can just multiply a number by one shot, but if I want to Fourier transform that thing then you have to in Fourier space I need to do this convolution. I will prove three theorems just to show you how to work with Fourier.

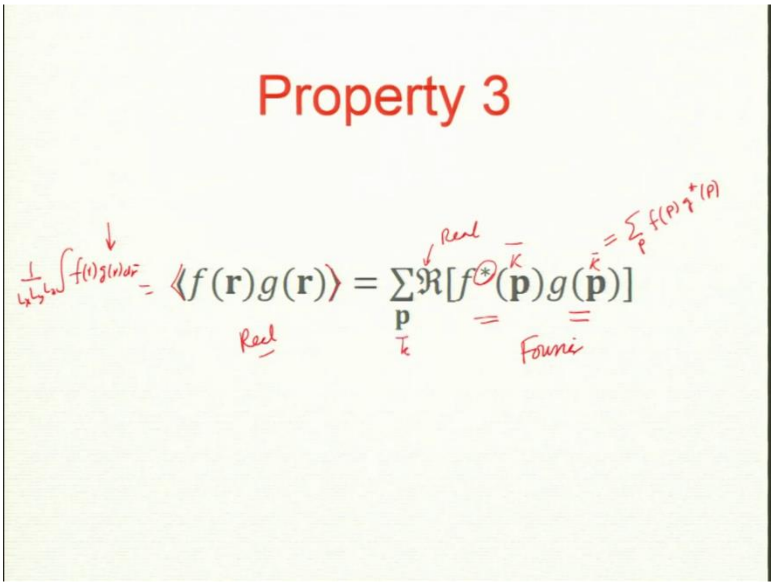
$\delta_{\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{k}}$ is a delta function, is 0 unless $\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{k}$.

So if you have product in real space, in Fourier space that will be sum and if this is called convolution.

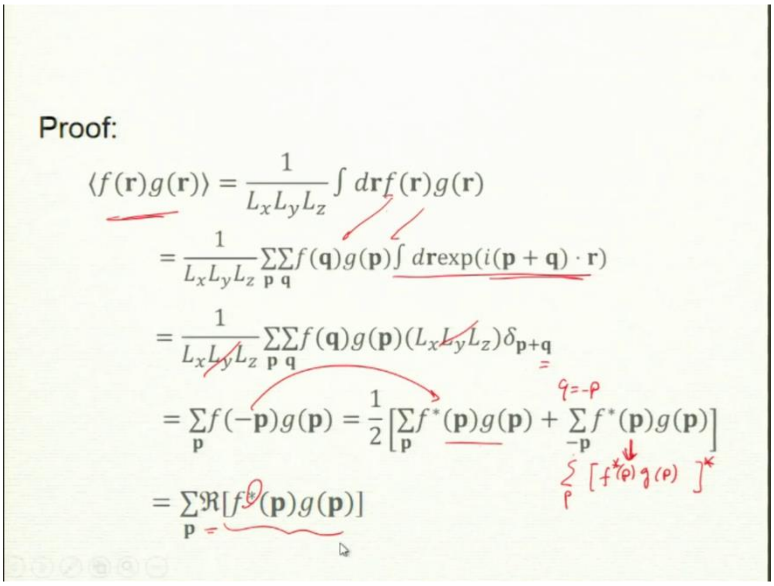


Property 2: derivative of a function. See the above figure.

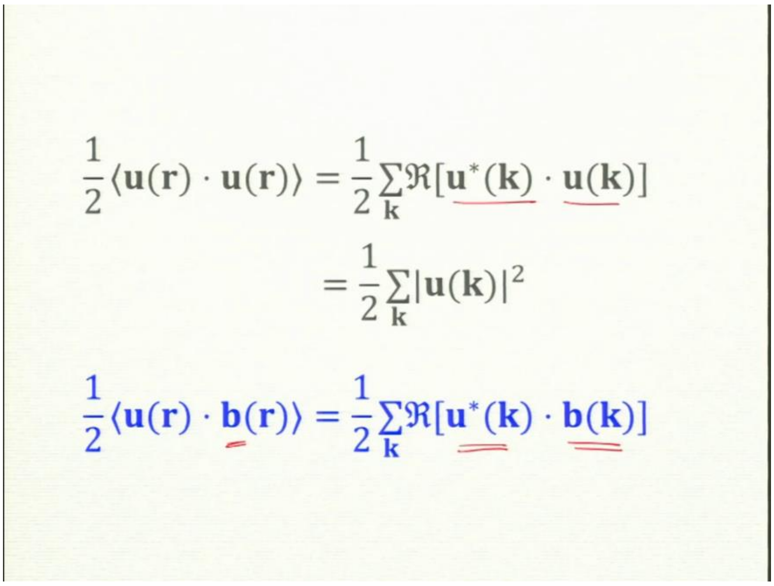
So, we need to remember that product becomes convolution and the derivatives becomes ik_j of that function.



Property 3: Volume average in real space. See the above figure. Right hand side is Fourier, left hand side is real space



For the proof see the above figure. The proof is very easy. We need this property for energy: see the below figure.


$\frac{1}{2}\langle \mathbf{u}(\mathbf{r}) \cdot \mathbf{u}(\mathbf{r}) \rangle = \frac{1}{2}\sum_{\mathbf{k}} \Re[\mathbf{u}^*(\mathbf{k}) \cdot \mathbf{u}(\mathbf{k})]$
$= \frac{1}{2}\sum_{\mathbf{k}} |\mathbf{u}(\mathbf{k})|^2$
$\frac{1}{2}\langle \mathbf{u}(\mathbf{r}) \cdot \mathbf{b}(\mathbf{r}) \rangle = \frac{1}{2}\sum_{\mathbf{k}} \Re[\mathbf{u}^*(\mathbf{k}) \cdot \mathbf{b}(\mathbf{k})]$

Thank you.