ELEMENTS OF MODERN PHYSICS

Prof. Saurabh Basu
Department of Physics
IIT Guwahati
Week-01
Lec 4: Postulates of Quantam Mechanics, Old Quantam Theory

Let me tell you a few things about the course, which is probably clear from the content of the course. It's going to give you a bird's-eye view of many topics in physics that are of interest, that are of contemporary interest, and that one should learn to understand physics. So, we have talked about classical mechanics earlier and have given you a very brief introduction to Newtonian mechanics, Lagrangian mechanics, and then we discussed classical relativity or the special theory of relativity, mainly focusing on time dilation and length contraction. However, you know, in no way are these topics that have been taught complete. They need to be supplemented by a lot of reading and solving many problems. A few of them, or rather part of those requirements, will be addressed in the assignments, but you should also do additional reading because the scope I could cover on any of these topics is limited.

So, with that introduction, let me move on to quantum mechanics or these quantum ideas, and we will talk about atomic structure. So, let me give you a brief introduction to quantum mechanics or how it begins. So, many of the known concepts of classical mechanics had to be revised or even some of them discarded to build quantum physics. And we are slowly entering the realm of microscopic particles, for example, atoms, electrons, protons, photons, etc. And these signatures of these particles that is, subatomic particles, atomic particles are broadly referred to as quantum mechanical particles.

They are realized through discrete energy levels or sometimes observed intensities, such as those seen in mass spectrographs and so on. But these particles can also be visualized in high-resolution microscopes. For example, high-resolution transmission microscopy or scanning electron microscopy, known as SEM/TEM, or even field emission scanning electron microscopy, called FESEM. So, we won't discuss these experiments, but it's in your interest to know how the signatures of these subatomic particles are observed or can be realized. So, let me ask a few questions. Is quantization a proprietary property of quantum mechanics, like the quantization of energy levels or the quantization of angular momentum?

And the answer is no, because the frequencies of the organ pipes that you are aware of are quantized as well. Now, this simply stems from the fact that when you are trying to solve a differential equation pertaining to certain boundary conditions or initial conditions, you would get quantization. For example, these open organ pipes have frequencies quantized in V over 2L. And in closed organ pipes, the frequencies are quantized in units of V over 4L. So V over 2L and V over 4L, which means, say, let's talk about a closed organ pipe, closed at both ends.

And they have frequencies which are V by 4L and 3V by 4L, which means that there is no frequency that's possible. So the first harmonic comes at 3V over 4L. Between the fundamental frequency and the first harmonic, there is no frequency that's available or that can be generated. And this purely stems from the fact that you're solving a wave equation. Here we are talking about sound waves. So wave equations pertaining to boundary conditions that are closed must have nodes at the extremities or at the boundaries. So one gets quantization there. But what is the speciality of this quantization that we are going to see in quantum mechanics? The speciality is that the quantization carries a scale of Planck's constant, which we write as either h or h cross.

So this is something new. All these other things that you have talked about, just like what we have discussed the frequencies of these organ pipes are not quantized in the scale of h or h cross. Remember that h or h cross have the dimension of angular momentum. So h cross is h over 2 pi. It's simply a factor that divides. So we'll use them later, interchangeably with this factor of 2 pi at times. So, is there any other similarity or dissimilarity between classical and quantum physics? And the answer is the superposition principle.

Q. Does quantization only happen only in Quantum Mechanics?

A. No. The frequencies of the organ pipes are quantized.

Q. So what is the speciality of quantum physics?

A. The quantization carries a scale of Planck's constant. $\hbar = \frac{h}{2\pi}$

Q. Any other similarity/dissimilarity between Classical and Quantum Physics?

A. Superposition principle.

Once again, this is due to solving the linear differential equation. We would mainly be talking about the second-order linear differential equation in quantum mechanics. And when we talk about linear differential equations, there's no term such as X and Y, or there's no term which is a product of the two variables or rather, the dependent and the independent variable and so on. So, in these linear differential equations, the superposition principle works, and you know we can talk about Laplace's equation in electromagnetics, where we solve these equations pertaining to certain boundary conditions. And in a similar spirit, we'll be talking about the solution of the Schrödinger equation in certain kinds of potentials. Both have a nature that's similar. As I said, it's a second-order linear differential equation.

And if, say, psi 1 and psi 2 or y1 and y2 are solutions of these equations, this psi is just a function of x, and, say, y is also a function of x. Then, C1 psi 1 plus C2 psi 2 is also a solution, and this is called linear superposition. This holds in quantum mechanics or rather, the solution of the Schrödinger equation and also holds in classical electromagnetism. Now, while talking about relativity, we have seen that the non-relativistic or Newtonian mechanics that we have learned, in a nutshell, is really a limiting case of relativistic mechanics and valid for v much, much smaller than c. And so, if v becomes of the order of c or at least, you know, one-tenth of that or one-hundredth of that, you need to go to relativistic mechanics because space and time would be coupled there.

While studying Relativity, we saw that, non-relativistic (Newtonian) Mechanics is a limiting case of Relativistic Mechanics, and valid for $v \ll c$.

Similarly, Classical Mechanics is the limiting case of Quantum Mechanics.

Classical mechanics provides description of macroscopic bodies only, While quantum mechanics is in principle applicable to microscopic objects.

Further, there is also a difference between cases $v \sim c$ and $v \ll c$.

And you have all these lengths and times; they would not look the same as they would in a stationary frame. Okay, so similarly, classical mechanics is also a limiting case of quantum mechanics. Okay. So just like relativistic mechanics should be the description of all particles moving with any velocity I mean, which could be a velocity close to the speed of light or it could be, you know, much smaller than the speed of light. In fact, there is a problem with Newtonian mechanics: it doesn't give any special consideration to the speed of light. And that's why it fails or rather, it is an inappropriate description, which is what we have seen.

Similarly, classical mechanics also works in certain domain and quantum mechanics is, should be universally applied to all particles to, you know, decipher the properties of these systems, microscopic or macroscopic. However, for macroscopic particles or macroscopic bodies, classical mechanics works fine and this is what we have learned. So, classical mechanics is a limiting case of quantum mechanics, okay. Classical mechanics provides description of macroscopic bodies only, which is what I have said. Quantum mechanics is in principle applicable to microscopic object, but there is no embargo on quantum mechanics to, you know, to describe even macroscopic bodies as well.

So, there is also a difference in quantum mechanics for this case when v is of the order of c and v is much, much smaller than c. c, the velocity of a particle, v is the velocity of a particle. Velocity of a particle can never be larger than c, but it can be of that order means that, you know, 10% of that c is also very large velocity, knowing the speed of light, which has been talked about many a times while teaching relativity. So these are the cases that also have to be discerned or they have to be distinguished even in quantum mechanics. So here I present a table. The table has four quadrants and the ultimate description is in the first quadrant that is on the left top which I wrote as relativistic quantum mechanics. And on the right top, there is non-relativistic quantum mechanics, which is what we are going to talk about now.

At the bottom right is a non-relativistic classical mechanics, is the Newtonian mechanics and Lagrangian mechanics that we have learned. And relativistic classical mechanics is what we have learned earlier. You know, that is in the last module, we have learned that. So this has been done a little. Again, this has been done in short. This is what we are going to do now. And the one that's on the first quadrant would be stored for a later use. Okay, so let me go into the atomic structure. And while talking about atomic structure, there are many theories that have stemmed from very early on.

And now we know what the structure of the atom is. But it's important to know about the history and the way it developed. So there are Dalton's atomic theory, Thomson's model, Rutherford's model, Bohr's model, and then the quantum mechanical ideas, which give rise to the wave-particle duality. We won't talk about Dalton's atomic theory or Thomson's model, but we'll talk about Rutherford's experiment of backscattering by the alpha particles. And then we'll talk about Bohr's model a little later. And we'll, of course, talk about the wave-particle duality, which says that there are experiments in which electromagnetic radiation or photons show up as particles because we use the energy and momentum conservation relations.

Whereas in other experiments, they show up as waves, displaying diffraction and interference phenomena that are observed in waves. So, early on, let us talk about that story as well. The earliest concept of atomic structure was arguably given by an Indian philosopher, Maharishi Kanad, whose concept of matter constitutes building blocks, which he called Parmanu. The Greek philosopher Democritus also claimed that matter is formed of indivisible elements called atoms and is credited with the early proposal of the atomic structure. Of course, now we know that atoms are the building blocks of matter, with a nucleus composed of neutrons and protons, and electrons orbiting around the nucleus. Electrons have a negative charge.

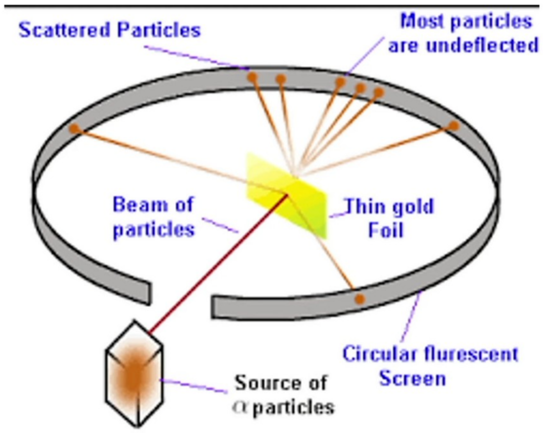
Protons have an equal but positive charge, and neutrons are chargeless. Though neutrons have a magnetic moment, which is important for certain kinds of experiments. The nucleus accounts for 99.94% of the atomic mass. This is strikingly similar to our universe, where the sun occupies a similar percentage of the mass, and planets orbit around it, held by gravitational force. Here, of course, they are held by the centripetal force. So, what is Rutherford's model? Rutherford lived between 1871 and 1937.

And most of these experiments were done in the early, you know, second decade of the 1900s, so around 1911 or so. Most of these experiments were performed, and he was assisted by Geiger and Marsden. And what they did was a scattering experiment where they bombarded heavy atoms with alpha particles, which have a mass of four units. And so these are two He4. So these are the bombarding particles or the projectiles. In fact, to his surprise, he found that these are like, you know, shooting bullets at thin tissue paper or something. And to his surprise, he found that some of them actually returned back along the direction in which they were fired.

So this sort of gives a feeling that there is something very strongly positive, so that these alpha particles are repelled and they return back. So that's how he tried to visualize the structure inside an atom. And this is what the main discovery is all about. And in that experiment, the effect of scattering by electrons is neglected. And these are really, you know, they may have some effect, but they are not taken into account. Also, there is no appreciable movement of the nucleus or the atom due to collision. And that's why he chose heavy atoms.

And this is probably one of the corrections that need to be done on the Rutherford scattering experiment, where you also need to talk about lighter nuclei. Okay, so the corrections for light nuclei need to be accounted for. And also the relativistic corrections, because the speed of these alpha particles is comparable to that of light. It's one over 20 that may also need to be taken into account. And so let us, you know, try to tell you what the experiment is like. So there is a source of alpha particles that you see here, and that's shooting alpha particles at this thin gold foil. How thin is it? I believe it's probably something around 500 to 600 Angstroms, maybe, and they are deposited along this ring that you see, this ash-colored ring.

Rutherford did detailed calculations on the angular distribution of the scattered α-particles

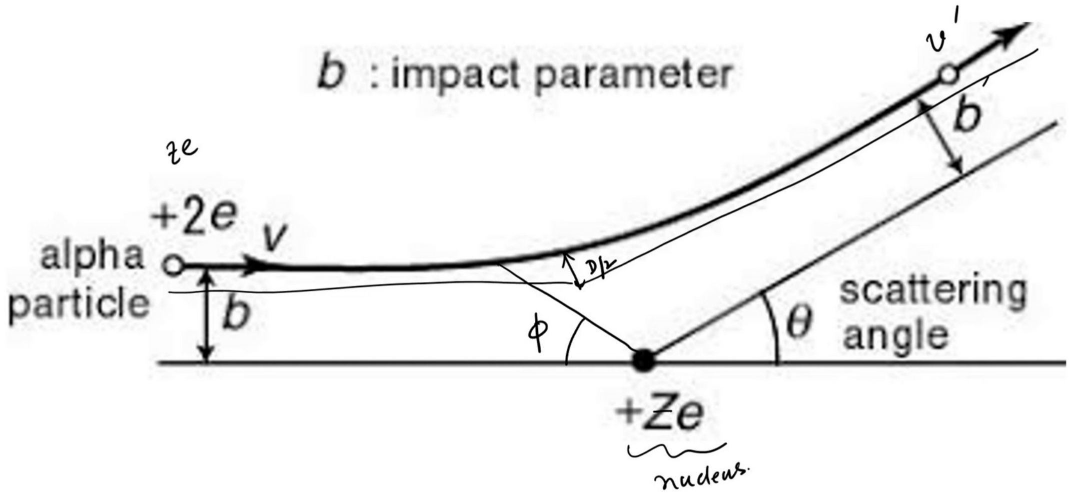


Rutherford realized that the backward scattering of the α-particles indicate that A small region of positive charge is responsible for this.

It's actually to give you a feeling that the angular distribution of the scattered alpha particles is measured, okay. And as you see, some of them are really reflected close to the direction, and some of them could almost trace back the direction of the incident alpha particles, you know, okay. So the scattered alpha particles are on the detector or they're captured in the detector, which is placed in some kind of angular array. And so most of the particles are found to be undeflected. So they go pretty much in the direction they are fired, and there could be a little deflection. And as you see, there is a circular fluorescent screen, and whenever an alpha particle impinges on the screen, a spot is formed, and the spot is noted, along with its angle with respect to the incident direction.

So, Rutherford realized that the backward scattering of the alpha particles indicates that there is a small region of positive charge, which is responsible for this. And this led to the discovery of protons and also neutrons. They are there inside this positive region. So let me show you a schematic picture of the scattering process, and we'll talk a little about the scattering. It's important to find out that this is what Rutherford did. He actually worked out the trajectory of these alpha particles. Okay. And so there is this alpha particle, and this plus Ze is the nucleus. Okay.

Schematic diagram of the scattering process



And this is 2 e , but we can call it as Z e with a small z, and this is a capital Z. So capital Z corresponds to the atomic number of the nucleus, and E is the electronic charge. And so the alpha particles are bombarded in this direction and get scattered in this direction. Let us call this velocity as v prime, and maybe this as b prime as well, but we will show that v equals v prime. And b equals b prime. You see that there's a scattering angle, theta, which is measured with respect to the incident direction. And if you let's talk about another angle, let's call it Phi, and so this is that angle, and maybe there is And maybe this perpendicular distance, let us call it as d by 2, all of which we will use in the calculation and so on. And so this is the schematic diagram, and let us try to work out the form of this trajectory so that we have an idea that there is a small region of space.

which deflects all the alpha particles and hence they should house positively charged particles which are of course protons. So we can show that v equal to v prime. I'm writing it as, you know, a scalar quantity and b equal to b prime. That is, now this v is, of course, the speed of the particle and b is called as the distance of closest approach. That is, so this is the distance over which it can approach the, you know, the location of this nucleus. So this is called b. And after it gets scattered and on its way to the detector, let's call that as b prime. That's it's called either called as the distance of closest approach or it is also called as the impact parameter. And it plays an important role in, you know, in this scattering theory, Okay.

So we'll prove this, that the initial and the final velocities are same and the initial and final, the impact parameters are same and which can be understood by the conservation of angular momentum because there's no torque that is acting, the angular momentum will be conserved. So it's m v b equal to m v prime b prime. So, that tells us that VB equal to v prime b prime where m is the mass of the alpha particles, Okay. And also the kinetic energy is conserved, so this is half m v square is equal to half m v prime square and that tells us that v equal to v prime and b equal to b prime. So, v equal to v prime here and then if you put it on the equation above then it gives you a b equal to b prime. So, which means that the initial and the final velocities are same and the impact parameters are same before collision and after collision.

So, after this alpha particles collide with the nucleus, I mean they do not collide head on, but they feel the presence of the nucleus and get deviated Because of this, there is a repulsion that they feel and this also tells you that there is a positive concentration or positive charge somewhere which is at this position which we have shown as Z e, Z e, the nucleus, okay. All right. So we need to know the trajectory. So the trajectory is needed and how you can get the trajectory is the following. Okay, so the force balance equation can be written. So what the trajectory means is that the relationship between some r and phi where r is equal to some, you know, some distance that is. So, it is like this kind of a distance. So, or you can say that it is a b and phi that is also equally possible to get your trajectory that you are looking for.

<u>Show that</u> $v = v'$ and $b = b'$

$$L = mvb = mv'b' \Rightarrow vb = v'b'$$

$$\frac{1}{2}mv^2 = \frac{1}{2}mv'^2 \Rightarrow v = v', \quad b = b'$$

<u>Trajectory</u>: Relationship between r & ϕ

$_2He^4$

$$F = \frac{(2e)(Ze)}{4\pi\epsilon_0 r^2} = m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2\right]$$

$\frac{d^2r}{dt^2}$ → radial acceleration

$r\left(\frac{d\phi}{dt}\right)^2$ → Centripetal acceleration; $\frac{d\phi}{dt}$ → ω

Define $r = \frac{1}{u}$

And the trajectory is important because we need to know how they are deflected and what the conditions are on various parameters. That is, if it comes very close, b becomes 0. That is, if it is heading directly towards this Z e, that is the nucleus, then it will, of course, be reflected back. So, how does the force balance equation look? The force balance equation, the force is equal to nothing but equal to Z e, it is 2 e, but we will write it as Z e because these alpha particles are nothing but this 2 He 4. So, that is why Z is 2, but we will just keep writing it as Z e and distinguish between the small z and capital Z, which is the nucleus, divided by 4 pi epsilon 0 r square, where r is the distance between the alpha particle and the nucleus. And it is a variable; I mean, this r is variable. And this is the force, okay? So, how do we write force in some generalized form? In the r, phi coordinate, phi is shown there.

So, it is equal to m, and we have this as d2 r dt2 and then it is a minus r d phi dt whole square. This is called the radial acceleration. And this is nothing but omega, and this whole thing is called the centripetal acceleration, basically r omega square. So, this force is equal to mass into acceleration and so on, and in order to find out the trajectory, define r equal to 1 over u. So, r is the inverse of u or u is the inverse of r equivalently, and this is done even in the gravitational case when you want to calculate whether the trajectory of a planet is bounded or it is a hyperbola which sort of escapes. So, this kind of redefinition of this angular or rather radial variable is done as well, okay. So, why did we do that? Because we need to calculate each of these radial acceleration and the centripetal acceleration separately. And so, we can write this as dr dt, which is equal to dr d phi and d phi dt,

And so, this is like dr du du d phi and d phi dt. And this is nothing but minus 1 over u square because dr du is minus 1 over u square, and then we have du d phi, and then we can write this as d phi dt is nothing but omega. And we can write this in terms of the angular momentum, which is equal to m v r and v equal to r omega. So, this is equal to m omega r square. So now we need omega. So, omega is equal to L over m r square or this is equal to L u square by m. So, in the case of d phi dt, we will write L u square over m. And the u square will cancel, and we have this as minus L over m du d phi, okay. So, that is your dr dt, but that is not where you stop because you need d2 r dt2. So, that is the second derivative of this. So, this is equal to d d phi of dr dt and d phi dt again, which is nothing but omega. So, this is equal to minus L over m d2 u d phi2.

So, we are, you know, slowly cruising at this d2 u d phi 2, that is, you know, this U is nothing but 1 over r. So, there is a relationship that is emerging between r and phi. If we can find out from this force equation, an equation which gives you a trajectory, that is what we are looking for, okay. So, this is, can be, you know, simplified as L square u square by m square d2 u d phi 2 and this M is nothing but the mass of the alpha particles, okay. So, if you put 2 in 1 that is this equation and let us define the equation that we have written here as 1. So, now we have everything we have a d2 or dt2. So, this is dt2 okay and you have d phi dt which is nothing but omega. So, we can simply put 2 in 1. And what one gets is the following.

We get an equation which is L square U square by M square. And there is a d 2 u d phi 2 minus 1 over u, l u square over m whole square equal to small z capital Z, e square u by 4 pi epsilon 0 m, Okay? So, that is the equation that we get, this is just the force balance equation. Now, we have put all the radial and this centripetal acceleration and their forms in terms of u, phi, L, etc. L is a constant of motion.

$$\frac{dr}{dt} = \frac{dr}{d\phi} \cdot \frac{d\phi}{dt} = \frac{dr}{du} \frac{du}{d\phi} \frac{d\phi}{dt}.$$

$$= -\frac{1}{u^2} \frac{du}{d\phi} \cdot \frac{Lu^2}{m} = -\frac{L}{m} \frac{du}{d\phi}.$$

$$\frac{d\phi}{dt} = \omega.$$

$$L = mvr = m\omega r^2.$$

$$\omega = \frac{L}{mr^2} = \frac{Lu^2}{m}$$

$$\frac{d^2r}{dt^2} = \frac{d}{d\phi}\left(\frac{dr}{dt}\right)\frac{d\phi}{dt} = -\frac{L}{m}\frac{d^2u}{d\phi^2}\frac{Lu^2}{m}.$$

$$= -\frac{L^2u^2}{m^2}\frac{d^2u}{d\phi^2} \qquad (2)$$

Putting (2) in (1).

$$\frac{L^2u^2}{m^2}\frac{d^2u}{d\phi^2} - \frac{1}{u}\left(\frac{Lu^2}{m}\right)^2 = \frac{zZe^2u}{4\pi\epsilon_0 m}. \qquad (3)$$

So, we can, you know, write this as instead of defining this as 3, we can define the next one where we make a, so we do a bit of simplification and write this as d2u d phi 2 plus a u, this is equal to minus z. Z and capital Z and this divided by 4 pi epsilon 0 L square and we use this impact parameter L writing it in terms of m, v and B. So this is minus small z capital Z e square m divided by 4 pi epsilon 0 m square v square b square, one of them will cancel and one will have a, so this m will cancel with 1 m here and you get this. So, let us call this as equation 3. Now, define d, d by 2 is shown there that is basically a length, scale of length. So, that is like 4 pi epsilon 0 divided by mv square by 2. So, that is the kinetic energy, Okay. So, one can write again this, you know, in a simpler fashion, this is equal to a minus d over 2b square and this is nothing but a differential equation for u, a linear differential equation where the second derivative is plus the variable itself is equal to some constant, constant for a given problem, okay.

So a general solution for this is u equal to A cos phi plus B sin phi minus d over 2 b square where we have used this trial solution as well as this the specific solution I mean the general solution is written as this for any phi. A and B are constants which need to be determined from the conditions that we have. And what are the conditions? The conditions are that phi goes to 0 for r going to infinity. So, as you see that this is the phi, the angle is phi. So, if it is infinity, this phi will go to 0. And the second one is dr dt, it goes to minus v for r going to infinity, Okay.

$$\frac{d^2u}{d\phi^2} + u = -\frac{zZe^2 m}{4\pi\epsilon_0 L^2} = -\frac{zZe^2}{4\pi\epsilon_0 m v^2 b^2} \quad (3).$$

Define $D = \frac{zZe^2}{4\pi\epsilon_0} \Big/ \left(\frac{mv^2}{2}\right)$.

$$\frac{d^2u}{d\phi^2} + u = -\frac{D}{2b^2} \quad (4).$$

A general solution:

$$u = A\cos\phi + B\sin\phi - \frac{D}{2b^2}.$$

(i) $\phi \to 0$ for $r \to \infty$

(ii) $\frac{dr}{dt} \to -v$ for $r \to \infty$

So, we are talking about v in the positive direction, that is why it is written as minus v. And that is how at very large distances, we have this v, which is the velocity of the approaching particles, alpha particles, okay. So, given these two, we can actually find out from one, we can find out that A is equal to D over 2B squared and from two, we can find out B equal to m v over L, which is equal to m v over m v b, which is equal to 1 over b, okay? So, this is say 5 and 6 which are the two unknown constants that we have determined from the boundary conditions there. So, if you put 5 and 6, we get u equal to d over 2 b square cosine of phi plus 1 over b sine phi That is minus d by 2 b square and then replacing or rather going back to our very own r variable. So, this is equal to 1 over b sin phi this plus D over 2 b square cosine phi minus 1 and that is equation 7 and that is equation of the orbit.

And this is what we have been looking for because we need to know how the alpha particle is scattered by that presence of there is a nuclear charge that is there. Of course, Rutherford did not know there is a positive charge, but he just knows that that if it has to, because the experiments say that it is deflected, most of them are undeflected, but even if there is a deflection, slight deflection, he wanted to find the trajectory of the alpha particles after collision. All right. So, this is quite simple. This is the equation of hyperbola. And it should be, I mean, in the sense that this should actually vanish to infinity unless you detect it by a detector. So, to remind ourselves that D is equal to 1 over 4 pi epsilon 0, it is a ZE square over m v square over 2. So, it is basically a parameter equal to the distance of the closest approach to the nucleus for a head-on collision.

(i) $$A = \frac{D}{2b^2} \qquad (5)$$

(ii) $$B = \frac{mv}{L} = \frac{mv}{mvb} = \frac{1}{b}. \qquad (6).$$

$$u = \frac{D}{2b^2}\cos\phi + \frac{1}{b}\sin\phi - \frac{D}{2b^2}.$$

$$\frac{1}{r} = \frac{1}{b}\sin\phi + \frac{D}{2b^2}(\cos\phi - 1) \qquad (7).$$

→ Equation of the orbit.

→ Hyperbola.

$$D = \frac{1}{4\pi\epsilon_0}\frac{zZe^2}{(mv^2/2)}.$$

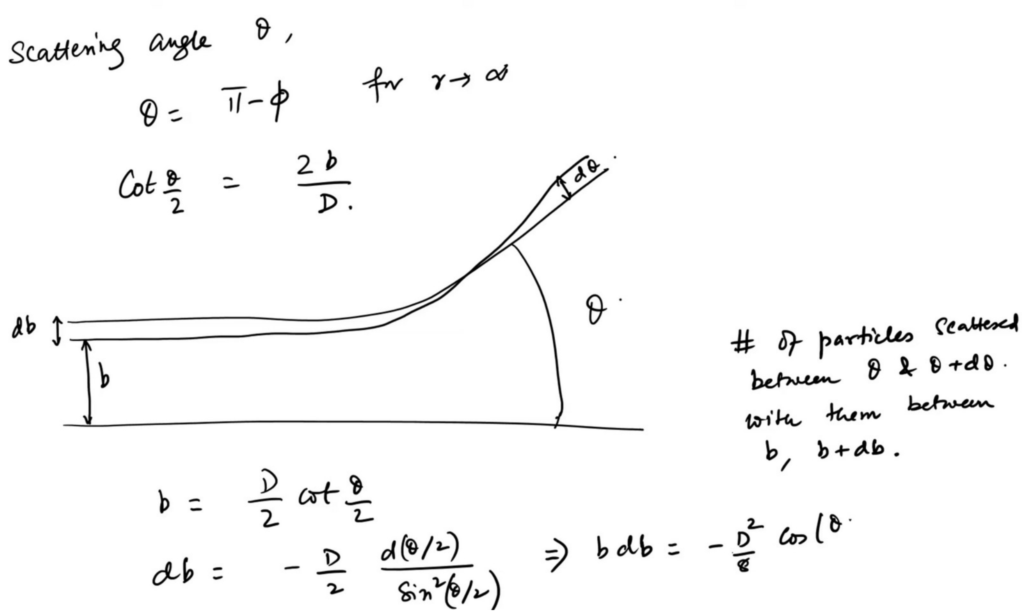
: Dist. of closest approach for head-on collision $(b = 0)$.

Also: the distance at which PE is equal to the initial KE.

So, the distance of closest approach for a head-on collision and a head-on collision would mean that B is equal to 0 also refers to the distance at which the potential energy, because this is the potential energy, is equal to the initial kinetic energy. Also, the distance at which the potential energy, which is nothing but z Z e squared over R (where R is equal to D here), is equal to the kinetic energy. So, the initial kinetic energy, which is half m v squared, okay. So, it is easy to see if you put D equal to R that is what happens, okay. Now, basically, in this particular case, the particle would stop because the potential energy and the kinetic energy would be the same, and then it will reverse back in the incoming direction. So, the scattering angle can be found, which is theta. So, theta can be found using theta equal to pi minus phi for, you know, r tending to infinity.

So, when you capture the particle at very large distances from this gold foil that you saw, that is where the scattering angle is. And so, cot of theta by 2, which is equal to 2b over d that is the relation. And so, you want to calculate this thing. Let us just redraw that picture. Alright, so this is basically theta. That is the scattering angle, and we want to know how many alpha particles are scattered between theta and theta plus d theta, so that we can get what is called a scattering cross-section. So, this is B, and this is dB.

So, between the impact parameter B and B plus dB, what is this? How many particles are scattered between that? So, we will use this to find the number of particles scattered between theta and theta plus d theta, and so, with particles between b and b plus db, okay. So, we use this b equal to d by 2 cot theta, and db becomes equal to minus d by d theta by 2 sin squared theta by 2. Okay, so basically d of theta by 2 sin squared theta by 2, and now we can calculate the BDB, which is given by so this BDB we have both and that is minus d squared by 8 cos of theta by 2 d theta and sin cubed theta by 2.



This is all algebra; you can fill in the steps, or they are all there actually. So, b d b is equal to minus d square over 16, and we have a sine theta d theta, this divided by sine of 4 theta by 2, okay. So, this is really proportional to the particles, you know, scattered between angle theta and theta plus d theta. So, we have a P b d b. This is equal to n theta d theta, where n theta is the number of particles; n theta d theta is the number of particles scattered between theta and theta plus d theta. They are proportional, but I am writing it as equality, and this is proportional to a sine theta d theta and divided by a sine of theta by 2. Without worrying about the prefactors, the scattering cross section can be obtained as d sigma d omega scattering cross section.

So, this d sigma d omega, which is equal to n theta d theta d omega, is a solid angle, and this divided by d omega, and this d omega is nothing but 2 pi sine theta d theta. So, that sine theta d theta will cancel with this. So, this is 1 by sine 4 theta by 2. There is a famous result in the sense that it is cosec for theta by 2, where theta is the angle that is shown, is a scattering angle, so it is half of that scattering angle, and this is what gives you the trajectory of this particle, okay? And this trajectory is important because this trajectory is, or rather these number of alpha particles scattered, can be experimentally determined, and so the scattering angle dependency of that scattering cross section comes as cosec 4 theta by 2 or 1 by sine to the power 4 theta by 2. So this is roughly what Rutherford's model is. So he understood that the nucleus is actually concentrated there.

$$b\,db = -\frac{D^2}{16}\,\frac{\sin\theta\,d\theta}{\sin^4\theta/2}.$$

$$P(b)\,db = N(\theta)\,d\theta \propto \frac{\sin\theta\,d\theta}{\sin^4\theta/2}$$

Scattering cross section: $$\frac{d\sigma}{d\Omega} = \frac{N(\theta)\,d\theta}{d\Omega}$$

$$d\Omega \to 2\pi\sin\theta\,d\theta$$

$$\sim \frac{1}{\sin^4\left(\frac{\theta}{2}\right)}$$

And, of course, there are electrons also. This really does not make too much of a comment about the electrons that are present in the orbiting part or which orbit about the nucleus, but that is more succinctly told by Bohr's model. Let us just do Bohr's model. So Bohr was more concerned with the electrons and about the stability of atoms, and if the electrons are moving around the nucleus in certain orbits and if they are classical particles, it is easy to understand that these would lose energy and effectively would crash onto the nucleus. And if that happens, of course, nothing can exist. So, there has to be a plausible argument that there is stability; these atoms are stable with the nucleus at the center of the atom occupying very little space.

But the most massive thing in the sense that it sort of comprises of more than 99% of the atomic mass and is consisting of positively charged particles which are protons there. And the electrons actually orbit in this around the nucleus and these parts or these orbits or these you know the over which they move the orbits over which they move around the nucleus they have to be you know specified. So, Bohr came and said that look I mean that these orbits are they do not the electrons when they move in this orbit they do not emit electromagnetic radiation and these orbits are characterized by the angular momentum taking on values which are integral multiples of the Planck's constant. So, this n is equal to n 1, I mean 2, 1, 2, 3 and so on and so this angular momentum of these particles would really be you know integer multiples or integral multiples of this H which is a Planck's constant. That seemed a little abrupt and what could be the reason for that.

Let us try to give some explanations how he arrived at that and why when he said that the angular momenta are quantized for these particles in those orbits and those are the orbits which are stable orbits and they do not emit electromagnetic radiation. In fact, they can jump from one orbit to another orbit either by absorbing electromagnetic radiation, which are photons, or they can emit electromagnetic radiation and can go from a higher orbit to a lower orbit and so on, Okay. So, for these electrons, the force is Z e square where Z e is again the charge. So, Z e into e if that helps I will write it as Z e into e divided by 4 pi r epsilon 0 r square, r is typically the atomic radius and this is equal to m v square by r. This is just like, you know, if you are taking, I mean, a stone at the end of a string and you're wielding it, you know, in a circular orbit, till you cut the string, the stone will keep rotating in that, revolving in that circular orbit.

And the tension actually provides the centripetal force. Here, the tension, the role of the tension is played by the Coulomb repulsion between the positively charged nucleus and the electron, okay. So, this is fine and then of course, we have this other thing that is m v r is equal to n h. So, if you eliminate v, So we have a Z e square is equal to 4 pi epsilon 0 m v square r, this is equal to 4 pi epsilon 0 n square h cross square by m r. So, this gives r is equal to that is the atomic radius is equal to n square h cross square and m Z e square, m is of course, the mass of the electron. I mean, please do not confuse, we have done the last problem in which we said m is the mass of the alpha particles, here of course, m is the mass of the electron.

Bohr's model.

$$L = n\hbar \qquad n = 1, 2, 3 \ldots$$

$$\frac{(Ze)e}{4\pi\epsilon_0 r^2} = \frac{mv^2}{r} \qquad (1)$$

$$mvr = n\hbar. \qquad (2)$$

r: atomic radius

Eliminate v,

$$Ze^2 = 4\pi\epsilon_0 m v^2 r = \frac{4\pi\epsilon_0 n^2\hbar^2}{mr}.$$

$$\text{or,} \quad r = \frac{4\pi\epsilon_0 n^2\hbar^2}{mZe^2} \qquad (3)$$

m: mass of the electron

So, this is the radius, the atomic radius, and you see that very nicely. This atomic radius depends on the quantum number n, where we have said n equals 1, 2, 3, etc. So, then v becomes equal to that's the velocity becomes equal to n h cross by m r, which is equal to 1 over 4 pi epsilon 0, that's Z e squared over n h cross, and that's 4. That's the velocity or the speed of the electrons. Okay. So, one is that the application of the angular momentum quantization, this is proportional to n squared. So, this is what we have seen here. So, R is proportional to n squared, and this comes from the quantization rule.

So, the atomic radius would depend upon n, where n equals 1, 2, 3, etc. Okay, so let us, you know, for a moment, specify it for a given problem. So, for Z equal to 1, we talk about H, the hydrogen atom, which is the simplest atom that we have. So, Z equals 1, and then if you put in, I mean, all these values m, e, h, etc. and one gets r equals 0.5 angstrom, and this is really in agreement with the atomic radii, which is of the order of 1 angstrom.

Usually, you know, so this is so how do we find the energy from this simple consideration? So, this energy is equal to so let us talk about the potential energy first with this radius, the potential energy of the electrons. So, this is equal to minus r to infinity, and we have a Z e squared by 4 pi epsilon 0 r squared dr, and this is equal to minus Z e squared by 4 pi epsilon 0 r. And the kinetic energy as well can be found. So, this is k equals half m v squared, this is equal to Z e squared by 4 pi epsilon 0 r, okay. So, they are the same, of course, but with opposite signs. So, the total energy is equal to I made a mistake somewhere. So, this is r. So, this would be 2 r because your v is given by this, so that is so it is half m v squared, so you can put all these things here, and this is a 2 r here, okay, not r.

$$v = \frac{n\hbar}{mr} = \frac{1}{4\pi\epsilon_0}\frac{Ze^2}{n\hbar} \qquad (4)$$

For $Z = 1$ (H-atom)

Putting $m, e, \hbar$ etc.

$r = 0.5\ \mathring{A}$

Usually, $r \sim 1\ \mathring{A}$.

Energy

Potential energy $\quad V = -\int_r^\infty \frac{Ze^2}{4\pi\epsilon_0 r^2}\,dr = -\frac{Ze^2}{4\pi\epsilon_0 r}. \qquad (5)$

Kinetic energy $\quad K = \frac{1}{2}mv^2 = \frac{Ze^2}{4\pi\epsilon_0 (2r)}. \qquad (6)$

$$E = K + V = -\frac{Ze^2}{4\pi\epsilon_0 (2r)}. \qquad (7)$$

And so the total energy is equal to K plus V, which is equal to minus Z e squared divided by 4 pi epsilon 0 into 2 r. So if you use the r from equation 3, let's call this as 5, 6, and 7. So the total energy of this hydrogen atom comes out to be E is equal to minus m z squared e to the power 4, 4 pi epsilon 0 squared, 2 h cross squared and 1 over n squared. And this is exactly what you get by solving the 3D Schrödinger equation, which is the Coulomb potential corresponding to the Schrödinger equation for the Coulomb potential.

You get exactly this, and this has a value which is 13.6 with this minus sign there. So let us remove this minus sign. So it is minus 13.6 divided by n squared electron volt. That is the energy.

But let us, you know, sort of this quantization of the angular momentum, which is the main postulate of Bohr. What does it mean for angular momentum? Okay, so for that, we start with the Bohr-Sommerfeld quantization condition. Now, this is stated it is also called, I think, the Wilson-Sommerfeld quantization condition. And it can be kind of derived from the WKB approximation. We will probably not have too much time to talk about that, but that is an approximate method which gives you solutions of the Schrödinger equation for linear, slowly varying potentials. So, this condition states that p q d q is equal to n q h, h or h cross does not matter. So, n q is a quantum number which has integer values, and q is a periodic coordinate of time.

Which means it is time-periodic, so q of t plus T is equal to q of t, okay. p q is the corresponding momentum, and we have touched upon the generalized coordinates and generalized momentum. So, these are really the generalized coordinates and generalized momentum. So, for a particle undergoing circular motion, as is the case for an electron. So, your p q is L and q is really theta. So, if we apply the condition, then we have L d theta, which is equal to n into h. Now, we are not writing q, but n is a sort of quantum number which corresponds to this coordinate.

$$E = \underbrace{\frac{-mz^2e^4}{(4\pi\epsilon_0)^2 2\hbar^2}}_{13.6} \cdot \frac{1}{n^2}.$$

Quantization of Angular Momentum — What does it mean?

Bohr-Sommerfeld quantization condition.

$$\oint p_q \, dq = n_q h$$

n_q : 1, 2, 3 . . . . .

q is a periodic coordinate of time.

$$q(t+T) = q(t).$$

p_q : Corresponding momentum

Circular Motion

$p_q \to L$

$q \to \theta$.

$$\oint L \, d\theta = nh$$

$$L \times 2\pi = nh \Rightarrow \boxed{L = \frac{nh}{2\pi}}$$

Bohr Condition.

So, that tells you that L into 2 pi becomes equal to n h, and this tells you that L equals n h over 2 pi. Sorry, this is L, and this is the Bohr condition. Of course, this is mathematically consistent; there is nothing inconsistent about it. But we are still looking for some physical arguments and trying to understand this condition that Bohr gave about the conservation of angular momentum or constancy of the angular momentum, which is the integral multiple of Planck's constant what it really means physically. So, a physical argument comes from the de Broglie principle. The de Broglie principle states that p equals h over lambda. We will talk about that, but just a priori, p and lambda are the momentum of the particle and the wavelength of the wave. So, we will get into this quantum mechanical model of matter or quantum mechanical interpretation of matter. Matter can exhibit both wave and particle nature if they display both these features that of matter. Particle and that of a wave, then there has to be a correspondence because it is the same system. In some experiments, it appears as a particle; in some systems, it appears as a wave.

So, the particle aspect is shown by its momentum because there is a mass and a velocity, and so on, and the wave aspect is shown by its wavelength. This is a relation that connects the momentum of the particle and the wavelength of the wave. So, if you write this as m v r, which is the angular momentum, this is nothing but p r, and this is equal to n h over 2 pi. As we said, h and h-cross are the same. Initially, Bohr's postulate was simply n h, but then we just got it here as n h over 2 pi. It is just a division by 2 pi; it would not matter physically at all. So, if you apply p equals h over lambda, then this gives you h r over lambda, which is equal to n h over 2 pi. The h will cancel, and it gives 2 pi r equal to n lambda. Now, this is an important thing and can be physically interpreted.

This tells you that the wavelength of the particle should have an integer number, or rather, an orbit of the particle should accommodate an integer number of wavelengths. It cannot happen that n is not a number. So, let me give you an example. This is a sort of orbit, okay? Badly drawn, but you understand that. So this and you know, sort of one, So, it is just showing that this is the orbit, and I am just showing it a little off because you need to sort of show that this one complete orbit so this n equals 1, okay. So, this one corresponds to, for example, there are kind of two orbits that are there so n equals 2 and another orbit should have this, probably the most, you know. So, let us.

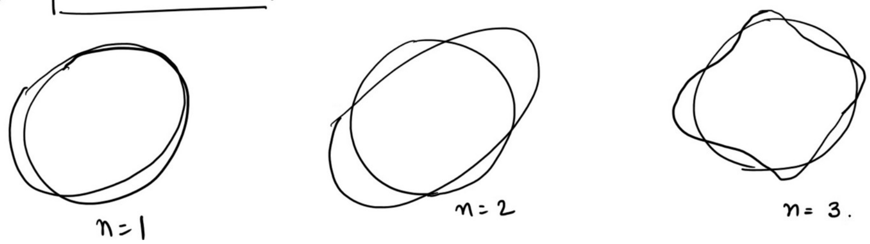
Debrogrie principle.

$$p = \frac{h}{\lambda}$$

$$mvr = pr = \frac{nh}{2\pi}$$

$$\Rightarrow \frac{hr}{\lambda} = \frac{nh}{2\pi}$$

$$\Rightarrow \boxed{2\pi r = n\lambda}$$



So, this is n equal to 3 and so on maybe I have drawn not very well, but what I want to say is that this accommodates the orbit is such that it accommodates integer number of wavelengths of the particle and where is this condition coming from this condition is coming from the Bohr's postulate which is about the conservation or rather the quantization of the angular momentum. And so the quantization of the angular momentum means that a particle with its you know with its equivalent wave description would have wavelengths the orbit of the wave would accommodate exactly integer number of wavelengths. and that so they should be consistent and that talks about the stability of the orbit. So, it cannot happen that they do not join you know I mean say I sort of draw another one where you have this and this and so on that cannot be true okay.

So, it does not have this integer number of wavelengths whereas the other ones have okay. So, this is something that will physically connect the reason that for the stability of these atoms Bohr had proposed this postulate of the concept or the quantization of the angular momentum. So, we will stop here for today and we will continue next class. Thank you.