Physics of Turbulence
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Lecture - 17
Route to Turbulence Rotating Convection Instability & Patterns

So we will do another instability which is on rotating convection. So, I will show you patterns. Do not worry about patterns, it is just instability. So, I have put rotation. So, my Navier-Stokes equation will have one more term - it is the coriolis force.

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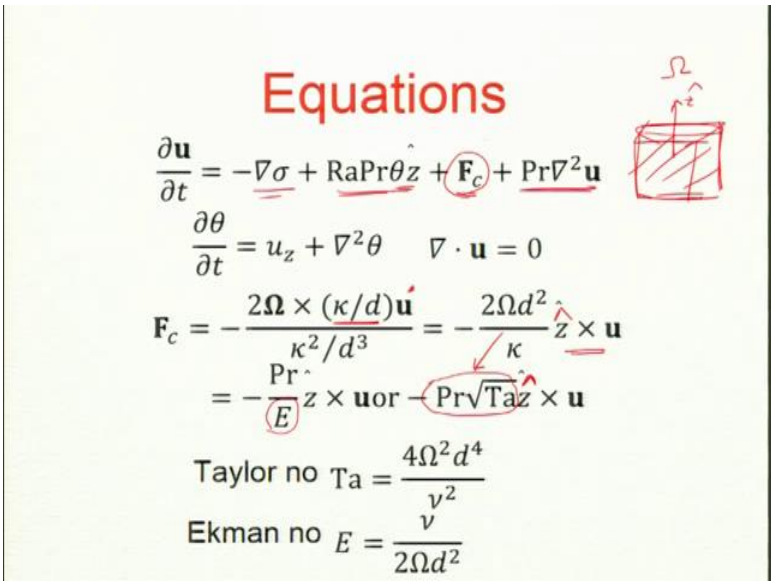

It is a linear system. So, our non-linear term is off and we got these 2 terms and these 3 terms are exactly the same as before. I am adding coriolis F_c, where c is for coriolis and θ does not get any other contribution for rotation only the velocity field gets contribution F_c. So, θ remains the same. So, I am adding new term F_c, coriolis force, and what that is? This is Boussinesq $\nabla \cdot u = 0$. So, F_c I need to non dimensionalize. Remember the whole equation was non dimensionalized.

So, what is coriolis force? $-2\omega \times u$. So, and this is the derivation you can look at. So, my velocity field is non dimensionalized with $\frac{\kappa}{d}$. So, this u is non dimensional and I divide by $\frac{\kappa^2}{d^3}$. So, this whole thing is divided. So, what I finally get is that $\frac{2\Omega d^2}{\kappa}\hat{z} \times \boldsymbol{u}$, because omega is along z direction. So, it is a fluid is rotating about z axis with rotation frequency omega.

You can think of a cylinder which is heated from the bottom and cooled from the top. We are doing a box only, not a cylinder. This is the box which is rotating with Ω frequency about z axis and gravity is also along z, downward. So, now, it is in fact, is very straightforward derivation. This is beauty of Craya-Herring.

So, you do not need to do much, I need to resolve this one along Craya-Herring. Because this one, I already have done it. I mean this you have seen in your notes, this we have already done. So, I need to resolve this one. Now $\hat{z}$, I know in Craya-Herring how it looks like, right? It has component on e_2 and e_3. So, but before I go on, I just want to define one nondimensional number.

It is $Pr\sqrt{Ta}$; Ta is called Taylor number. Or if I use another definition, it is called Ekman number. So, Taylor and Ekman are related with rotation. Now for this lecture I will not really go too much into detail of this stuff. So, this number is exactly same as Prandtl square root Taylor. So, Taylor is square of Ω, not Ω. So, I have Ω, that is why I take square root. So, $\frac{4\Omega^2 d^4}{\nu^2}$. So, that is called Taylor number. So, non dimmensional numbers are useful. I will probably emphasize towards the end of this lecture. Well, let me just tell you right now why nondimensional numbers?

So, Rayleigh critical, the 657, is fixed irrespective of what fluid you use or what temperature I am heating at the bottom, top. So, for any experiment you can compute Rayleigh number and see whether it is above Rayleigh critical. So, it is independent of your experimental conditions of fluid or plates if free slip boundary conditions work, great. Rayleigh is what is important ok.

So, if I increase the box size by double, but my temperature difference is tripled, does not matter you do not need to compute, just compute Rayleigh critical. Same thing for Taylor number, you do not need to you know worry about what Ω, what box size. Compute the Taylor number and I will get a condition for instability. So, you can see the power of non dimensionalization. You do not need to work out this stability condition for system specific values. And Ekman number is defined as this, these are just quantities.

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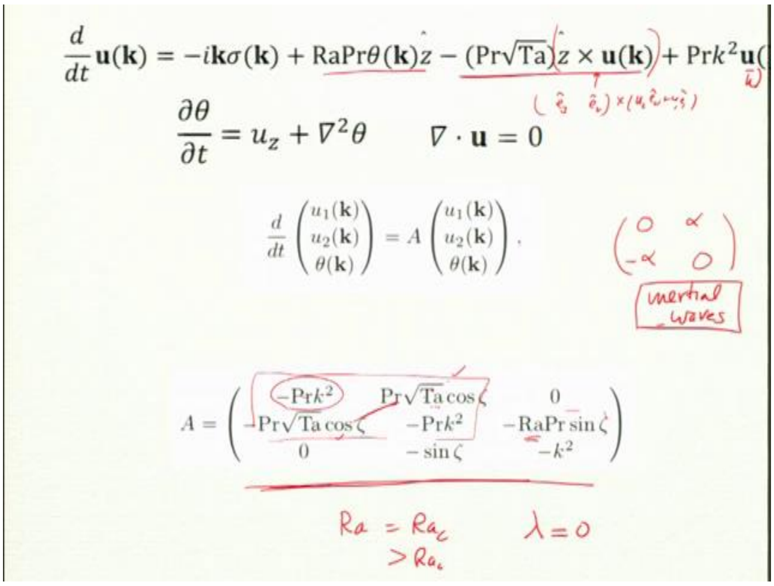

Now so, now, this is my equation in terms of $\boldsymbol{u}(\boldsymbol{k})$. So, this equation is not fitting here. Now; so, this one I have already computed in terms of Craya-Herring, now this one can also be computed on Craya-Herring, but it is cross.

So, this needs some labor. I will not do it here but this we need to resolve. So, this will be e_3 and e_2 and cross u_2 and u_1. So, this I will leave it for you to do it; right. This is just cross products, and just remember that e_1, e_2, e_3 are cyclic and straightforward. It is bit of algebra but it can be done easily.

So, these equations we get. Now u_1 will not decay. So, now of course, I have to tell you what is A; A is this matrix. So, I have u_1, u_2, θ all 3 of them they are now and A is this matrix. Earlier my u_1 was basically.. These two guys were 0.

So, Taylor number 0 will give you.. This one is decoupled. You have set Taylor as 0, this will decouple. So, u_1 will decay. Now because of Taylor number being nonzero, this will not decay. It is a positive feedback. Now; so, the Taylor number is coupling u_1 and u_2, do you see that? Because without Taylor number u_1 was decoupled and decayed because of viscosity. But now this will be playing a role in instability. I will just ask you if I remove Taylor number you can easily see that you will get these as same as before.

Now, one important point I want to tell that if I had only rotation but no convection then which terms will go away? So, by the way Pr will remain there as a viscous term.

So these two terms will non zero and this term. Well, θ of course, does not exist, only rotation. So, I just take a fluid and rotate it. So, these will be u_1 and u_2. So, the diagonal term will be coming from viscosity term, and they will decay, and these two terms are what? What will they give you? So, let us put viscosity to be 0 and this α and $-\alpha$. What does it give you?

So, what are the eigenvalues for this? So, trace is 0. And determinant is α^2. So, trace 0 means? There must be imaginary complex conjugate of each other. And your determinant is positive, α^2. So, it is $i\omega$ and $-i\omega$. So, these are oscillations. I mean you can easily see this from the structure of the matrix. So, these are called waves. In fact, they have a name, it is called inertial waves.

For rotation this for gravity; this is called inertial waves. For rotation it is called inertial waves. And the property of the waves, I am not asking you to do it for this course but do can read in this book; very briefly I discussed about these waves. It has a role in the instability. Well I am going to give that as a homework. So, we can disentangle from the matrix.

So, there is a part which is oscillations. Convection will give you instability. It will try to create rolls, but because of oscillations, it may try to make oscillatory instability. Now I need to worry about eigenvalues and eigenvectors of this. Let us look at the eigenvalues. Now this is a 3 by 3 matrix. Now I cannot get all the 3 eigenvectors and eigenvalues by hand, which I could do for 2 by 2, but this one I cannot.

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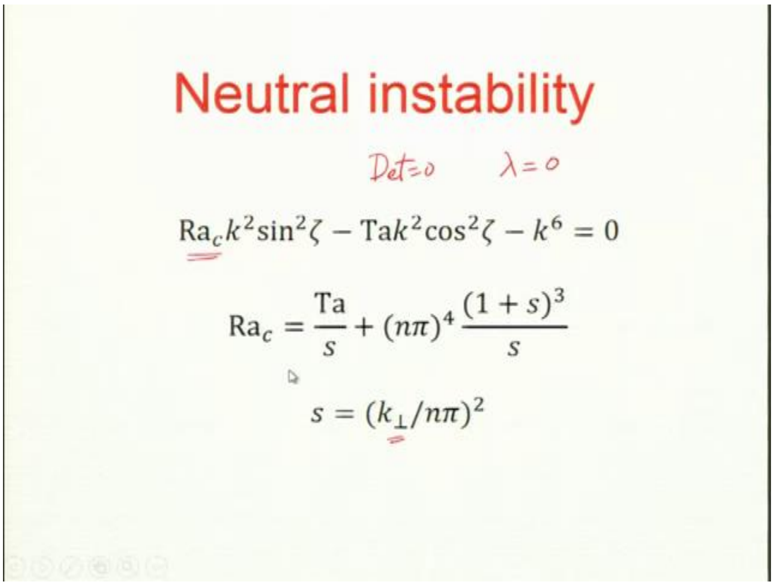

So, I can look only for neutral stability. So, how do you get neural stability? So, 0 eigenvalue. So, what do I get this 0 eigen value by the way? The condition for 0 eigenvalue is - determinant is 0; the easy part determinant is product of the eigenvalues.

So, determinant being 0 will give you one of the eigenvalues must be 0, and I do not even know what the other two guys are, but I know one of them are 0. The trace is negative.

So, at least one of them is negative for sure ok? But this is what I have left it as a very crude analysis, right now, that I just set determinant to be 0. Will give you one of the eigenvalues is 0. One of them is at least has a negative part but this is a condition for $\lambda = 0$, and this will give you Rayleigh critical given Taylor number. By the way, I immediately put Rayleigh as Rayleigh critical for $\lambda = 0$.

So, when I see Rayleigh critical; that means, $\lambda = 0$. So, that gives you in terms of Taylor number. Now s is non dimensional number, $\frac{k_\perp}{(n\pi)^2}$. So, given $k_\perp$ and Taylor number, I can get Rayleigh critical. Rayleigh critical is not for given n, it is not some $\frac{27\pi^4}{4}$. It depends on Taylor number, and if you want to get for what $k_\perp$ will Rayleigh critical be minimum? We need to analyze this further.

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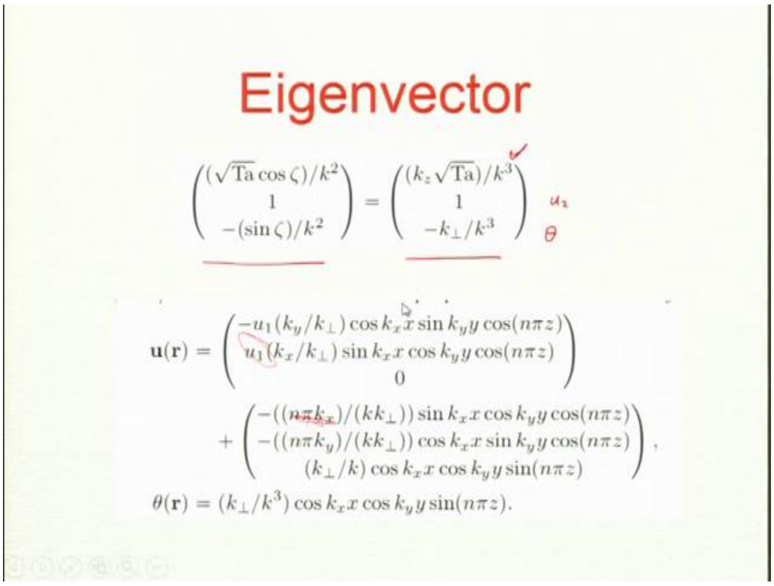

I am going to leave that as a homework. So, eigenvector for $\lambda = 0$; I can find. I cannot find for others by hand. So, eigenvector for $\lambda = 0$ is this. So, remember this was u_2. I should not write.. u_2 and θ. Before these two were nonzero, everyone was going to 0, but now u_1 is not zero.

So, u_1 and u_2 both are non-zero. So, that means it is a 3D structure; it is not a 2D structure like earlier case. Now I have plotted for a very crude example. For one special case, this is coming from Chandrasekhar's book. Now it has both u_1 and u_2, you see there is u_1 and there is u_2. This comes from these conditions.

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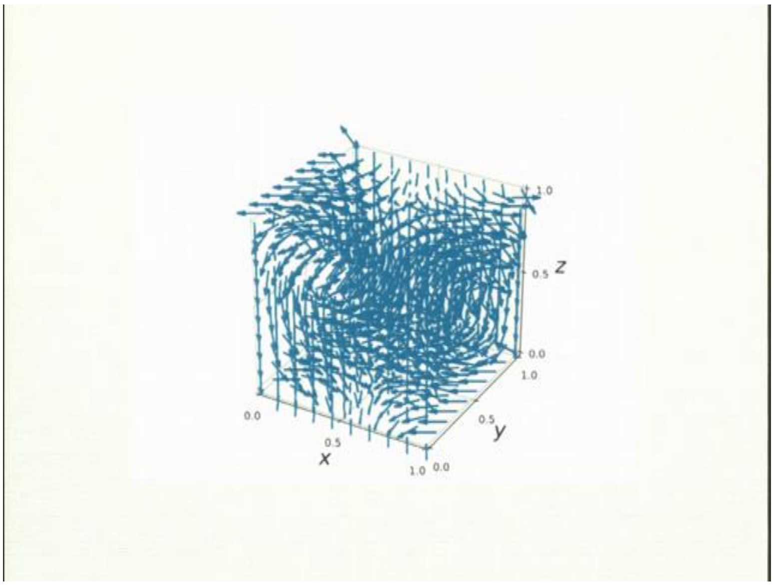

It is a 3D structure. It is like some kind of roll which is twisting. So, what I would like you to do is using computer program, either Mathematica or Maple or MATLAB or SymPy, get all the 3 eigenvalues, and 3 eigenvectors.

So, we have our $\lambda = 0$. Now what are the other eigenvalues? Now this $\lambda = 0$ is at the transition. Now if you go slightly above Rayleigh critical. So, Rayleigh is increased to beyond Rayleigh critical. Then does it remain real or does it become complex? Is the question clear?

So, this matrix A; so, I know that for Rayleigh equal to Rayleigh critical, one of the eigenvalues, which is likely to become unstable is $\lambda = 0$. So, when Rayleigh becomes greater than Rayleigh critical, I can plug in this. So, compute the eigenvalues. Now will it become complex, or will it remain real you can do it on computer, you know?

So, I would like to see the nature of eigenvalue, and that will tell you whether it is oscillatory or stationary. And get the eigenvector and we should analyze it. This structure better. So, I will stop here.

Thank you.