Introduction to Electromagnetism
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Lecture - 70
Assignment 3
Problems 1-5

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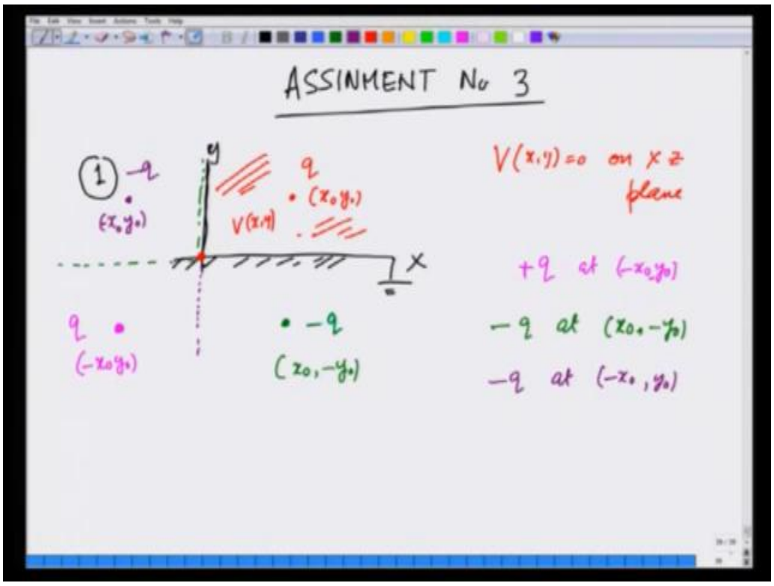

This and the next tutorial we are going to solve assignment three. The first problem of the assignment is on the image charge, and it says if we have the x z and y z plane, which are grounded; that means, there are at potential zero. And we take a charge at the position charge q at the position x naught y naught. What should be the image charge for this problem, if I want to solve for the potential in x y in this region shown by red. So, we take the image charge is such that the boundary condition is satisfied. If we wish to make the potential zero, we wish to make potential zero on x z plane. You must be wondering where is z is coming from, z is here, it is coming out of the paper. So, if I want to make the potential zero on this x z plane. Then I should be putting a minus q charge. Let me make it slightly different color. So, there is darker minus q charge, at the position x naught and minus y 0. This makes potential zero all over this plane, exact plane; however, this does not make it zero on the y z plane; that is this plane.

To make it zero on that plane, I will now put a charge minus q at minus x naught and y 0. That will make potential zero all over this, but this then creates a potential on the x z plane again, and I got to neutralize them. To neutralize this I take third image charge, which I put at this point which is plus q at minus x 0 y 0. Now you will notice that, the charge in pink and charge in green make the potential zero on the y z plane. The real charge in red and green make the potential zero on the x z plane. And minus q in purple and pink again make it zero on the x z plane. Minus q and red q the real charge make it zero on the y z plane. So, all these set of charges make potential zero at the desired boundary. So, the images charges are; minus q at minus x naught y naught, and I have... This is plus q plus q at minus x naught minus y naught minus q at x naught minus y naught and I have minus q at minus x naught y naught.

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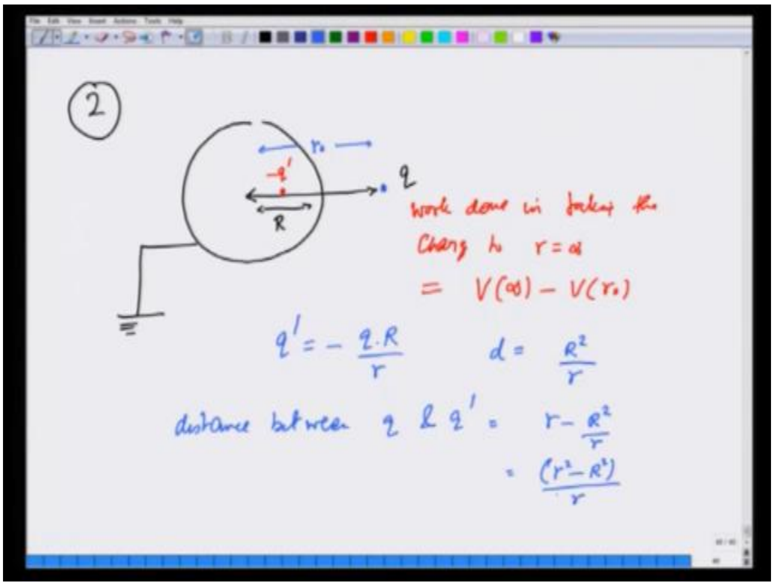

Problem number two; a point charge is located at a distance of 7 centimetres from the centre of a grounded conducting sphere of 5 centimetres. So, I have a grounded conducting sphere; that means the potential zero on this. And I have a charge point charge q which is located at a distance from the centre which is larger than the centre, the radius of this sphere. Let the radius be R, and let this distance initial distance be r 0 from the centre. So, let me show it on the top. Let this distance from the centre be r 0. Then we wish to calculate the potential of this charge q at this point. Why is there a potential. The potential is there because this q gives a negative image charge here q prime, and that attracts it. Now, what we will do to calculate the potential, calculate the work done, in taking the charge to r equal to infinity, and that work done will be equal to the potential energy at infinity minus potential energy at r naught. How do I calculate the work done? I will calculate by calculating the force on this charge. The force on this charge arises, from the image charge q prime, which is equal to minus q r over r, if its distance is r. Notice that I am taking the distance to be r, because I will be moving the image charge at from r naught to infinity. And q prime is at a distance which is equal to; q prime is at a distance d which is equal to r square over r. Therefore, the distance between q and is equal to r minus r square over r which is equal to r square minus r square over r.

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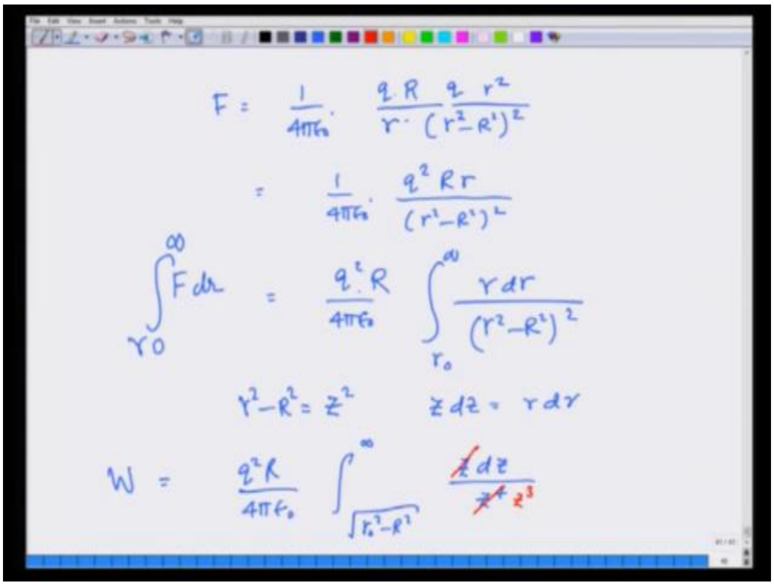

And therefore, the attractive force between the charge and the image charge, the attractive force f, is going to be equal to 1 over 4 pi epsilon 0 q prime which is q r over r times q over the distance square, which is nothing, but r square minus r square square and r square will go on top. So, this force comes out to be 1 over 4 pi epsilon 0 q square r r over r square minus r square square. So, work done in moving. This is attractive force; I will be applying a force in the opposite direction. So, work done by me in moving this charge from r naught to infinity, is going to be r 0 to infinity, which is equal to q square over 4 pi epsilon 0. Even this r comes out and I have r d r over r square minus r square square r 0 to infinity. To integrate this, let us take r square minus r square to be equal to z square, so that z d z is r d r, and I have therefore, work done is equal to q square r over 4 pi epsilon 0 integration r 0 is going to be, z is going to be equal to square root of r 0 square minus r square up to infinity. I have z d z over z raise to 4, this gives me z cubed.

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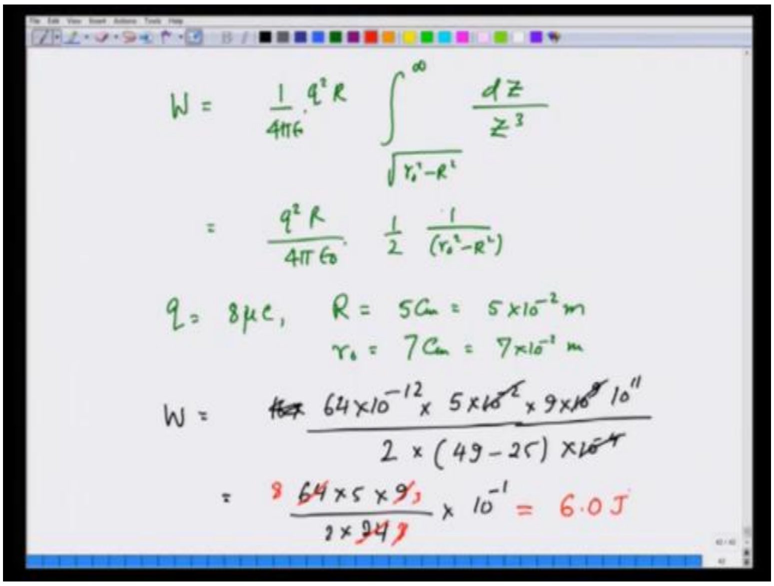

And therefore, the work done, is going to be equal to 1 over 4 pi epsilon 0 q square r integration square root of r naught square minus r square to infinity d z over z cubed, which upon integration gives me q square r over 4 pi epsilon 0 1 over 2 1 over r naught square minus r square. Now we are given that q is equal to 8 micro coulomb; r is equal to 5 centimetres, which is nothing, but 5 times 10 raise to minus 2 meters; r naught is given to be 7 centimetres, which is 7 times 10 raise to minus 2 meters, and therefore, the answer I get, is work done is equal to q square which is 64 times 10 raise to minus 12 times r which is 5 times 10 raise to minus 2 divided by 4 pi 1 over 4 pi epsilon 0, which is going to be nine times 10 raise to 9 divided by 2 times r square r naught square which is going to be 49 minus 25 times 10 raise to minus 4. So, this comes out to be 64 times 5 times 9 divided by 2 times 24 times 10 raise to. Let us see, this comes out to be 10 raise to minus 4, 10 raise to 2, 10 raise to 11, 10 raise to minus 1, so 3 8 3. So, which comes out to be 6.0 joules.

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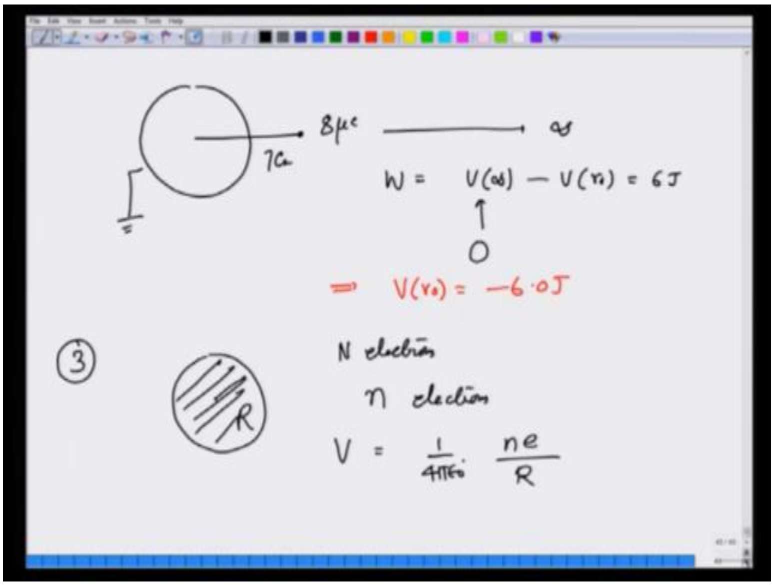

So, what we have shown, is that if I have this metallic sphere, which is grounded, and if I take this charge of 8 micro coulombs from 7 centimetres to infinity. The work done which should be equal to v at infinity, potential energy at infinity minus v at this point r 0 is equal to 6 joules. Taking v infinity to v 0 I get the answer v r 0 is equal to minus 6.0 joules, and that is the answer. Next question is, if we take a metallic sphere of radius r, and it is charged by assembling electrons on it. The total work done is, what we are going to assume in this that when the electron comes, it is kind of the charge of an electron is spread over this sphere or we are going to take that this charge is spread over capital radius R. So, if I have n electrons; small n electrons, the potential they are going to give rise to, is going to be 1 over 4 pi epsilon 0 n e over R for the metallic sphere. If I bring an extra electron now, so the work done bringing that extra electron, is going to be v times e, which is going to b 1 over 4 pi epsilon 0 n e square over r. This is from going from n equal to, number of electron n to n plus 1.

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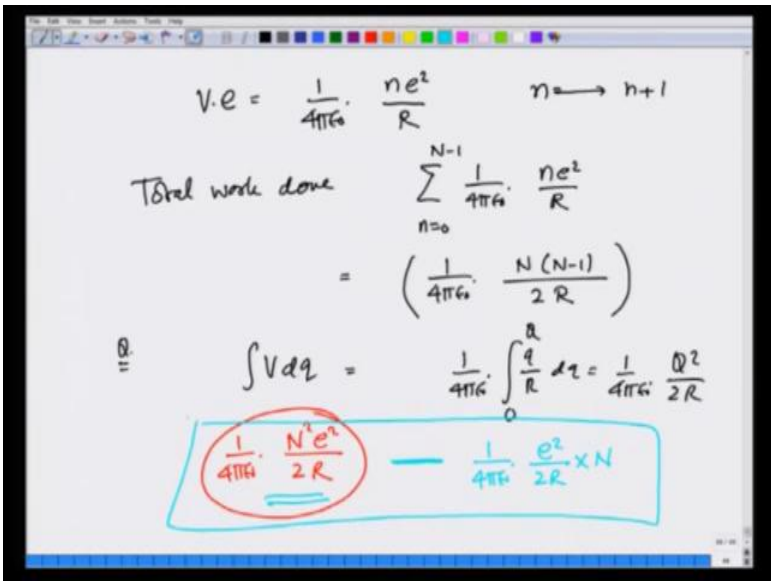

So, the net work done, total work done, when we increase n from 0 to N is going to be the sum of all these individual works 4 pi epsilon 0 n e square over r n equal to 0 to N, which gives me 1 over 4 pi epsilon 0 n n minus 1 over 2 r, and that is the answer. There is a nice interpretation to it. You see conventionally when I charge a sphere which with charge q what we do is, we do v d q which is nothing, but 1 over 4 pi epsilon 0 q over r d q, q going from 0 to Q, and that answer we get is 1 over 4 pi epsilon 0 q square over 2 r.

So, if I could bring the electrons with the infinitesimal charge; that is not bring 1 electron at a time, but even break that electron into infinitesimal charge, then the net work done would have been; 1 over 4 pi epsilon 0 n square e square over 2 r; however, we are not breaking up electrons. We are bringing in them one electron at a time. So, I am over counting in this, the energy required to assemble each electron n times. Each electron assembly would require the work of 1 over 4 pi epsilon 0 e square over 2 r, if I were to assemble each electron bit by bringing n by bringing it in infinitesimal charges, an electrons would require this much work; n times e square over 2 r 1 over 4 pi epsilon 0. This is the amount that I am over counting in calculating the energy that I am showing in the left. If I subtract this I should get the right answer, and that is my answer; that is the interpretation of our answer.

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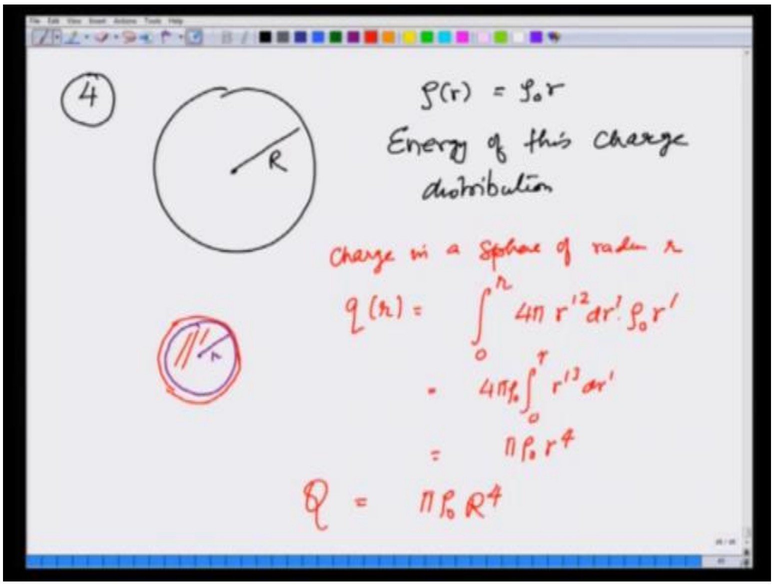

Next question number four. We are asking if a sphere of capital radius R has a charge density which depends on the distance from the centre, and is equal to rho 0 r, then what will be the energy of this charge distribution. So, I will think of assembling this charge as if I have a sphere of radius r and on top of it I bring a shell of thickness d r. Calculate the work done in this, and let r small r vary from 0 to R; that is how I will calculate the energy of this charge distribution. So, if I calculate the charge in a sphere of radius r; this is going to be q. Let us call it q r is going to be equal to integration; since this is spherically symmetric, I can write the volume element as 4 pi r prime square d r prime, and the charge density is rho 0 r prime, which then comes out to be 4 pi integration rho 0 r prime cubed d r prime 0 to r, which is equal to pi rho 0 r raise to 4; that is the charge which is contained in this sphere of radius r. So, total charge on this big sphere is pi rho 0 R raise to 4.

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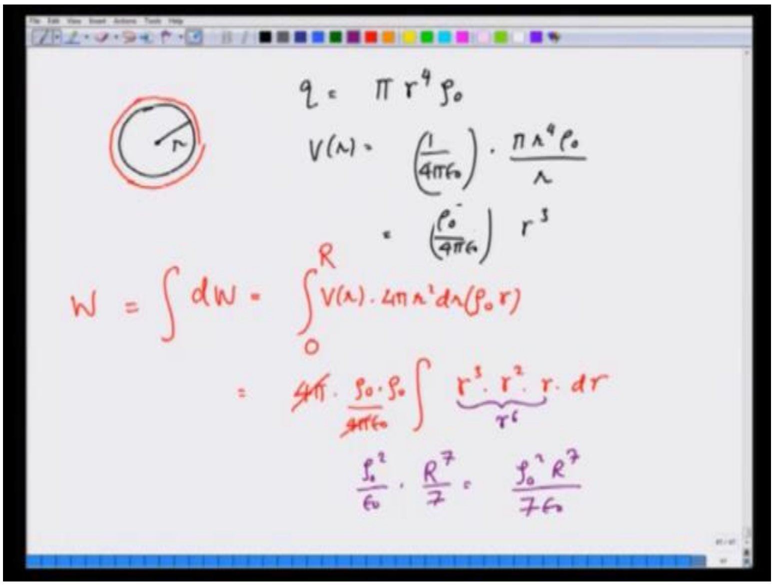

Now, due to this small sphere of radius small r, the charge is pi r raise to 4 rho 0, and therefore, the potential on the surface, is going to be 1 over 4 pi epsilon 0 pi r raise to 4 rho 0 over r, which is nothing, but 1 over 4 epsilon 0 rho 0 over 4 pi epsilon 0 r cubed. Now if I bring in that shell of thickness d r, the work done, let us call it d w, is going to be equal to the potential v r times 4 pi r square d r is the volume times rho 0 r; that is the density. And if I integrate this, from 0 to R, I will get the total work done. So, this comes out to be 4 pi comes out v r is nothing, but rho naught over 4 pi epsilon 0 integration r cubed times r squared times r d r, and there is an extra rho naught coming from the charge. So, this becomes 4 pi 4 pi cancels. I get rho naught square over epsilon zero times. This is r raise to 6 so I get r raise to 7 over 7, which is rho naught square r raise to 7 over 7 epsilon 0 is the total work done. Let us express this in terms of the total charge.

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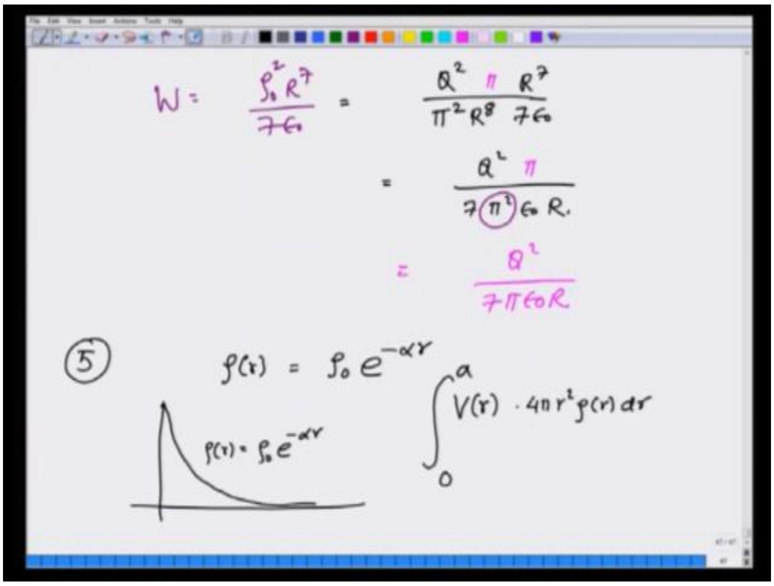

So, work done is rho naught r raise to 7 over 7 epsilon 0 rho naught square, which is equal to q square over pi square r raise to 8 times r raise to 7 over 7 epsilon 0. So, this answer comes out to be. So, this is nothing, but q square over 7 pi square epsilon 0 r and that is the answer. Just a correction, I am getting this extra factor of pi square here, let me check where I get it from. This is coming from here, this pi I forgot to take in. So, there will be a pi here, there will be a pi here, there will be a pi here, there will be a pi here, and therefore, there will be a pi here, there will be a pi here. So, final answer is q square over 7 pi epsilon 0 r; that is the right answer. Next we want to calculate the energy of a charge distribution rho r, which is equal to rho 0 e raise to minus alpha r. So, this charge density looks like this. It is rho 0 at r equal to 0 and then goes exponentially down, which is rho r equals rho naught e raise to minus alpha r. The trick used will be same as in the earlier problem; that is we will calculate v at r multiply this by 4 pi r square rho r d r and integrate it from zero to infinity.

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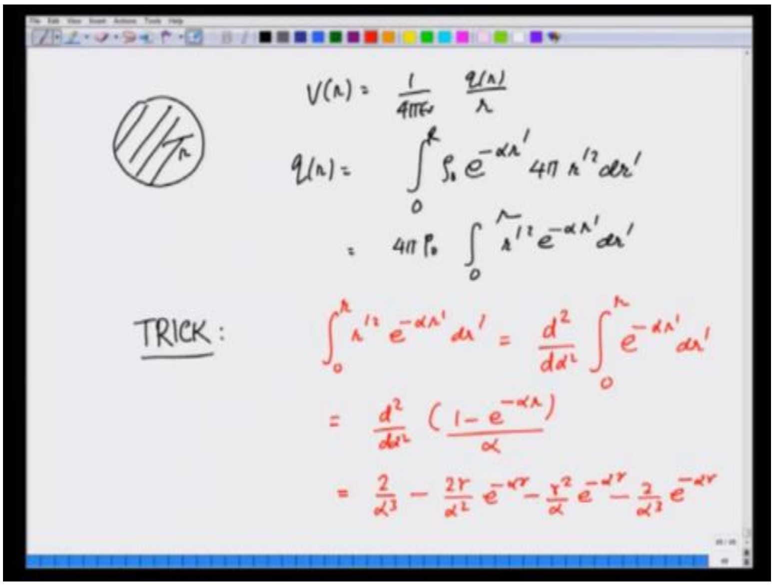

To calculate v r due to this charge in the region up to radius r v r is nothing, but 1 over 4 pi epsilon 0 q r over r. q r is going to be integral rho 0 e raise to minus alpha r prime 4 pi r prime square d r prime from 0 to r. So, this is 4 pi rho 0 integral r prime square e raise to minus alpha r prime d r prime from 0 r. I am going use a small trick here to calculate this integral, which is if I want to calculate 0 to r r prime square e raise to minus alpha r prime d r prime. I can write this as the second derivative with respect to alpha of integral e raise to minus alpha r prime d r prime 0 to r. Notice that when I differentiate twice with respect to alpha, the exponential function gives me an r prime square, and therefore, this is going to be equal to d 2 by d alpha square 1 minus e raise to minus alpha r over alpha, which when differentiated gives me 2 over alpha cubed minus 2 r over alpha square e raise to minus alpha r minus r square over alpha e raise to minus alpha r minus 2 over alpha cube e raise to minus alpha r. So, that is the charge contained in this sphere upto radius r.

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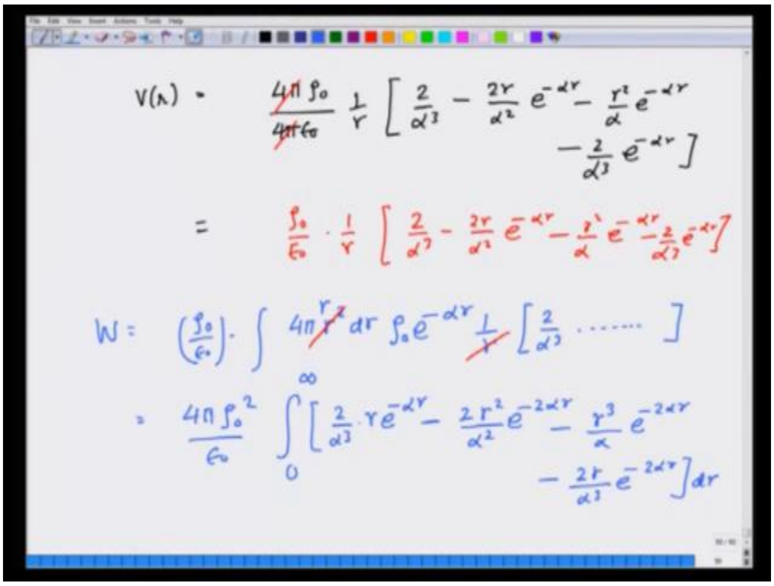

And therefore, the potential v r is equal to 4 pi rho 0 over 4 pi epsilon 0 1 over r 2 over alpha cubed minus 2 r over alpha square e raise to minus alpha r minus r square over alpha e raise to minus alpha r minus 2 over alpha cubed e raise to minus alpha r, which is equal to. I cancel this 4 pi rho 0 over epsilon 0 1 over r inside I have this whole thing; 2 over alpha cube minus 2 r over alpha square e raise to minus alpha r minus r square over alpha e raise to minus alpha r minus 2 over alpha cubed e raise to minus alpha r. Therefore the work done, is going to be rho 0 over epsilon 0 times integral 4 pi r square d r times, the charge density which is nothing, but rho 0 e raise to minus alpha r 1 over r 2 over alpha r cubed minus this whole thing. Let us cancel a few terms. This r cancels here gives me r alone. So, I get 4 pi rho 0 square over epsilon 0 integral 0 to infinity. I will take this r inside and I get 2 over alpha cubed r e raise to minus alpha r minus 2 r square over alpha square e raise to minus 2 alpha r minus r cubed over alpha e raise to minus 2 alpha r minus 2 r over alpha cubed e raise to minus 2 alpha r d r from 0 to infinity.

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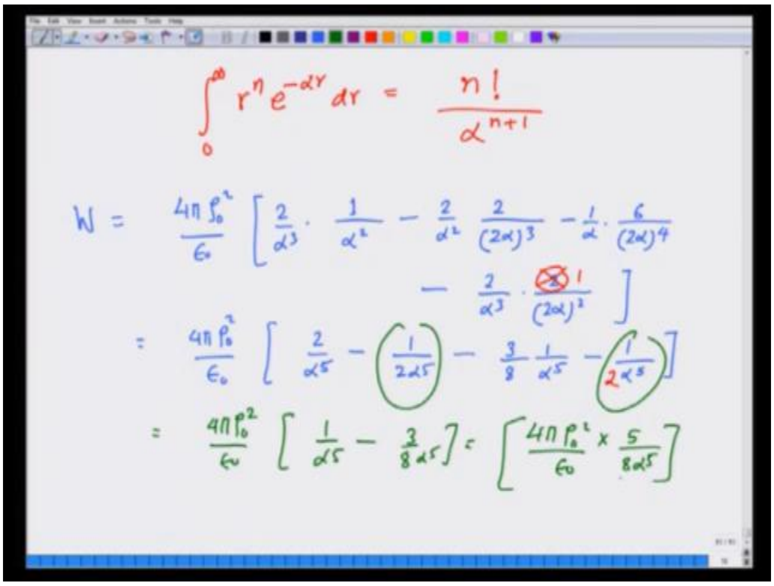

Now, we are going to use result that zero to infinity r raise to n e raise to minus alpha r d r is equal to n factorial over alpha raise to n plus 1, and that when applied to. Let me write this term by term, work done is equal to 4 pi rho 0 square over epsilon 0. The first term gives me 2 over alpha cubed times. Its r e raise to minus alpha r and therefore, it is 1 over alpha square minus 2 over alpha square r square e raise to minus 2 alpha r. So, that is going to be 2 factorial divided by 2 alpha cubed. Then next term is minus r cubed over alpha so I have minus 1 over alpha. r cubed will give me three factorial which is 6 over e raise to minus 2 alpha r.

So, 2 alpha raise to 4, minus the last term is 2 over alpha cubed r again will give me 2 factorial divided by 2 alpha square, and that is going to be my answer, which is 4 pi rho square over epsilon 0 2 over alpha raise to 5 minus 1 over 2 alpha raise to 5 minus 6 over 16. So, that is 3 over 8 1 over alpha raise to 5 minus 1 over alpha raise to 5. There is a mistake here I think I should not have this 2 here, because this was a integration of r, so it should be one factorial. So, this is only one and therefore, this would be a factor of 2 here. So, now this term, the term here and term here give me one over alpha raise to 5. So, this becomes 4 pi rho 0 square over epsilon 0 1 over alpha raise to 5 minus 3 eighths alpha raise to 5 which gives me 4 pi rho 0 square times 5 eighths alpha raise to 5 over epsilon 0; that is my answer. What about rho 0 in terms of the total charge.

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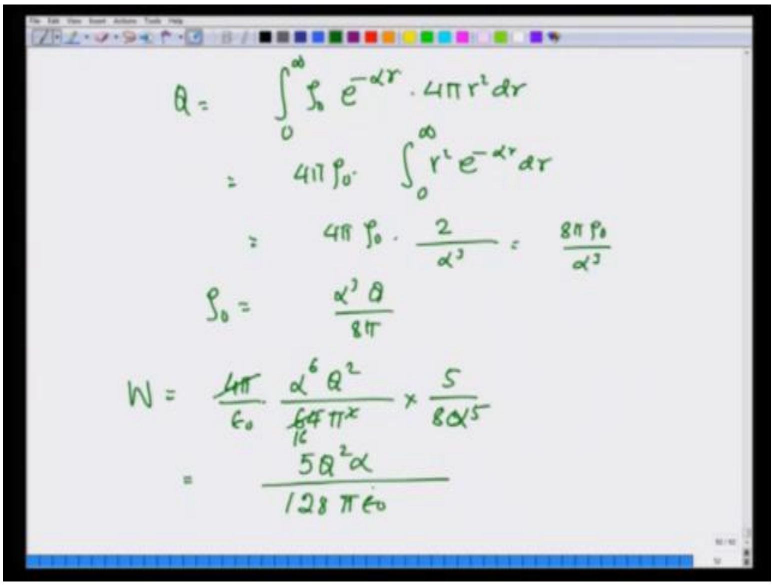

Now total charge q, is equal to integration rho 0 e raise to minus alpha r times 4 pi r square d r 0 to infinity, which is 4 pi rho 0 integration of r square e raise to minus alpha r d r 0 to infinity, which is 4 pi rho 0 times 2 factorial divided by alpha cubed. So, this is 8 pi rho 0 over alpha cubed, is the total charge. So, we can substitute rho 0 equals alpha cubed q over 8 pi and get the energy of the distribution to be w equals pi over epsilon 0, rho square which will be alpha raise to 6 times q square over 64 pi square times alpha raise to 5. So, that gives me the answer as 5 q square alpha divided by. Let us see 4, this becomes 16 pi cancels. So, 128 pi epsilon 0; that is the answer. That is the energy of this charge distribution.