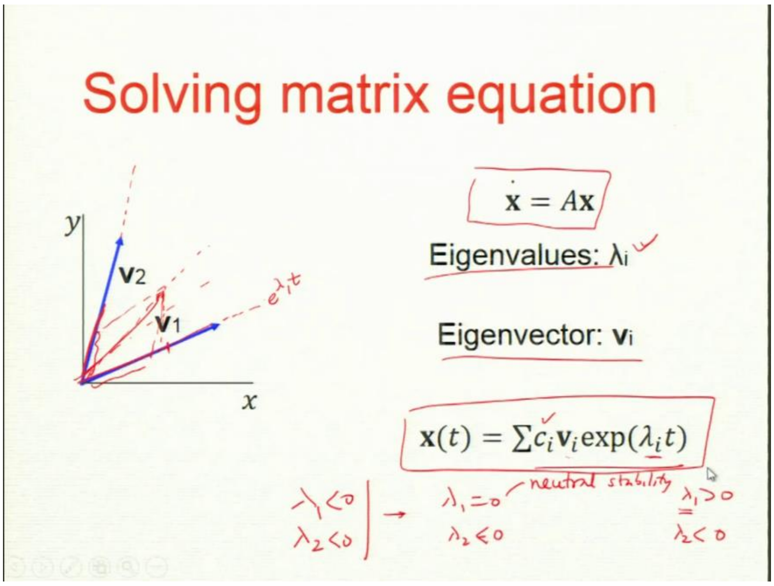
Physics of Turbulence
Prof. Mahendra K. Verma
Department of Physics
Indian Institute of Technology, Kanpur

Lecture - 16
Route to Turbulence Thermal Instability Continued

(Refer Slide Time: 00:18)

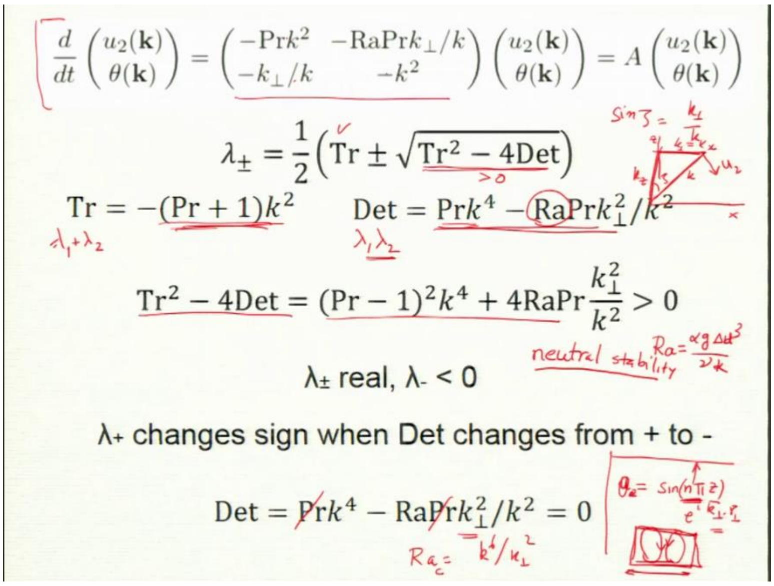


So, in the previous discussion, we had the equation which was of the form $\dot{x} = Ax$. Now, this is a linear equation, so it has eigenvalues and eigenvectors. So for our case the equation has 2 degree - u_2 and θ. So, it has 2 eigenvalues and 2 or less eigenvectors, but I will show that there are 2 eigenvectors. Now, the eigenvalues will tell us whether system will grow or not. I mean the velocity field will grow or not. How does eigenvalue tell you? So, we will look at the general solution.

So these are the 2 eigenvectors. Now, if my initial condition is along this eigenvector then depending on the eigenvalue, it will either increase in length or decrease in length. Same thing for if an initial condition is along v_2, it will decrease or increase. λ being negative is decay, λ being positive is growth, λ being complex could have oscillations.

Now, if it is combination of the two, so my initial condition let us say somewhere here. Then it will not grow along that line. It can go in some different direction depending on how much contribution it gets from v_1 and v_2. So, the general solution is this. From the initial condition you can get c_i's and once you have the c_i's then you can get this. x is a function of time. So, this is how we solve in linear algebra. So, our objective is to find the eigenvalues and eigenvectors. So, eigenvectors will tell you how the flow profile will look like. So this part I will tell you how.

(Refer Slide Time: 02:18)



So, let us look at our equation, so this equation which I had written ok. Now, please remember that $\sin(\zeta) = k_\perp/k$. Why? Because this is my angle ζ and this is $k_\perp$ and this is k. So, this is z axis and this is x axis. Now, so what we should look for is eigenvalues of this. So for 2 by 2 matrix, there is a simple formula for eigenvalues. Now, this, I memorize it - $\lambda_\pm = (Tr \pm \sqrt{Tr^2 - 4Det})/2$. Now, trace (Tr) is you know, the sum of the diagonal elements.

So, this one is $-Prk^2 - k^2$, this is trace. What is determinant? It is $Prk^4 - RaPrk_\perp^2/k^2$. So can you say that one of the eigenvalue is always negative? If you must know this theorem. $\lambda_1 + \lambda_2 = Tr$. Sum of the eigenvalues is equal to trace; that is a theorem of linear algebra.

And the product of eigenvalues is the determinant. Now let us look at first. So, if the sum is negative and of course, we need to check whether they are real or not. If they are complex then it has to be complex conjugate of each. The must cancel because this is real, Prandtl number is real, k^2 is real ok.

So, let us first see whether the eigenvalues are real and how do I know it is real? So, this is real I need to check these quantities. If $Tr^2 - 4Det$ is positive, then I know that my eigenvalues will be real. So, this is a bit of algebra, but you can easily do it. This is equal to that and is always positive. This algebra I will not do it right now, but you can easily show this.

So, if this is greater than 0 then my eigenvalues are always real. Now if it is real, then and the sum is negative that means, at least one of them must be negative ok. Now, so what happens really, so you can just check. Now if they are negative, both are negative let us say. So, what happens to the determinant? It becomes positive. If one of them becomes negative then? One of them negative the other one becomes positive then determinant becomes negative.

So, it changes from positive to negative and it goes to 0, so you look for the transition. So when is it? When will that transition take place? When determinant is 0. Now, do you recall the formula for Rayleigh number? $Ra = \alpha g \Delta d^3/\nu\kappa$. So you can see this is positive. Now this Rayleigh, if it is 0, then determinant is positive.

So, if the temperature difference is 0, then determinant is positive. That means, both the eigenvalues will be negative. Because you see, the determinant can become positive. Both have same sign and one of them is definitely negative, the other one must be negative. You keep increasing Rayleigh. At some point the determinant will become 0, because Rayleigh is the only parameter which I am tweaking. My tweak parameter is Rayleigh, and when Rayleigh crosses some critical value, then determinant becomes negative. That means, one of the eigenvalue has become positive, and eigenvalue becoming positive is what? Growth. I mean it is quite clear that the eigenvalue positive is growth. So $\lambda = 0$ is a transition when determinant is 0, and that is called neutral stability.

In this situation my, say the u_2 and θ will not grow. They will just stay there, but immediately after Rayleigh beyond Rayleigh critical, the growth will start.

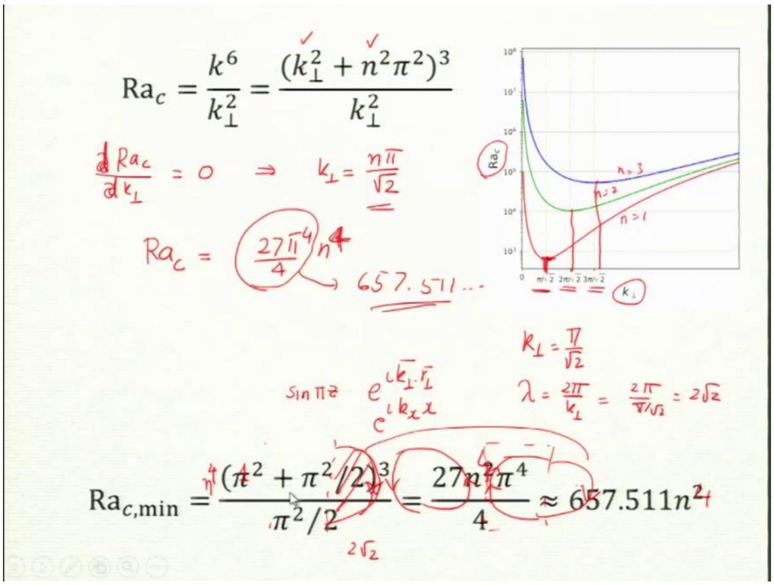
So, the condition I know now it is easy. When I increase Rayleigh it becomes positive and negative that is correct? So, the transition happens when determinant is 0, and I can easily see what is the value of Rayleigh? It is $k^6/k_\perp^2$.

So, what is wave number? k is the wave number and $k_\perp$ is component of the wave number. So given wave number, I know what Rayleigh critical should be. Rayleigh, so I define my Rayleigh, Ra_c. So k is related to the roll size. Now recall my definition of the Fourier modes. So I am going to write it here. I will not go back to those definitions. So let us look at θ. So $\theta = \sin(n\pi z) \exp(\mathrm{i}\mathrm{k}_\perp \cdot r_\perp)$.

So basically at the vertical direction my roll will be like that. In horizontal direction, it is periodic. So, in fact I need to worry about the size of a pair of rolls. So my box size will determine $k_\perp$, so I am going to come to that. I am going to show you a bit of it. But you can easily see that the dimension of the roll will tell you $k_\perp$ and k. Well $k_\parallel = n\pi$. That is k_z, and $k_\perp$is in the horizontal direction wave number. It is coming from k_x. One more point - just make sure you understand this one - that my $u_1 = 0$; that means, it is a 2D system. So, I am going to choose my 2D system, that means, my velocity field is u_2 and I am going to choose xz-plane.

So, I say my velocity field lies in xz plane. So my $k_\perp$ is nothing but k_x. This is $k_\parallel$ which is k_z and this is always $n\pi$. Now, let us do some more analysis. So, by the way, this is where I really have almost derived my condition. I am almost done. I have this Rayleigh number at which the stability will start. Now, you can easily see that there are many, many possibilities of k. Right? Because many n's are possible, many k's are possible. So, we need to see which one among them grows fastest or grows first. So this is my Rayleigh critical.

(Refer Slide Time: 10:39)



So, given n and $k_\perp$, I have this formula. Let us say I fix n. Which value of $k_\perp$, Rayleigh critical will be lowest? Now, as I said you know, the noisiest student obviously he the person, you know.. You can make that person.. So if some chance that noisy student will start being noise. So, you will looking for the most unstable system. I do not mean to say noisiest student means unstable, but I am just trying to make an analogy.

So, the most unstable mode, so we keep our k general but now we are looking for which among them will get unstable earliest. So, I plot this Ra_c. So now, for $n = 1$, the red curve. I have plotted for different n's. You can easily see from here that if I keep $k_\perp$ fixed and increase n, then $n = 2$ has larger value than $n = 1$.

But for given n, so you see right now if I keep more n then it will be even bigger. This y axis is Ra_c, and this is $k_\perp$. Now, given n there is a minima at some $k_\perp$. So for $n = 1$ and for some $k_\perp$, in fact, this value is a minimum. So, that will tell me which $k_\perp$ will get excited first, so how do I find this? It is easy.

So, I fix n and I take total derivative. This is equal to 0. Now, this is only set of algebra. This gives you condition that $k_\perp = n\pi/\sqrt{2}$. In fact from the figure you can see there is $\pi/\sqrt{2}$, $2\pi/\sqrt{2}$, $3\pi/\sqrt{2}$. These are minimum.

So, $n = 2$ is here, $n = 3$ is here but the figure does not prove it, but you can easily show from this derivation. So, now, once I put this $k_\perp$, then I can get Ra_c for given n. Formula is $\frac{27\pi^4}{4}n^4$. So, that is the Rayleigh critical for any n. I have derived this for given n. For $n = 1$, it is $\frac{27\pi^4}{4}$, which is 657 by 511. You can just plug it in.

So, this is my Rayleigh critical. So, for this Rayleigh critical, $n = 1$ mode will become unstable, for this $k_\perp$. Now, this $k_\perp$ also tells you that size of the convection roll.

So, $k_\perp = n\pi/\sqrt{2}$. So, what should be my wavelength? Wavelength is $2\pi/k_\perp$. The wavelength is x axis; remember it is exp $(ik_\perp \cdot r_\perp)$ which is $\exp(ik_x x)$. So, wavelength will be that - $\frac{2\pi}{\frac{\pi}{\sqrt{2}}}$, which is $2\sqrt{2}$. So, my horizontal dimension is $2\sqrt{2}$.

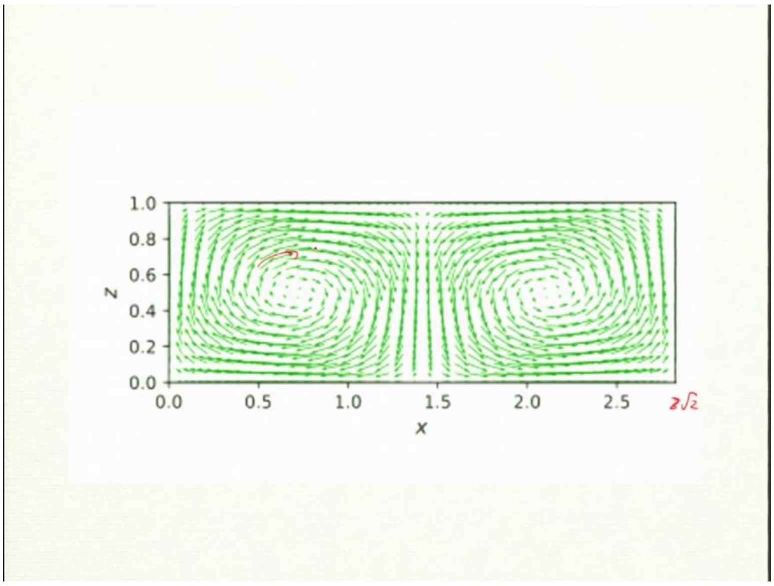
And the vertical dimension is $n = 1$, so in vertical direction I have $\sin(n\pi z)$. So remember, it is a non dimensional box, so $z = 0$ and $z = 1$. So, vertical dimension is 1. So, my box is basically like this, so this dimension is 1 and this is $2\sqrt{2}$, and the rolls will be a pair. So, I have better picture in the next slide, but it goes like this, and it is periodic.

So, same thing reappear after this. If I choose my origin somewhere here, then I will get half the roll here, and other half will come. So this half if I remove this half.. So, it is a possibility that we may get something like this know like this, so this must complete. So, what will I get here?

Coming up like that; so, this roll is periodic boundary condition.

Coming up like that; so, this roll. It is periodic boundary condition. So, you please remember that this guy must be replicated. This box comes here and it will come here. Is that clear? So, pair of rolls will be contained in dimension $2\sqrt{2}$ along horizontal.

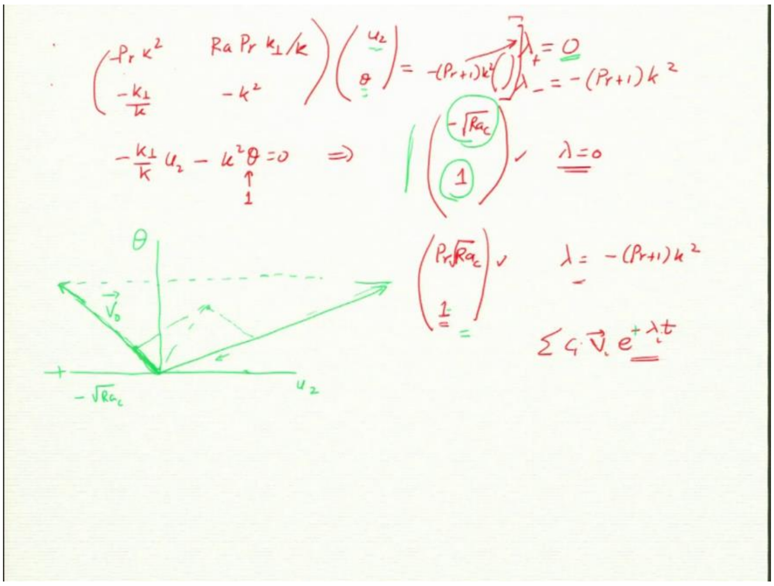
(Refer Slide Time: 17:20)



So, this is just generated from the computer. So, this is $2\sqrt{2}$ and the 2 rolls you can see the arrows are going like this, arrows are going like this and so, my origin is where the arrow is starting to go up. Of course, it could have been shifted, and vertical dimension is 1. So, in summary, I started with equation for that velocity field and temperature field. Remove the non linear term, then we linearized equation, go to Craya-Herring basis.

Then I get 2 linear equations - u_1 becomes 0, u_2 remains non zero, and these two equations, I find the eigenvalues and eigenvectors. Well, I have not yet found the eigenvector. So, I need one slide. Okay, so, we got the velocity profile. So, this was the basis function. I just plotted there, but I want to know exactly what are the eigen directions. What is relationship between θ and u? Velocity amplitude is fine, and velocity amplitude will grow in time. So, I want to derive that stuff. So, just pay attention to this part.

(Refer Slide Time: 19:06)



So, my equations were over - $\dot{u}_2$, $\dot{\theta}$ as shown in above slide. So, how do I find the eigenvector? So, there are 2 eigenvalues. So what are the 2 eigenvalues? For neutral stability, let us look at a neutral stability. So, neutral stability has one of the eigenvalue is 0. So, I call it λ_+. Other one is always negative - λ_-.

So, the sum is.. All we do is $(\mathrm{Pr}+1)\, k^2$. So that must be λ_-, because sum must be $(\mathrm{Pr}+1)k^2$. So, I can find the eigenvectors for these. It is easy no? In fact, for 0 eigenvalue I erase this one. This must be 0 for zero eigenvalue. So that gives you eigenvectors. So, let us work out for the eigenvector $-\frac{k_\perp}{k}u_2 - k^2\theta = 0$. I am going to make θ as 1. So that gives me u_2 which is $k^3/k_\perp$.

So, my eigenvector is $-\frac{k^3}{k_\perp}$ and 1. So, this corresponds to $\lambda = 0$. So, other one, how do I find the other one? Instead of 0, I put $-(\mathrm{Pr}+1)k^2$ times this vector.

So, $\lambda = -(\mathrm{Pr}+1)k^2$. So, by the way, the two equations are not independent. They are the same here. One of them is enough because the length of the eigenvector is not unique. Any length is fine. I can multiply this by any number, still it is eigenvector, and that is $Prk^3/k_\perp$. What is $k^3/k_\perp$? Now, we just had the formula. So $k^6/k_\perp^2$ was Rayleigh critical; right?

So, this nothing but $\sqrt{Ra_c}$. Why do I need to write all this? So, now we can plot them. So, by the way, now this is the answer. So, my temperature and velocity field are related. If the temperature amplitude is 1, then my velocity amplitude will be what?

So, by the solution is, if you recall, it was c_i then v_i eigen direction. This is a vector, $e^{-\lambda it}$. So, this eigenvector in that direction, my amplitude will go to 0 because my λ is negative, so I have this vector. So, if I choose any arbitrary direction, so let us first plot the eigen direction ok, so please keep in mind that this will go to 0.

So, let us plot them. So this is my u_2 direction and the θ direction; θ is y axis then u_2 is the x axis. So, what is the first one? So, y axis is 1 right? I am choosing θ to be 1 for both. First one is $-\sqrt{Ra_c}$. So, this is $-Ra_c$. Now the x component is proportional to Pr. So, $\Pr > 1$ will go in that direction; $\Pr < 1$ will go here, $\Pr = 0$ will be here.

So, this is the first eigenvector which is v_0. This is not velocity field, this is eigenvector, and the other eigenvector depends on prandtl number. It is somewhere here; depending upon Prandtl number. Now, if my u_2 and θ, they are real numbers. Well, I choose amplitude is real here. Now only this component will survive, this component will.. component along this direction will decrease to 0.

So, if I choose any initial condition here. This is linear algebra I do not need to emphasize it; you take a component of this along this direction how do I take the.. This direction and that direction; they are all non-orthogonal like this. So, this component will go to 0, this component will in fact, it does not increase; it will remain, so this will remain like that; is that clear?

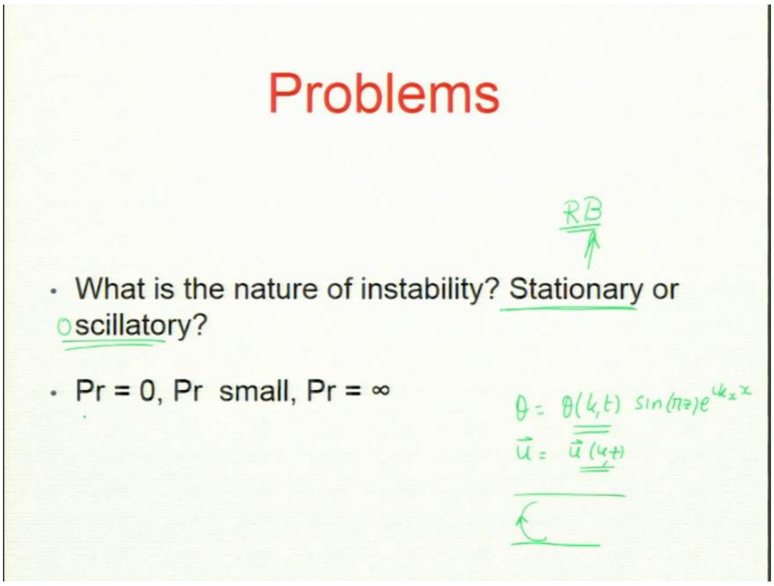
So, I know the what is a steady state for this Rayleigh critical or Rayleigh, Rayleigh number Rayleigh critical it should be just this ok. So, I plotted the only the velocity field from u_2 you can compute u_x and u_z right? By that formula which I gave you. And once you know u_x and u_z you can plot the figure I showed you. And once you have the u_x and u_z you can also compute temperature precisely because a neutral stability gives you that the velocity field and temperature field are related. So this is what how we compute it both.

Now, what happens when I increase my Rayleigh number a bit more? So, my eigenvector will change or eigenvector will remain same? So, my eigenvalue will become positive right? So, my eigenvalue becomes positive, so my eigen vector will change, it won't remain exactly same eigenvector.

But near the onset it will be approximately in the direction. Approximately. So, near the onset means my Rayleigh number is 1.1 Rayleigh critical and still my rolls will be behaving something like the original roll. But, if you make it too large and of course, things will not work. So, we can analyze all this using this linear stability analysis. So, eigenvectors will give you very useful information. I am going to show you in the next example that it will be more interesting. So any question on this?

So basically, the eigenvector and eigenvalue tell you about the roll structure near the onset, both temperature and velocity field.

(Refer Slide Time: 27:14)



Now, since my eigenvalues are real, and it changes from negative to positive. So, it will be basically 0 before and becomes only growing rolls. It will not be oscillating roles, so these are called stationary. So, what we get for Rayleigh-Benard is stationary, so it will grow but it is not oscillating. It is non oscillating. but sometimes you can get complex eigenvalue, my eigenvalue is real right now. It could become complex, then the imaginary part will give you oscillations.

So, first thing I know is that it is going to be a stationary roll, stationary not in the sense of time, time independent. It will the amplitude will keep growing, but no oscillations. So, what does it mean by amplitude growing? So, if you recall my θ was, $\theta(k,t)$. I did not write sometimes but $\sin(\pi z)$ for primary mode $e^{ik_x x}$.

So, this is growing. That means our velocity field is $\boldsymbol{u}(k,t)$ and of course, it depends on sin and cos and so on. So, if my amplitude is changing then it is growing faster and faster. Now, so that is what is instability. So, this will keep growing faster and faster but will it saturate at some point? So, can any real system just keep growing? So, somewhere it should saturate or something should stop growing and what will make it to stop to grow?

It is a nonlinear saturation, please keep in mind that if you had viscosity equal to 0 and diffusivity is 0 then any temperature will make it grow. You must wait till Rayleigh critical because of viscosity. You can look at the equation if you turn off the ν and κ, it is unstable at the onset itself. Well, any temperature if you give; it will just grow. So, Rayleigh critical is nonzero because of viscosity and diffusivity. So, you need non-linear saturation, non-linear term to kick in for growth to stop; ok.

So, we will do that bit later not today but we will we do it tomorrow possibly. Now, this was a finite Prandtl number. What I showed you is the finite Prandtl number, but we can do interesting cases. Interesting study of the extreme cases - Pr 0 and Pr infinity, or Pr small, but please keep in mind that my Rayleigh number was independent of Prandtl number; Prandtl and Prandtl got canceled.

So, my Rayleigh critical does not depend on the Prandtl number. Only thing it depends is.. Well, I will not answer this question but I would want you to think about what happens with Prandtl number 0; ok. This is interesting case - Prandtl number 0. Rest all is basically not so complicated, but that is an interesting case ok?

So, I will stop here, so I hope I answered the questions which was asked.

Thank you.