Physics of Turbulence
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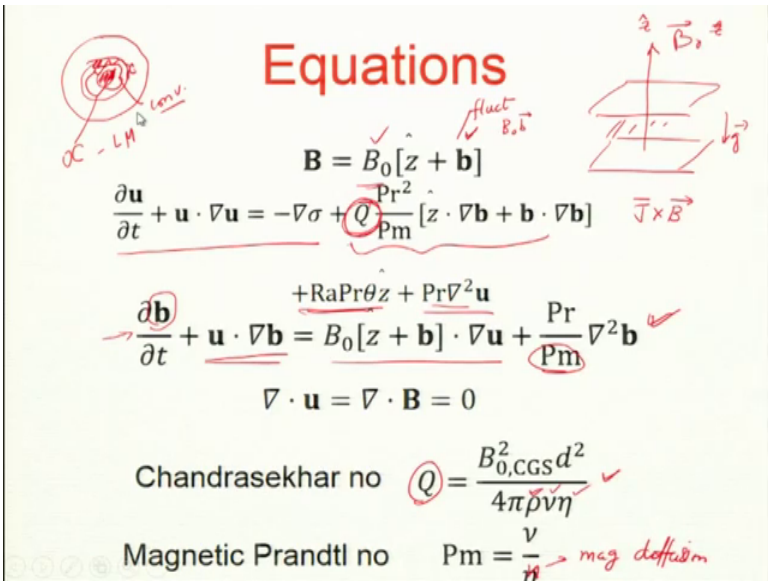
Lecture – 18
Route to Turbulence Magnetoconvection, Instability & Patterns

So, far we discussed instabilities in two systems - Rayleigh Benard and rotating convection ok. And the first system was 2D and second system was 3D. Now, today I am going to work with another system which is like rotating convection, but it is magnetoconvection. So, we will put magnetic field and temperature.



So, the idea is we have this fluid contained between two plates like this and apply a mean magnetic field B_0. It is along z-direction. So, I will assume along z-direction. This is along z-direction. And of course, gravity is downward like this, and there is a fluid in between. Now, of course, there would be magnetic field fluctuations. So, we will do that bit later in MHD.

So, fluid can generate magnetic field, local magnetic fields, like earth magnetic field is generated by fluid motion. There is $\boldsymbol{b}$ which is fluctuation, but I normalized with B_0. So, B_0 is the magnitude of the mean magnetic field. So, $\boldsymbol{b}$ is normalized fluctuation. So, this is what is the notation used in literature.

So, the equation now very similar to what we had before. So, let us put all the terms. So, you are familiar with these three, right? Pressure gradient. This one also you know, and this one also you know. So, what is new, this equation, this term. And this term I will not derive it now, but we will do it bit later when we will do MHD. It comes from force, $J \times B$. So, there is current, and there is a magnetic field, so there is $J \times B$ force. Now, J is like curl of magnetic field under certain approximation. So, this is called MHD approximation.

Well, strictly speaking B_0 times $\boldsymbol{b}$ is a fluctuation, and the $B_0\hat{z}$ is a mean magnetic field. So, you apply this strong magnetic field to fluid motion, and that fluid motion we will generate fluctuations, and that is small, B_0 times $\boldsymbol{b}$, like earth magnetic field is dynamic. If you have seen my talk or somewhere, the earth magnetic field is changing all the time. And you will see that magnetic field here also will change. In fact, we will see consistent equation.

I am not deriving it right now, but you just treat that is a given to you. How many independent fields we got? We have got velocity field; we have got magnetic field and we have got temperature. So, temperature equation will be same. So, temperature equation I have not written here. It is same as before. So, it does not get affected by magnetic field, but we need to write down equation for the magnetic fields as well.

B_0 does not change with time, B_0 is constant magnetic field in space and time. We will assume that it does not change the space and time. Constant. So, the $\boldsymbol{b}$ is the fluctuation which is evolving by this equation. Again, it comes from Maxwell's equation, but we will not derive it right now. So, it has $\boldsymbol{u} \cdot \nabla\boldsymbol{b}$, it is like advection of the magnetic field, and this term is $\boldsymbol{b} \cdot \nabla\boldsymbol{u}$.

And this magnetic Prandtl number Pm, which I have it here, let us put all of them. Magnetic Prandtl number is ν by η, where η is magnetic diffusivity. It just like there is a temperature diffusion, similarly magnetic field we will diffuse because of conductivity of the material, electrical conductivity will make it diffuse.

So, and this Q is the parameter here, it is called Chandrasekhar's number. It is a strength of the $J \times B$ force. You can think of that $J \times B$ force as the Ampere's force. So, current carrying conductors in a magnetic field experiences force. So, instead of current carrying conductor, we have current carrying fluid which is like wires in fact, in some sense, and that is experiencing a force.

So, the strength of this term is given by the non-dimensional number Q which is Chandrasekhar's number and is given as $Q = \frac{B_{0,CGS}^2 d^2}{4\pi\rho\nu\eta}$. ρ is the density of the fluid; ν is kinematic viscosity, η is a magnetic diffusion. Now, we will treat that as a number, it is a non-dimensional number.

The derivation is simpler for me with CGS units. MKS can also be done. In fact, all the engineering books and dynamo community also uses MKS SI, but CGS is chosen by some plasma physicists and it is a choice. If you look at the book, I do also give the formula in SI as well, but right now we just treat as the parameter.

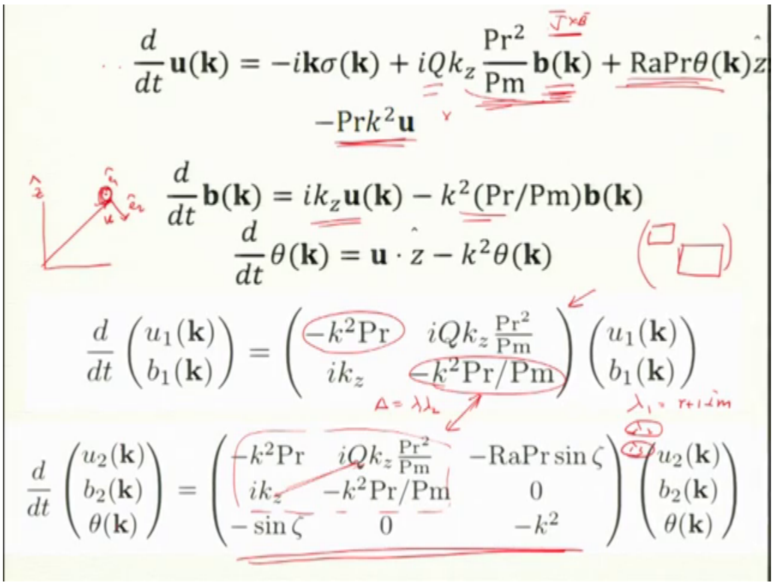
So, these equations are non-dimensionalized right now, it is non-dimensionalized equation. So, $\boldsymbol{u}$ and $\boldsymbol{b}$ are all non-dimensional. So, $\boldsymbol{u}$ is normalized by $\frac{\kappa}{d}$, $\boldsymbol{b}$ is normalized by B_0, and B_0 in turn has basically gone in here. So, B_0 does not appear in this equation you see well. This is the equation, $\boldsymbol{b}$ is also normalized with velocity fields.

So, presently just treat them as equations, because this will require some more discussion if I get into in that aspect. So, we want to see whether this system is unstable or not. So, in fact, earth magnetic field is generated by magneto-convection and rotation. So, you know how the earth magnetic field is generated? So, let us make a schematic diagram here. So, there are four layers, inner most layer is solid, called inner core and this is called outer core, OC - outer core.

And this part is liquid metal, molten iron, chromium and so on. So, this is basically liquid metal, LM - liquid metal, it is around 5000ºC temperature, variable, but it is of that order thousands of degrees. So, inner part is hot, while there is a radiation. there is heating at the bottom inside and this part is colder.

So, there is a convection in this zone. But it is liquid metal, so it can support current. Unlike water it can support current, this liquid metal, and that current generates magnetic field. So, earth's magnetic field is generated here. And the equations are what I have written here. So, if we include some rotation in this, and make it bit more complicated, then you have the equation for magnetic field generation in the earth.

Now, presently we want to see whether the system will have nonzero $\boldsymbol{u}$ or not. So, instability means we want to see convective motion. So, magnetic field can supress convection. This magnetic field can affect convection very strongly. But we just want to study it as a mathematics problem and see whether this will generate convection or not. This is an exercise on instability analysis.



And so, in Fourier space, I drop the non-linear term - linear stability, and work in Fourier space. In fact, we will work with Craya-Herring basis. So, first I write down the velocity field in Fourier space. So, this is Chandrasekhar number, this is that $\frac{Pr^2}{Pm}\boldsymbol{b}(\boldsymbol{k})$ and this is convection term, and we have one diffusion term - viscous term. So, this term is basically a new term, $J \times B$ term, coming from magnetic field. And this fluctuation $\boldsymbol{b}$ has come, the non-linear term $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ has been dropped, and $\boldsymbol{u}\cdot\nabla\boldsymbol{b}$. there was a term $\boldsymbol{b}\cdot\nabla\boldsymbol{b}$, that too has been dropped.

Now, we need equation for magnetic field. So, if you drop the non-linear term, we get these three terms. This is a diffusion term for the magnetic field. So, these are all positive definite in terms of energy. If I dot this with $\boldsymbol{u}^*(\boldsymbol{k})$, then this is negative always, negative definite quantity, because if I dot this with $\boldsymbol{u}^*$, what will I get?

$E(k)$, and it is always positive. $\boldsymbol{u}\cdot\boldsymbol{u}^*$ is always positive because that is mod square. So, these are all dissipative terms that dissipate energy, but these terms can increase velocity field. This is temperature is increasing. This can also increase or decrease. Now, we need equation for the temperature which is same as before. So, now, we do Craya-Herring. so how many components we will have? So, $\boldsymbol{u}$ will have two components u_1, u_2, and $\boldsymbol{b}$ also we will have two components, $\boldsymbol{b}$ is a vector.

So, $\boldsymbol{b}$ also we will have two components, b_1 and b_2. So, this is $\boldsymbol{k}$ vector. I will use $\hat{z}$ as $\hat{n}$, and this will be e_2 and e_1 is here. So, you could have both u_1 u_2 for velocity field and b_1 b_2 for the magnetic field. So, we can derive it. We do not have b_3 component as $\nabla \cdot \boldsymbol{b} = 0$. So, $\boldsymbol{b}$ is also divergence free. So, this is very interesting that $\boldsymbol{u}$ and $\boldsymbol{b}$ both satisfy divergence free condition. So, now, the derivation can be done, but I will not show you here, I will just state the equations. So, this is for b_1 and this is for b_2. So let us look at the eigenvalues of these. So, the trace is negative. It is of course, at least one of them is negative right.

Now, what about the determinant? Determinant is positive. So, what does it mean about the eigenvalues? Both must be negative. So, determinant is product of λ_1 and λ_2. So, I know at least one of this is negative. So, that one must also be negative ok. So, well one idea could be these are complex numbers, but. So, you can work it out. So, I can tell you right now that these are negative and they will decay, negative eigenvalues mean it will decay.

Now, one interesting point you see I am not deriving from the first principles. So, this I did it when I was writing this chapter of the book. You can do it yourself, but maybe no need to spend some time on it. Save your energy for more, more difficult task. So, u_1 b_1 and u_2 b_2 θ are decoupled. Since the equations are decoupled, it is, in fact, not a 5 by 5 matrix, but there is a 2 by 2 block and 3 by 3 block.

The b_1 and u_1 component along e_1 direction are waves; they are not affected by temperature. Now, these are, if you look at the equation more carefully, are called Alfven waves. There is something called Alfven wave for these MHD. And u_1 and b_1 constitute the Alfven waves, dissipative Alfven waves. So, there is i sitting here, that means, they are basically oscillations. But they dissipate because of these terms.

That is right. So, u_2 is affected and b_2 is affected by temperature gradient, but not u_1 and b_1. And this comes nicely from this Craya-Herring, that is nicest thing about it. You worked with Cartesian, it is clumsy, which I have done, not for this problem, but I have done with Cartesian, it is just not very transparent. Now so, we do not need to worry about

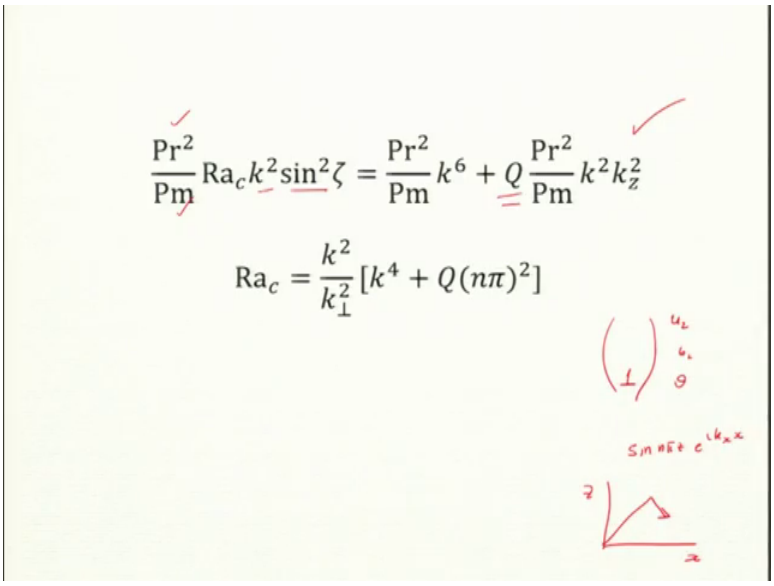
u_1 b_1, they will dissipate, in a dissipative system. If Prandtl number was 0, and then this will not dissipate, they support waves.

So, right now because of this dissipation already present in the system, they will just go to 0. So, u_2, b_2 θ this equation may generate instability. Now, that does look complicated. Now, in fact, this part is very similar to this part right. In fact, these are called Alfven waves. This i will support Alfven waves. And if you look at u_2 and θ, if I drop this q term and b_2 term, then we could recover the convection equation.

So, you can see that they are all consistent, by just seeing the structure. But since u_2 b_2 are coupled by this Alfven coupling, this is a 3 by 3 matrix, not 2 by 2 matrix, and that will complicate things a bit. This is derived from the linear equation. Now, this equation has 3 eigenvalues and 3 eigenvectors. Now, computer can tell us what the eigenvalues and eigenvectors are, but to see the instability or to look for neural stability I just set the determinant to zero. Simplest thing to do.

Because that is where the eigenvalue will cross the zero line from negative to it becomes positive, one eigenvalue, but there is there is no proof I mean this is what is by when I do it, I get the same thing what Chandrasekhar gets ok. So, I am going by faith, I am not really proving it that all the eigenvalues are negative before at least the real part, then one of them crosses over it becomes positive, is that clear to everyone?

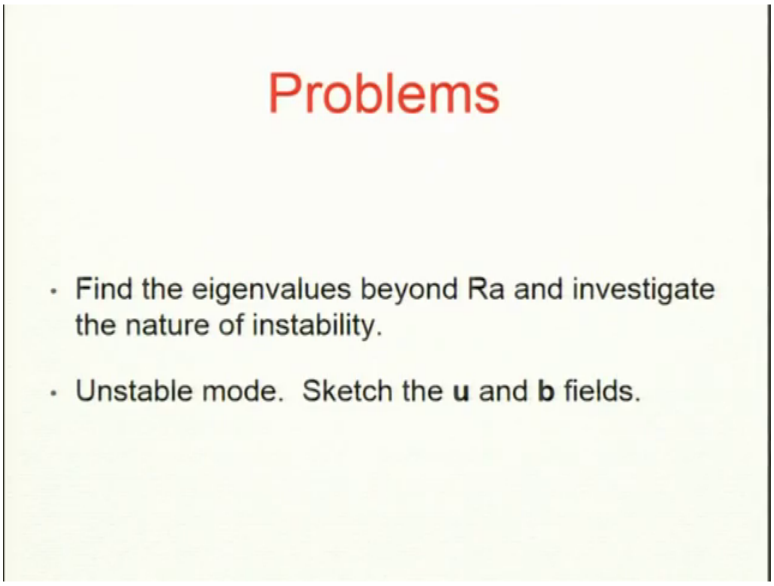
So, stable means all the real part, so there could be complex. So, λ_1, λ_2, λ_3 and it has a real part plus imaginary part. So, real part must be negative for stability. So, real part I have all these three must be negative if it is stable. But as soon as it crosses 0, so I say the λ_2 becomes 0 at some point and that is what I am curious to see. I have not done it myself. So, this is what I want you to do. This is another project, that compute the eigenvalues and see what are the eigenvalues, what is the nature of the eigenvalues?



So, I am going to set right now determine to 0 and the condition we get is this. Now, so we will fix parameters and my variable will be only Rayleigh number. So, I fix Rayleigh, Prandtl number, Pm. Also $k \sin\zeta$ is $\frac{k_\perp}{k}$, right? I mean that is why. So, Q is also fixed, that is, external magnetic field is fixed. Given this parameter, what is Rayleigh critical and from here. So, this is of course, I can compute Rayleigh critical.

Now, we can also compute the eigenvector and that eigenvector will tell us the structure of the flow - it is a 3D flow. Well it is usually a 2D flow, u_1 is 0. So, in fact, it is simpler than rotating convection and u_2 and b_2 are related from the eigenvalues.

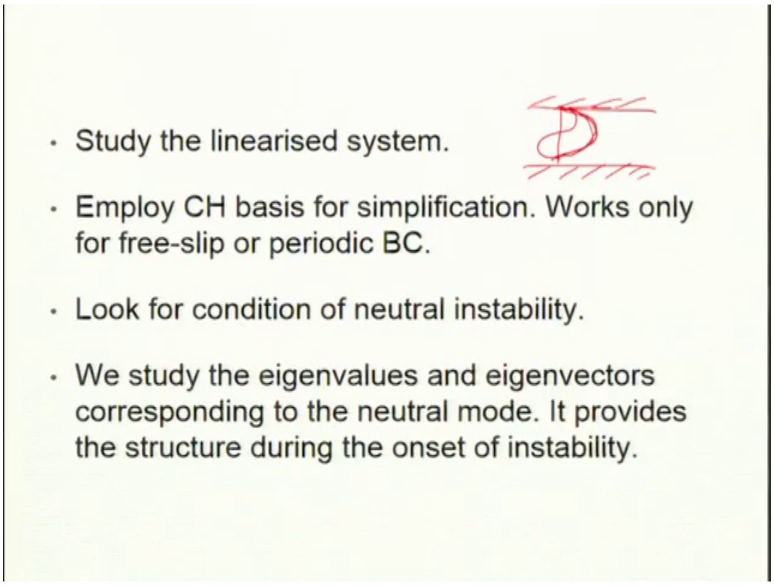
So, I will get eigenvector here. So, this is for u_2 b_2 and θ. So θ we can choose it to be one, these are wave modes. And please remember it is sin, θ will be $\sin(n\pi z)$ and $e^{ik_x x}$. In fact, it is not a 3D system, it is a 2D system. So, you can take the xz-plane. So, we can draw this plane and draw this structure. So, I have not done that work, but you can do it.



So, problem would be to find the eigenvalues beyond Rayleigh. So, just go from Rayleigh critical slightly above, see whether they are complex or real or how do they choose. So, this is what I want you to do, and sketch unstable modes, how does $\boldsymbol{u}$ and $\boldsymbol{b}$ look like. Now, here I applied the magnetic field in the direction of gravity, but you can also apply horizontal magnetic field, along x or y axis. So, when choose along x axis and you can do the analysis.

So, roll structure is important to see and so, if I do like this, it is easy to see that convection will set the rolls like that. And magnetic field does affect the flow, but it is somewhat happy to let it go like that. Magnetic field probably wants the flow to go like this and convection wants the flow to go like that. I mean this is a complex structure that one needs to understand. So, this what I want better understood.

So, I just sketched it, but you can see that we need some more work. And one more complex thing could be magnetic field is along some arbitrary direction. Then this will affect, and the structure will be more complicated. And this liquid metal flows are used by engineers heavily in industry. In fact, many metals you know like titanium, tungsten or even car sheets. So, you want to smooth sheets. So, apply strong magnetic field, that will smoothen the fluctuations or kill the fluctuations. So, then they want to apply magnetic field and control the convection and things like that.



So, let me summarize what we learnt from instability. So, we studied the linearized system. So, these are all part of linear stability analysis.

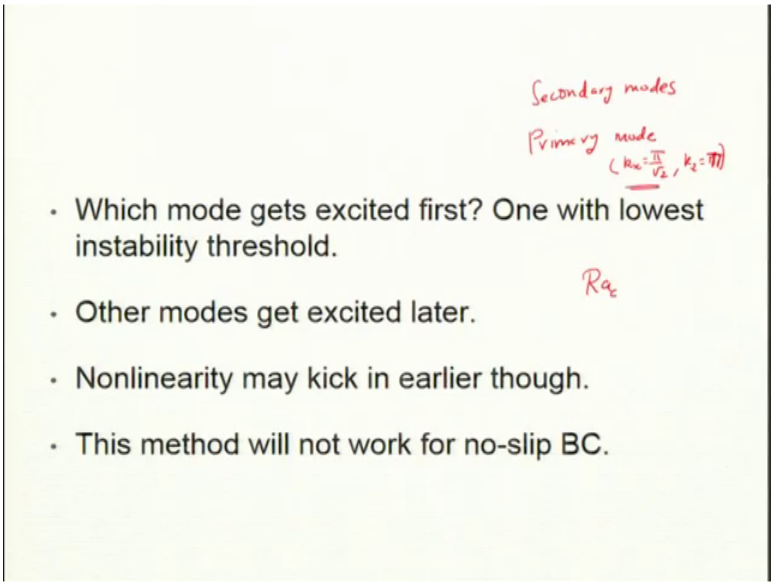
But I did tell you that there are systems that are nonlinearly stable, so nonlinearly unstable. So, linear system is always stable, here we found that linear system is stable for some parameters and becomes unstable for some other parameters. And there is a transition and that is called neutral stability. Pipe flow is one example, channel flow which is stable if you do linear analysis. In fact, dynamo is stable if you do linear analysis, it always remains stable. So, you need non-linear term to make it unstable.

So, we will do dynamo; dynamo is simpler to analyze. One difficulty with channel flow is that I cannot use no-slip boundary condition. Well, we can use free slip, but normally the engineering problems use no slip boundary condition. And there, Fourier analysis does not work. So, convection for free slip Fourier analysis is very good. I would like to in fact study channel flow with free slip boundary condition and see whether we can set it up and what happens.

Employ Craya-Herring. It simplifies your calculation, but Craya-Herring is not working. It does not work for all the time, right? I mean if we apply no slip, Craya-Herring would not work, and that is why in channel flow it does not work. But even for those flows, I would like to see what happens with the Craya-Herring. It means you just try some new stuff. Look for condition for neutral instability. So, we apply the Craya-Herring, then we get linear ordinary differential equations for the Fourier modes.

In channel flow, what is the structure? It is like that right? So, this one I would like to put a sine wave, I will treat this as a sine. It is a kind of modeling. So, I want that kind of model, that velocity field. So, instead of really looking for exact profile, but just model with this and with the higher modes will be that. So, just analyze that. I do not know somebody has done it or not, but it is good thing to try.

So, study the eigenvalues and eigenvectors once we get neutral stability and that will be the structure at the onset and near the onset not necessarily at the onset, but near the onset, structure does not change dramatically if I change the Rayleigh number slightly. So, it is a good starting point for also studying the later part of Rayleigh.



A few things which were asked during the class. So, which modes get excited first? So, there are many, many Fourier modes. So, our analysis tells you that among many modes, one of them is most unstable. So, one with the lowest instability threshold right. So, how would I get that? So, my parameter was $k_\perp$ and I varied it and saw that Rayleigh critical was lowest for some particular $k_\perp$. And for our problem it was $\frac{\pi}{\sqrt{2}}$, that was k_c. So, we can see among Fourier modes which become unstable.

But if I increase my temperature or Rayleigh number, then other modes also get excited. So, other mode also gets excited later. So, you can say when I increase the temperature, some more modes are getting excited. So, we will get mixture of modes, but non-linearity also is not going to sit there. So, one example which I would like to say is that instability means this person is suddenly becoming rich, getting more and more money. But other people cannot simply stop watching that person. They say, well I also want my money.

So, nonlinearity will transfer money from one mode to other mode and that is what nonlinearity does. So, this growth also saturates. It is not that people just wait for their turn to come, there is also exchange and that exchange is by non-linear coupling which I will do it after this presentation.

In fact, first we have primary mode and after that you we start putting in nonlinearity it starts growing. So, first make the primary mode, the first mode. So, this one word which is used quite often, it is called primary mode. So, primary mode wave number, what is the primary mode wave number for Rayleigh Benard? So, k_x is $\frac{\pi}{\sqrt{2}}$. You just give me the wave number, then I know exactly all of it, all these properties, and k_z is π. My box size is 1. So, this is my wave number it is 2D, the k_y is 0, and this is a primary mode.

Now, when primary mode starts to grow, so we will generate more modes, this is called secondary modes. A secondary mode can come by instability as I said here. When I increase the Rayleigh number, more modes can come because other modes also get a chance, but before other modes come, remember Rayleigh critical for n equal to 2 was n^4, n equal to 2 takes bit of time. If I increase my k.

So, to get higher harmonics my instability may be taking longer, higher temperature difference. But if I increase temperature a bit, nonlinearity will kick in and it will generate higher modes which I am going to show you after this. So, non-linearity kicks in and it just makes the patterns come, and then chaos comes. And this method will not go for no slip because Craya-Herring and Fourier is not good enough for no slip. So, we will stop here.

Thank you.