ELEMENTS OF MODERN PHYSICS

Prof. Saurabh Basu
Department of Physics
IIT Guwahati

Lec 30: Binding Energy, Shell Model, Liquid drop model

Welcome to this concluding class on nuclear physics. We will be talking about nuclear models. This is actually quite interesting and inspiring as well because nuclear forces are not as well understood as Coulomb forces. But still, just like atomic theory, one has been able to formulate nuclear theories as well. And there are nuclear models.

These two of them we are going to talk about briefly. But before that, we will talk about binding energy. And this is a very important topic in the study of nucleons and nuclear physics. So the question is that what binds these nucleons, which are nothing but neutrons and protons that we know of now? What binds these things together?

Binding energy

What binds the nucleons together?

Let's consider Deuterium ${}_1^2H$ which has a proton and a neutron in the nucleus.
Thus total mass of ${}_1^2H$ = ${}_1^1H + \text{neutron} = 2.016490\ \text{u}$.

However, the experimentally measured mass is 2.014102 u, which is 0.002388 u lower than the above value.

In terms of energy, this is $(0.002388\ \text{u})\left(931.49\,\frac{\text{MeV}}{\text{u}}\right) = 2.224\ \text{MeV}$

We could ask how much of energy is required to break apart the nucleus of a ${}_1^2H$ into a proton and a neutron? The value is indeed 2.224 MeV!

Any energy lesser than that, the nucleus is intact.

And we know that the it's a short range force and within a few for me meter, it's very, very large. And outside the nucleus, it has no existence or other. It is not important with regard to or other as compared to Coulomb forces. So let us consider deuterium as an example, which is 1H2, which has a proton and a neutron in the nucleus. So the total mass of deuterium is equal to 1H1, which is hydrogen, plus a neutron, which is equal to 2.016490 units.

However, the experimentally measured value is found out to be slightly less, which is 2.014102 atomic unit, which is about 0.002388 units lower than the value that we have quoted above. And this value, the difference between them is this. So this is the experimental value for deuterium nucleus. And so in terms of energy, if we multiply it by this 931.49 MeV per U. then it comes out as 2.224 MeV and this is a fairly large energy if you think and so we could ask this question that how much of energy is required to break apart the nucleus of a deuterium into a proton and a neutron and the value is indeed these value that we have got which is 2.224 MeV.

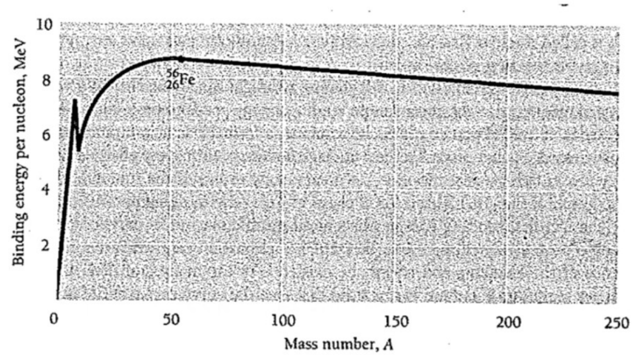
For a general element ${}^{A}_{Z}X$, the binding energy is:

$$E_B = [Zm({}^{1}_{1}\mathrm{H})\,\mathrm{u} + Nm(n)\mathrm{u} \; - m({}^{A}_{Z}\mathrm{X})\,\mathrm{u}\,](931.49\ \mathrm{MeV/u})$$

The nuclear binding energies are very high. Ranges from 2.224 MeV to 1640 MeV for ${}^{209}_{83}Bi$

Binding energy/Nucleon:

$$\frac{\text{Total Binding energy}}{\text{Total \# nucleons}}$$



Binding energy plot vs A.
The peak at $A = 4$ (He) is stable.

So any energy if you supply the nucleus will be intact and will not have any emission or rather the nucleus doesn't break apart and if you give this energy then of course the neutron and proton can be separated from the nucleus. And so this is for a general element, so we have given an example for deuterium and so for a general element which is given by X whose atomic number is Z and the mass number is A, the binding energy has this form which is Z is the number of protons which you multiplied by the mass of hydrogen in these atomic units and then Mn, which is the number of mass of a neutron, and so n is the number of neutrons, and just to remind you that A is equal to Z plus n, where Z is the number of protons, and capital N is the number of neutrons, so this is the neutron mass, and then there is this term that comes, which is the mass of this atom, and when you multiply by these atoms, you know, the 931.49 MeV per U, one gets, you know, some value. These values will be decided by these, all these things, all these knowing these masses and so on.

Okay, so this is the binding energy for any general element. And in fact, one important quantity that one needs to find or rather is interesting is that, This is, you know, the binding energy per nucleon. So, in fact, just to one step above, the nuclear binding energies are very high and it ranges from 224 MeV to 1640 MeV for these bismuth 83209. And so we plot here the binding energy per nucleon and is the total binding energy divided by the total number of nucleons.

I am sorry that this part of the text is not visible. You'll see that when you get these slides. So what is written here is that from this plot. This is the place. Let me use a color here.

So this is the most stable nuclei, which is that corresponds to, you know, 56 is a mass number corresponds to FE, which has got some, you know, value, which is about, you know, more than 8 MeV per nucleon. It's actually 8.8 MeV per nucleon. So this value is about 8.8 and that's what is written there. And you see a peak here that is also interesting and that corresponds to the helium which is 2He4. So this 2He4 and helium is stable, and we know that helium is stable.

There are alpha decays that happen, and helium nuclei do not undergo any further disintegration. So this corresponds to the first peak that corresponds to the helium nuclei. And then the one that you see here is actually the most stable one, which corresponds to a mass number of 56. And it's actually the iron. So now, let us talk about fission and fusion.

Let me just go back and to this binding energy curve and then you see that the binding energy is plotted as a function of this mass number A and after this 56 it sort of more or less remains constant it just goes down a little as the mass number is more but it sort of remains by and large a constant I mean I mean, the value that you see here is just about 8 MeV, which is the binding energy per nucleon. So that's the trend of the binding energy versus the mass number. Coming back to the fission and fusion, so when a heavy nucleus splits into two lighter nuclei, that is called as the process is called as a fission, okay. So, each of the daughter nucleons has a binding energy larger than that of the parent nucleus.

Fission and Fusion:

When a heavy nucleus splits into two lighter nuclei, the process is called Fission.

Each of the daughter nucleons have binding energy larger than that of the parent nucleus. The energy given up in the process can be huge.

For example, a Uranium nucleus $\left({}^{235}_{92}U\right)$ splits into two smaller nuclei, with a difference in binding energy of 0.8MeV/nucleon, the energy released is, $0.8 \times 235 \text{ MeV} = 188 \text{ MeV}$. This is indeed HUGE from a single event.

Heavier nucleus would yield more energy.

I mean, it is not really used in this context, but I am just telling you about the parent and daughter because this one nucleus may be a moderate or heavy nucleus that splits into two lighter nuclei. So, the lighter ones are called as daughter and the heavy ones which was initially there is called as a parent one. So, what can happen is that the energy given up in the process can be very large and for example, if you look at uranium nucleus it splits into two smaller nuclei. without worrying about what they split into, there is binding energy that appears, which is like point MeV per nucleon.

So you have 235 nucleons. If you multiply 235 by 0.8 MeV per nucleon, one gets an energy equal to 188 MeV. And this is actually a huge energy, you know, emerging out from a single event. So, even heavier nuclei would yield more energy. So, this is fission.

One actually gets very large amount of energy. On the other hand, when two light nuclei combine to form a heavier nucleus, the resulting process is called as a fusion. So, these words are somewhat familiar even in the context of English language that they fuse in order to become one. So, two lighter nuclei would combine to form a heavier nucleus and fusion is a notable source of energy production. So single nucleon of moderate size also means that more binding energy in the resultant nucleus, that is the one that's forming out of this fusion.

For example, if you have two deuterium nuclei that combine to form a helium nucleus, About 23 MeV energy is released. So, in fact, this nuclear fusion is a main source of energy for the sun and the stars. And the fact that nuclear binding energy is non-zero, it means that the nuclei, that the nuclei are more stable and they account for diverse forms of matter to exist. So this binding energy is an important and indispensable component of this, you know, this discussion of the nuclear structure.

Okay, so let us look at the binding energy curve in little more details and this gave rise to a liquid drop model. So, these short-range nuclear forces are completely different from the electromagnetic origin, which is what we have discussed earlier, and hence they are very different in nature from the Coulomb force. And they are not well understood either, because the Coulomb force from the classical perspective is very well understood. But of course, that did not stop the nuclear models to develop and, you know, tell us more about the nuclear structure and the nuclear levels. And to understand how strong the force is that exists inside the nucleus, which we have termed the strong force.

So up to a separation of three fermimeter and fermimeter is one fermimeter is like 10 to the power minus 15 fermimeter. So within this distance, the attractive force, say the attractive force between two protons is 100 times larger than the repulsive force between them. Otherwise—otherwise meaning if we are not talking about them being inside the nucleus. but they to be like two charged particles interacting by an electromagnetic force or Coulomb force, then this, within this nuclear range or nuclear radius, they are about 100 times larger, okay? This is the force between two protons, okay?

And similar forces exist between similar large forces, rather exist between neutrons or neutron and proton. So either we talk about the force or we talk about interaction energy, both mean the same thing. Now, thinking of nucleus with a drop of liquid was quite innovative and it was, you know, conceptualized by Gamow in 1929 and Weizsacker in 1935. So that accounts for the binding energy per nucleon versus A, which is basically the curve that you have seen here. Okay.

And similarity with that of a drop of liquid and not solid can be understood because even the solids actually, they interact, the ions interact by nearest neighbor spring-like force, but the vibration that it causes because of this spring-like motion, or rather they are connected by springs, is enormously large and will harm the stability of the nucleus. So the solids were kept out of the discussion and the only analogy that was found was with that of a liquid drop. And the nucleus is thought to be like all those nucleons. They are free to move as the molecules of a liquid, but they maintain the fixed intermolecular distance between them and such that these all the nucleons also stay within the nucleus.

And this is an important part of discussion for these liquid drop model. So if you consider the energy associated with each nucleon-nucleon bond is given by some U. And so this U is actually negative because that's like a binding energy. You know, it's a bound state energy, so it has to be negative. But then, you know, this is when this is quoted as a binding energy, which is what we have seen and we have quoted this binding energy to be positive.

So the when it's, you know, converted or rather talked about in terms of the binding energy, then, of course, we do not talk about this energy. energy to be negative rather we quote positive values for the binding energy. So, say if this is like there are you know there is a nucleon here and there is a nucleon here and then there is some you know bond between them and if the binding energy is U and which is shared by each one of them like this. So, then the binding energy is U by 2 it is shared by two nucleons. Now, assume the nucleus to be spherical and the nucleons to be spherical in shape as well.

So you see this dotted line, that's the nucleus. And then there are these open circles and these darkened circles or these colored circles. And these are the nucleons that are there inside. And if you think that these are like liquid drops which are interacting with each other, or rather these liquid molecules are interacting with each other, but maintaining a shape because the nucleus has to maintain its shape. It can go a slight bit of distortion, which eventually gives rise to the quadrupole moment, but it cannot completely change its shape.

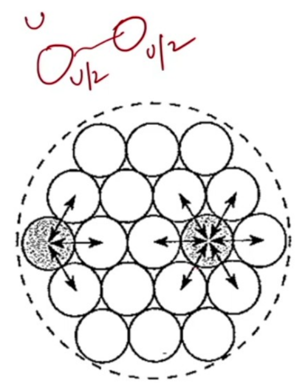
The binding energy is $U/2$ as it is shared by two nucleons.

Assuming the nucleus to be spherical, and the nucleons spherical in shape as well.

Each nucleon has 12 other spheres in contact with it. Thus, nucleus has binding energy of $6U$.

For all A nucleons, the total binding energy should be

$$E_B = 6AU$$



This is also called as the volume energy of the nucleus, $E_{vol} = a_1 A$.

The nucleons that are on the surface have lesser neighbours (< 12). An estimation of the Surface energy can be done.

So the shape would remain constant, and that is why it is like these intermolecular forces in liquid. And when you think about that each, you know, sort of nucleon which is inside this sphere, it is in contact with 12 other spheres. And these 12 other spheres, each one will have a binding energy of U by 2. So this nucleus has a binding energy of U, 6U. 6 times U. And for A number of nucleons, it is 6AU, where A is the mass number.

So, that is the binding energy. And this binding energy, because we are all talking about these nucleons to be enclosed within the nucleus. So, we are talking about this volume binding energy. So, we call it E volume, which is some constant times A. Of course, this constant here is equal to 6U. That constant is immaterial.

So, it is written as a1 into A, a small a1 into capital A, where capital A is the mass number of the element. So, now there are also nucleons which are at the surface. And these nucleons do not have, you know, neighbors, you know, on the other side, just like an inside one, say here has a neighbor on all sides. The one here doesn't have a neighbor on all sides. So they have, you know, less than 12 neighbors, as we have talked about for these ones that are inside.

So an estimation of the surface energy can be done. So the total binding energy gets reduced by the surface energy because of these, you know, the energy is less for these nucleons. And how do we make an estimate of this surface energy? We can take the surface to be 4 pi r squared, where capital R is the radius of the nucleus, which we can write it as, you know, some r naught squared, where r naught is a constant. It's r naught squared into a to the power two thirds.

The surface area ~ $4\pi R^2 \simeq 4\pi R_0^2 A^{2/3}$.
The surface nucleons have fewer than the maximum neighbours are proportional to $A^{2/3}$.

Thus, these peripheral nucleons reduce the binding energy by

$$E_{\text{surface}} = -a_2 A^{2/3}$$

E_{surface} is considerable for lighter nuclei, where most of them reside on the periphery.

There is always a competition between the two, yielding properties of a liquid.

Just like a liquid drop on a non-wetting surface assumes a spherical shape, because of surface tension, the energy of the configuration is minimized.

In a nucleus, it tries to maximize the binding energy to be more stable.

This justifies the name **liquid drop model.**

So, as opposed to A in the volume term, this involves a term, the surface term involves A to the power 2 by 3. So, this surface term, so this peripheral nucleons or which are on the surface, they reduce the binding energy by this amount which is A2 into A to the power 2 by 3. So, the binding energy becomes actually E b equal to E volume. and plus E surface and at this stage it becomes equal to A1A and minus A2A to the power 2 by 3. So E surface is actually considerably large for lighter nuclei because most of them, it's like very few of them are inside and most of them actually are lying on the surface.

So there's a competition between two yielding the surface or properties of a liquid. Now, how is this related to the property of the liquid? So if you drop a liquid onto a non-weighting surface, it doesn't wet the surface. And then you'll see that because of the surface tension, it tries to minimize its energy by making a sort of spherical in shape. You can see that when you boil water, so the saucepan that you boil the water on, if you look at when it started boiling, you will see that there are bubbles formed, but the bubbles are still stuck to the bottom surface of this pan.

And then when the thermal energy increases, these bubbles become bigger and then they get detached. So the surface tension is no longer able to, you know, hold them together. So they kind of detach from from the surface or from the bottom surface. So this is just like a liquid drop on a non-weighting surface assumes a spherical shape because of surface tension, the energy of this configuration is minimized. Configuration means liquid drop or the spherical shape is, you know, is forms because to minimize the energy.

And we see a similar thing as well here that these surface term is actually minimized. bringing down the binding energy so that you know stable nuclei are formed. So this justifies the name liquid drop model. And we have this electric repulsion or rather this Coulomb term also they contribute to this binding energy or lowering of the binding energy. And how do they do that?

Consider that the Coulomb energy to be the work done in bringing Z protons from infinity to the nucleus or this spherical region in the nucleus. So the potential energy for two protons to be at a distance r apart, r is any distance, so V of r becomes equal to minus e squared by 4 pi epsilon 0 r. Now if there are Z protons, there are Z into Z minus 1 by 2 pair of protons. What I mean is that these protons are all indistinguishable, so out of the Z, if you have to make two protons as pair, the possibility or rather the number that you can come up with for the number of pairs is Z into Z minus 1 by 2. So, these are the pair of protons and there is a pairwise interaction.

So, we have this Coulomb term to be equal to Z into Z minus 1 by 2 into V, where V is this minus E square by 4 pi epsilon 0 R. And if you write this R to be equal to 1 by R average, there could be, you know, these because you are considering all pairs of protons. Some of the protons can have different distance than others. But for a uniform packing, you know, this is 1 over R average would be a good approximation for R. these distance r. So you replace 1 over r by 1 over r average.

The electric repulsion between the protons also contribute to lowering of the binding energy.

Considering Coulomb energy to be the work done in bringing Z protons from infinity to the spherical region (nucleus). The potential energy for two protons to be a distance r apart is

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r}$$

There are $\frac{Z(Z-1)}{2}$ pair of protons, whose total energy would be

$$E_{\text{Coulom}b} = \frac{Z(Z-1)}{2}V = \frac{Z(Z-1)e^2}{8\pi\epsilon_0}\left(\frac{1}{r}\right)_{av}$$

where $\left(\frac{1}{r}\right)_{av}$ is the average distance pver all proton pairs.

Assuming protons to be uniformly distributed $\left(\frac{1}{r}\right)_{av} \sim \frac{1}{R} \sim A^{1/3}$

$$E_{\text{Coulom}b} = a_3\, Z(Z-1)/A^{\frac{1}{3}}$$

So that is the average distance over all these proton pairs. So assuming the protons to be uniformly distributed, that is 1 over r average is like 1 over r, which is r is the radius of the nucleus. That goes as a to the power one-third. Okay, so that is like because your 4 3rd pi r cube is like your a, that is the volume term which we have seen that it is equal to a. So, the Coulomb term comes out to be something like z into z minus 1 divided by a to the power one third. So, we have one term that goes as A, then another term goes as minus A to the power two-third and this also will be a minus and this is A to the power one-third.

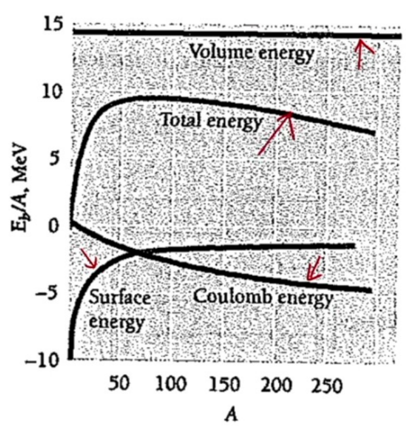
So, the binding energy becomes equal to A1, A, A2, A to the power two-third and A3Z into Z minus 1 divided by A to the power three-third. One third. And when you plot this, you see different energy scales. So this again, this binding energy per nucleon is plotted, which is what we have shown earlier. And you see that the surface energy grows like this.

So this is the surface energy curve. And then the Coulomb energy falls like this. And the volume energy is like this, which, you know, is linear in a. So that's like a constant. So a combination of that gives you this total energy, which gets reduced by the Coulomb term and the surface term. And you get a form which is very similar to what you have seen in experiments.

Thus, the total energy becomes,

$$E_B = E_{\text{vol}} + E_{\text{surface}} + E_{\text{Coulomb}} = a_1 A - a_2 A^{\frac{2}{3}} - a_3 \frac{Z(Z-1)}{A^{1/3}}$$

To find out the binding energy per nucleon is $\frac{E_B}{\text{A}}$.



So this is the triumph of the liquid drop model. And all these are in analogy with the liquid drop. And that's why this model is quite successful in explaining the properties of a nucleus or these energetics, rather the binding energy of the nucleons. It explains it well. But there is another model called as the shell model.

And there are emergence of certain magic numbers, which are like the nuclear shells. You know, all these nucleons, these are these magic numbers, which are the number of nucleons occupying various energy levels and so on. So we have seen that the assumption of the liquid drop model was that each nucleon interacts with its nearest neighbors as in a liquid. But there could be a more general kind of interaction between these nucleons, which is not sort of taken into account in the liquid drop model that we have seen. Now, let us try to, you know, bring back the atomic picture and these atomic pictures, you know, there are these, if you look at the periodic table, these inert atoms or inert elements such as krypton, xenon, neon, etc.

Shell Model: Magic numbers

Assumption of the liquid drop model was each nucleon interacts with its nearest neighbours, as in a liquid.

Now what if we try to bring back the atomic picture, where atoms with 2, 10, 18, 36, 54, 86 Electrons have all the shells completely full (inert elements). Chemical inertness asserts Stability.

Similar features are seen in nuclei as well. Nuclei having 2,8, 20, 28, 50, 82 and 126 nucleons are found to be exceptionally stable.

Since they are inert, they are found in abundance.

The numbers 2,8, 20, 28, 50, 82 and 126 are called as magic numbers.

They have, these atoms have these 2, 10, 18, 36, 54, 86 electrons whose all the shells are completely inert. And these inert elements and when we say inert, we mean that they do not actually interact with anybody or anything, any other element and make a compound. So they do not take part in any kind of chemical reaction and that is why they are called inert. And do we have a similar scenario in nucleus that do we have such nucleons, you know, specific number of nucleons which are inert, chemically inert? Now, we cannot say chemically inert, but we can always say that they are stable.

And if they are stable, they are abundantly available. So they are available in abundance. And this is what it says. So these are the numbers for nucleons. You know, for the nucleons like 2, 8, 20, 28, 50, 82, these nucleons are found to be exceptionally stable, just like these ones which are stable.

Stable means they are chemically inert. And here, stable means that they are found in abundance. For the nucleons or the nucleus, inert means they are found in abundance and they don't undergo any decay. And cannot be broken up. And these are subsequently called the magic numbers.

Okay. So this shell model actually explains the existence of these magic numbers. We won't go too much into detail because these nuclear interactions are not very well understood. So the precise form of U of R is unavailable. And if you think, you know, the form of the nucleus is either, you know, something like this.

Or ones with slightly curved edges, or they are like the ones that, you know, are like this and so on and so forth. And they are, I mean, what I mean to say is that they are like repulsive and then they are like attractive at large distances and so on. And if you solve Schrodinger equation or if you take something like a harmonic oscillator, they haven't yielded very well description of these magic numbers at least. And so if you solve Schrodinger equation, if you can solve at least for this case, it can be solved. But otherwise, this you'd find it very difficult.

But they don't give rise to any magic numbers. And what comes to the rescue is that the spin-orbit coupling or the LS coupling for the lighter nuclei and JJ coupling for larger nuclei. In fact, JJ coupling occurs in most of the nuclei. We have talked about this earlier, the LS coupling.

LS coupling, if there's a spin and orbital coupling, and that happens when you take these electron in the rest frame of the electron, the proton is moving around it and a moving charge because the nucleus is charged with the charge of a proton. So a moving charge is like a current and so the electron actually experiences a magnetic field due to this current. And the spin of the electron couples with the magnetic field via Zeeman coupling. And this will give rise to a L and this can be shown to have a L dot S kind of thing.

The proposed **Shell model** explains the existence of these magic numbers.

Since the nuclear interactions are not very well understood, a precise form of $U(r)$ is unavailable. A reasonable guess would be a square well potential (may be with rounded corners) or may be something like a harmonic oscillator.

The solution of Schroedinger equation corresponding to such potentials are solved.

However, they don't give rise to the Magic numbers.

The rescue comes in the form of considering
(i) Spin-Orbit LS coupling for lighter nuclei
(ii) jj coupling for larger nuclei.

In fact jj coupling occurs in most nuclei.

And one artifact of that is that you cannot use either L or S. but you can use a new quantum number or a new operator called J which is equal to L plus S. So this LS coupling that occurs for lighter nuclei and one has JJ coupling for larger nuclei. And just to tell you a little about this JJ coupling is that so S and L of each nucleon They couple to give a value of J, which is the total angular momentum and J is equal to L plus S. So, L is the orbital angular momentum and S is the spin angular momentum. So, the various J's can couple to give the J-J coupling.

And the spin orbit interaction that splits each of the state for a given value of J into this 2J plus 1 sublevels and with, you know, allowed orientations of the J vector. And the numbers available, if you carefully do all these calculations, then the numbers available in each of the nuclear shell is 2, 6, 12, 22, 32 and 44. I did that quite briefly because I didn't want to get into the details of that. But what you should know are these that there exists these two models, the liquid drop and the shell model, which can by and large explain all the nuclear properties that we are. familiar with.

jj coupling:

$\vec{J} = \vec{L} + \vec{S}$

$\vec{S}$ and $\vec{L}$ of each nucleon couple to give a $\vec{J}$. Various $\vec{J}$s can couple to give jj coupling.

The spin-orbit interaction splits each state of a given j into $(2j + 1)$ sub-levels with allowed orientations of $\vec{J}$.

The numbers available in each nuclear shell is 2, 6, 12, 22, 32 and 44.

Hence the nuclear levels are filled when there are 2, 8, 20, 28, 50, 82 and 126 neutrons Or protons. These are the magic numbers!!

So with these numbers that are available for this nuclear shell, we can have these many number of nucleons, either they are neutrons or protons, and these are the magic numbers that one is familiar with. So that's a triumph of this shell model, and the success is not only in explaining the magic numbers; it is also capable of explaining the nuclear magnetic moment. So for protons and neutrons to have even numbers—if it's even-even nuclei, both of them are even— the angular momentum will be you know integer or they may cancel out giving you so these are zero or integer values. And for even odd or odd even combination it yields half integral value of the angular momentum

And similarly, the odd-odd combinations yield integral angular momentum again. So, the observed values of angular momentum for the nucleus—these values are well explained by this shell model. So, the liquid drop model sort of explains the binding energy and various energetics of these like these various magic numbers that you get and as well as the observed angular momentum values of the nucleus are explained by this shell model. So, we shall stop here, and that concludes our discussion on nuclear physics. We have kept it very brief.

And the topics we have discussed are the nuclear structure—how historically it became clear that the nucleus really consists of protons and neutrons. And then we talked about radioactivity. And here briefly, we talked about this models to models and the binding energy of the nucleus. So I'll stop here. Thank you.

Amen.