ELEMENTS OF MODERN PHYSICS

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Lec 25: Free Electron Theory

Welcome back. We have talked about classical statistics, all the machinery to calculate thermodynamic quantities via the partition function and so on. We know that the partition function actually gets carried over to quantum statistics as well and we have used the grand canonical partition function. We could use the canonical partition function as well, but it is better to use the grand canonical because of these number, the constraint on the total number of particles and that the ensembles are allowed to exchange number of particles in addition to energy. So we have looked at Bose-Einstein statistics, and the important thing that comes along with is the Bose-Einstein condensation, which happens in 3D for non-relativistic dispersion, like a K-square dispersion, E going as K-square.

And we have seen that the ground state is macroscopically occupied below a certain temperature, and this is called as a Bose-Einstein condensate. And there are certain properties associated with it, such as, you know, the pressure is entirely due to the excited state occupancies or the particles that occupy the excited states. And the ground state exerts no pressure at all. And for a Bose-Einstein condensate, it is only half of the classical pressure, which is very low. And then we have seen other quantities or other physical properties, such as internal energy.

We have looked at the specific heat and observed discontinuity in specific heat and so on. Now we will talk about Fermi-Dirac statistics. That is the other set of quantum particles which obey Pauli exclusion principle and They have antisymmetric wave functions and correspond to spin angular momentum being half-integer and so on. Okay.

So they come in this Fermi-Dirac statistics. We have introduced that. Now we'll talk about various properties of this, and we'll try to cover these few things that are written here in this class. We'll talk about degenerate free Fermi gas.

So what's the degeneracy limit, and what does degeneracy mean actually? Then we'll talk about the Fermi wave vector, Fermi energy. The ground state energy and pressure of a free Fermi gas will of course talk about the Fermi Dirac distribution, how it looks like, that is the average number of particles occupying these single particle states, how they look like and talk about the chemical potential, derivation of specific heat and so on so forth. So there are a number of things that are on the cards. And once we are done with this, we will get into a little more involved calculations—one is that we will calculate the paramagnetic susceptibility of these free electrons.

This is distinct than the Curie's law, which you are aware of. which goes as 1 over T and will show that the susceptibility is actually independent of temperature and depends on the density of states at the Fermi level. In addition to that, we'll talk about stars, particularly white dwarf stars, which can be seen as examples of heavily degenerate Fermi gas. So these are very important applications of Fermi-Dirac statistics. So we would still be talking about non-infracting fermions and their properties.

So let us see what this degeneracy is and how it plays a role. And so this whole idea of studying this Fermi gas is that we should be able to apply it to metals. So metals are, there are a lot of free electrons which are not bound to the atoms and they are ready to take part in the conduction phenomena when you apply voltage on the material. So for metals, typical mass density. So mass density is, you know, of the order of few grams per cc.

You can convert that into a kilogram per meter cube and so on. And so that tells you that with this kind of a mass density, the number of electrons per unit volume is of the order of 10 to the power 23 to 24. So let's write 10 to the power 24. And this tells you that if you have this kind of density, so this is like a density and the density is like your total number of electrons divided by volume.

So if you take them to be spherical and consider the radius of those electrons to be around, say, the volume is four-third pi r cube, where r is the radius, r is also the mean distance or the average distance between two electrons, assuming that they are just, you know, touching each other. So that gives you an average, you know, a distance between electrons. So, we are simply trying to calculate the order of magnitude and this let us call it as R average and this should be of the order of like Angstrom or 10 to the power minus 8 centimeter. So, this is a typical inter electron distance in a metal. Now, you know when you need the classical limit to dominate, so the classical limit would dominate

if this condition is made. That is, our average should be much, much greater than lambda. And this lambda is nothing but the thermal de Broglie wavelength, which we have written several times earlier. And it looks like this at a given temperature T, where T is in the denominator inside the square root. So this tells you that this is equivalent to T being much, much greater than H divided by 2π and m k and r average square.

And if you put r average, etc. So if you put mass to be the mass of an electron and r average to be around 10 to the power minus 8 centimeters, with the value of k and this Boltzmann constant and H. Then what you get is that you get R average to be at, you know, so R average is of course 10 to the power minus 8. So you get this T to be even much greater than 10 to the power 5 Kelvin, okay, or at least we write T to be 10 to the power 5 Kelvin. So even at room temperature, which is known to be our classical limit from the discussion that we had so far, it looks like the room temperature is far lower than this temperature.

Degeneracy of a fermi gas.

For Metal   mass density ~ few gm/cc

No. of electrons/vol ~ 10^{24} $= n = \frac{N}{V}$.

av. distance between electrons - $r_{av} \sim 10^{-8}$ cm

Classical limit dominates $r_{av} >> \lambda$ $\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$

$$T >> \frac{h}{2\pi m k_B r_{av}^2}$$

$$\boxed{T > 10^5 \text{ K}} \qquad \boxed{\text{The ground state is enormously degenerate.}}$$

So what you would have imagined as a classical limit, so can be, you know, the room temperature that is, it should still be able to be taken as T equal to 0. So, this is like saying that the ground state is enormously degenerate. Okay, let me sort of make a box of this. It's an important statement. So what we have said is that if you really want the classical limit that we have applied to, say, bosons or any quantum particle, if you apply that to fermions and particularly to electrons, then what will happen is that the temperature at which you want the classical limit to set in

That is, the classical limit would be valid at a temperature which is 10 to the power 5 Kelvin, more than that, in fact. Which means that room temperature, which is 300 Kelvin, which is just about 10 square, that is 10 to the power 2 Kelvin, can still be taken as room temperature. So all temperatures all the way up to say room temperature can be considered as zero temperature because the scale that you consider with or the scale that you want your quantum gas to go into the classical regime is extremely high. And we'll show that, in fact, the Tf, called as the Fermi temperature, is precisely of this order for all metals, okay, all known metals like aluminium, copper, and various other things that you are, iron, nickel, etc. This Fermi temperature is very large, it's 60, 70, 80,000 Kelvin.

For example, for copper, it's around 60. maybe around 70 to 80,000 Kelvin by putting in all these values that we have just done. So I think the argument is clear that almost all temperatures that are under consideration in which experiments take place can be taken as zero temperature, which means very low temperature. And in very low temperature, the quantum effects become very important. And we should consider these electronic systems as quantum systems, even at room temperature or even a little more than that.

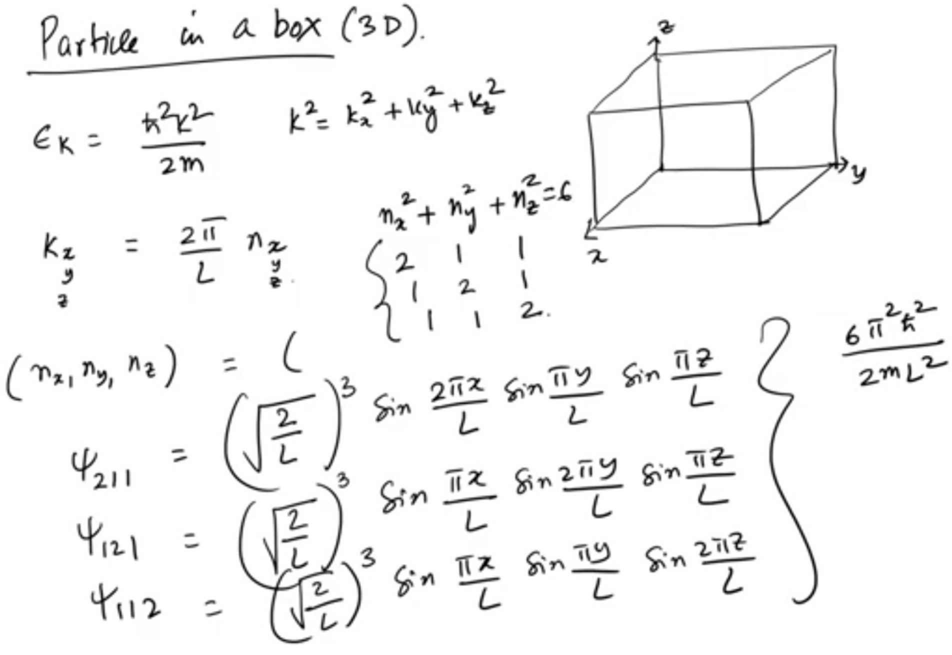
We talk about room temperatures to be 300 Kelvin. Even if you talk about 350, 400 Kelvin, it really doesn't matter. Even 1000 Kelvin is like three orders of magnitude, two to three orders of magnitude lower than the temperature scale that we have found out where the classical limit would be valid. So this is something that we need to understand and we are sort of let me do a simple minded analysis of this Fermi systems and we can start with say a particle in a box and you are all well aware of this problem

Now, I will just talk about a 3D box. OK, so the box is a three-dimensional box with X going from zero to L, Y going from zero to L, and Z going from zero to L. OK, so you can imagine a box, you know, a cubic box, but with all its dimensions. The sides, you know, are infinitely large so that the particle is really contained between 0 and L. So the potential is infinitely large. So we still have a box like this. Okay.

Call it X, call it Y. And call it Z, and the particle is contained in this, and it cannot escape because the walls in all directions have infinite potential. So this problem is well known. I mean, you have these epsilon k equal to h cross square k square over 2m, and when you solve this, k square is equal to k x square plus k y square plus k z square. And you have kx going as, so this kx, etc., quantized as 2 pi over L nx, and similarly for ky with ny and kz for nz. Okay, so the quantization of this problem can be expressed in terms of these quantum numbers, which can take values, all integer values starting from 1.

There can't be 0 because 0 would mean that there is no particle in the box at all in any of these nx, ny, or nz, and we start from, say, 111, so 111 would be the ground state, and then 2 1 1 or 1 2 1 or 1 1 2 is the first excited state, and so on. And you see that apart from the ground state, now there is a degeneracy, and this degeneracy is what we are talking about, that is, corresponding to different quantum numbers which correspond to different quantum states or wave functions. They have the same energy. For example, say these n x square plus n y square plus n z square, it corresponds to 6. Is equal to 6, then we can have 2, 1, 1, 1, 2, 1, 1, 1, 2.

So, these three are degenerate, and this degeneracy occurs because they correspond to different wave functions. What wave functions do they correspond to? So, psi for this particular thing, say 2, 1, 1, corresponds to the square root of 2 over L whole cube, and sine 2 pi X by L, sine pi Y by L, and sine pi Z by L. And similarly, the psi 1, 2, 1 would correspond to root over 2 by L whole cube sine of pi X by L sine of 2 pi Y by L and sine of pi Z by L. Similarly, the last one which is 1, 1, 2 would correspond to root over 2 by L whole cube, I am so sorry. should be whole cube, and sine pi x over L, sine pi y over L, and sine 2 pi z over L, okay.



Particle in a box (3D).

$$\epsilon_K = \frac{\hbar^2 k^2}{2m} \qquad k^2 = k_x^2 + k_y^2 + k_z^2$$

$$k_{x,y,z} = \frac{2\pi}{L} n_{x,y,z}$$

$$n_x^2 + n_y^2 + n_z^2 = 6$$

$$\left\{\begin{array}{ccc} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{array}\right.$$

$$(n_x, n_y, n_z) = ($$

$$\left.\begin{aligned} \psi_{211} &= \left(\sqrt{\frac{2}{L}}\right)^3 \sin\frac{2\pi x}{L} \sin\frac{\pi y}{L} \sin\frac{\pi z}{L} \\ \psi_{121} &= \left(\sqrt{\frac{2}{L}}\right)^3 \sin\frac{\pi x}{L} \sin\frac{2\pi y}{L} \sin\frac{\pi z}{L} \\ \psi_{112} &= \left(\sqrt{\frac{2}{L}}\right)^3 \sin\frac{\pi x}{L} \sin\frac{\pi y}{L} \sin\frac{2\pi z}{L} \end{aligned}\right\} \quad \frac{6\pi^2\hbar^2}{2mL^2}$$

So, they have the same energy which is given by 6 pi square h cross square by L. So, they have the same energy, and that is why they are degenerate. Now, if you have a large number of particles which are now fermions and you cannot put two fermions in the same quantum state, one electron can only occupy one state. So that way, if you start filling up and if you think of, you know, homogeneously filling it up, starting from zero, that is nx equal to or rather, you know, the lowest energy, say, for example. Okay, let us take nx equal to zero, ny equal to zero, nz equal to zero.

That does not correspond to anything physical. physical, but say that's the origin of this coordinate system in terms of nx, ny, and nz. So you fill them up, uh, homogeneously from the center or from the origin, and this will take the shape of a sphere. So, in 3D, it will be like a sphere. So, and this sphere, if you have n particles in a volume V, you can fill it up all those n particles within a sphere of radius, which is called as a Kf or the Fermi momentum.

So, we are filling it up in momentum space because this n really connects to the momentum variable, okay. So, we, you know, number of allowed K points Within a radius Kf and this radius has a special name called as a Fermi wave vector and will be used heavily in all, I mean all discussions of the Fermi system. So, this is called the Fermi wave vector. And how many allowed values are there?

So we have a V over 2 pi whole cube and a 4 by 3 pi kf cube. So V over 2 pi, if you remember that if you walk a distance 2 pi over L, you get 1 k point. So, if you walk unit distance, you get L over 2 pi k points and in 3D, it is V by 2 pi whole cube if you want to get 1 k point and you multiply that by the total volume of the sphere and this gives you kf cube equal to divided by 6 pi square into V. So, if you equate this N equal to, you know, KF cube divided by 6 pi square into V, that will give you the density. And I will just by hand multiply a factor of 2, which stands for spin degeneracy, which means that there are spin up particles and spin down particles, okay.

We talk about spin half particles. So, the spin angular momentum is half. So, this is the spin degeneracy which is nothing but equal to 2s plus 1. So, if you do a simple calculation of this, that is in order to find Kf, you find Kf to be equal to 3 pi square n whole to the power one-third. So, that tells you that if you want more particles to be accommodated, it is just that the Fermi momentum or the radius of the Fermi sphere that just increases as n to the power one-third.

And we can also define—so this is that Fermi wave vector expression—now we define the Fermi energy. We'll show in the distribution what Fermi energy is. So, Fermi energy is nothing but h cross square kf square over 2m, where m is the mass of these electrons. So, this is equal to h square over 2m, and we have a 3 pi square n whole to the power two-third. So, that is the Fermi energy.

So, the Fermi energy grows as n to the power two-thirds, which means that if you have higher electron densities or more electrons, you have larger Fermi energy and so on. So, if you plug in typical values of this N, which is, as I said, about 10 to the power 24, and the mass of the electrons, the Fermi energy comes out as 5 to 6 electron volts. We will see that, okay? And that is a very large energy. In solid state physics, 1 or 2 volts of energy, electron volts of energy is very, very large and large.

That is because one electron volt is approximately equal to 11,600 Kelvin. You can understand how large the temperature is because this is not even, you know, the core of the sun probably would have a temperature something similar to that. Okay, it's only the periphery of the sun is about 6000 Kelvin. Okay, and what's room temperature? So room temperature is nearly 0.025 electron volt, which is 300 Kelvin.

So what I do is simply take the electron volt and connect it to KT. K is the Boltzmann constant and find T and this is just the hand waving or rather this is almost learning by heart that these are some of the numbers that we are interested in. So while high-energy physics talks about very high energies such as GeV and TeV and all that, we are very content with this scale of energies, like one electron volt, which is a very large energy. And, you know, typical semiconductors have an energy gap which is of the order of one electron volt or a little less than that or a little more than that in some other semiconductors. So, you can never make a material, such as a semiconductor, conducting by applying temperature because you will melt everything in the universe if you apply this kind of temperature.

No. of allowed K-points with a radius k_f (fermi wavevector).

$$\frac{V}{(2\pi)^3} \times \frac{4}{3}\pi k_f^3 = \frac{k_f^3}{6\pi^2} V.$$

$$N = \underset{\substack{\uparrow \\ \text{Spin degeneracy} \\ (2s+1).}}{2} \times \frac{k_f^3}{6\pi^2} V \Rightarrow k_f = (3\pi^2 n)^{1/3}$$

$$\boxed{1ev = 11,600k}$$

Fermi energy $$\varepsilon_f = \frac{\hbar^2 k_f^2}{2m} = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}.$$

Room Temp

$\sim 0.025\,eV$

$(300k)$

So, one electron volt of energy difference is a very large energy difference, okay? So, this is, you know, the Fermi energy and so on. So, these we can find out—the ground state energy, for example, for a degenerate Fermi gas. So what we are trying to calculate is, you call it E0 or EG—up to you.

You can just simply, you know, get this 8 by V over HQ and 0 to PF. We are not writing in terms of KF, but you can write it in terms of KF. We simply do this. p square by 2m, and we have a p square dp. So, this 4 pi p square dp—the 4 pi has been taken out.

This 8 comes because there is a 4π and then there is a 2 factor due to spin degeneracy, and that is how this is calculated. So, this is a very simple integral, like a P4 integral. So, you will get a P to the power 5 and so on. So, if you simplify this, this ground state energy per particle comes out as 3 by 5 epsilon F, okay. So, that is the Fermi energy.

We probably have written down, yeah, Fermi energy is EF epsilon F. so epsilon F, so it's the ground state energy of free Fermi gas is very very large okay and it's independent of temperature because there's no temperature that is you know equivalent to it. If you convert that into a temperature scale so the Tf gives the temperature scale and which as I said is of the order of 10 to the power 5 Kelvin okay. So, if you want to calculate the pressure, So pressure due to the Fermi gas is, call it a P0, so at T equal to 0 or we call it, you know, the very low temperature because I said any temperature is like 0 temperature, any accessible experimental temperature is like 0 temperature.

So we write T equal to 0 nevertheless. So, this if you calculate it and use this formula that we have derived a number of times at least for the non-relativistic case we have derived it earlier which was two-third U by V where U is nothing but the internal energy is nothing but the ground state energy here and this is equal to 2 by 3 into E0 by V which is 3 by 5 and EF and then this goes and it is something like that. So, it is 2 by 5 epsilon F and things like that. So, this is that pressure due to a Fermi gas.

Ground state energy of a degenerate fermi gas

$$E_0 = \frac{8\pi V}{h^3}\int_0^{p_f}\left(\frac{p^2}{2m}\right)p^2\,dp$$

$$\boxed{\frac{E_0}{N} = \frac{3}{5}\epsilon_f.} \qquad \epsilon_f \sim n^{2/3}.$$

Pressure due to the fermi gas at T = 0.

$$P_0 = \frac{2}{3}\frac{E_0}{V} = \frac{2}{\cancel{3}}\times\frac{\cancel{3}}{5}\epsilon_f = \frac{2}{5}\epsilon_f.$$

And as you can see, this is very large, and this pressure is due to the hard-core nature of the particles; that is, they exert enormous pressure on another electron coming into the same quantum state. and it doesn't allow at all it's like infinite pressure so that no other particle can come and occupy the same quantum state and that's why this this energy is so large and so on okay so this epsilon F really goes as n to the power two-third and so on and similarly we can find out that how this this pressure ground state pressure with density goes as okay this is very simple Now let us talk about the Fermi Dirac distribution and in short we write as a FD distribution. And this distribution is very distinct from the Bose distribution, and it is at t equal to 0. So this is at T equal to 0.

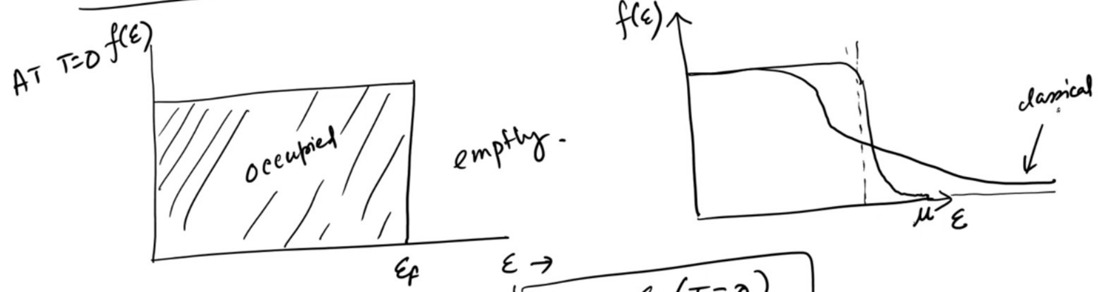
This is f of epsilon, and then you have epsilon, and so on. So it is just a step function. So for all epsilon lesser than epsilon f, all the states are occupied. So these are all occupied states.

And all states here are empty. So this is at T equal to 0. And what happens at t slightly different than 0 is that, so this f of epsilon to be, and now what happens is that this one slightly broadens and goes like this. So say here there used to be a sharp drop and now it got bent from there.

Let me draw a little better here. So it gets bent there, and a little bit of spectral weight gets transferred to larger energies. And now we call this: all the states below mu are occupied. So we have two things, which are mu, called the chemical potential, which we have defined before many times in various contexts—this chemical potential. And we have talked about Fermi energy, which is in the recent past.

So, Fermi energy. How are they related? Mu is equal to epsilon F at T equal to 0. And at T not equal to 0—that is, the temperature not equal to 0—Fermi energy is not well defined. Okay.

F D distribution.



μ : chemical potential

ε_F : Fermi energy.

$\boxed{\mu = \varepsilon_F (T=0)}$

At $T \neq 0$, Fermi energy is not well defined.

Chemical is still " "

And so basically the chemical potential is still well defined and which tells you or rather which defines the highest energy state that is occupied at any temperature. Still well defined. OK. So now, as you understand that this really deviation from T equal to zero, that is this distribution becoming slightly curved from the top and then a bit of spectral weight is appearing at the bottom of the tail. This happens at a large temperature.

And in fact, if you simply talk about, you know, increasing the temperature, it sort of goes. I have to be a little careful because it goes like this. this and so on and in the limiting case of course we know that the classical statistics would prevail but we know that now the classical statistics in order to for it to prevail we need to go to very very large temperature okay so that kind of temperature like 10 to the power 5 Kelvin 10 to the power 6 Kelvin is not achievable in under any experimental conditions So that is why we said that for all this thing, all practical purposes, this degenerate Fermi gas is considered as zero temperature. We still are not, this has to be smooth and so on.

This is probably slightly better. And so that's how the Fermi distribution looks. That's how the Fermi energy and the chemical potential are defined. They both mean the same thing, but the Fermi energy is strictly defined at T equal to zero. The chemical potential is defined at any finite temperature.

They mean the same thing. They denote... the maximum the energy up to which the electrons occupy and above which they are empty or you can say that it is the energy required to add one particle to the system because you see that the maximum energy levels which is occupied all the energy levels that are here And here they all have one electron each. And if you are thinking of spin, then they have two electrons each with one with up spin and one with down spin.

And all these are completely filled. And for you to add one electron to the system, it cannot go anywhere. But it has to go to the higher energies because by creating more quantum states, Because there is nothing that is available to them because they are completely filled. Think of a room that is completely filled and if someone comes that more space has to be created because this room would not accommodate an extra person.

So the walls have to be moved. It is just like that—the walls have to be moved. The walls of these step functions have to be moved. In order to accommodate extra particles. So, that is the same meaning that we have.

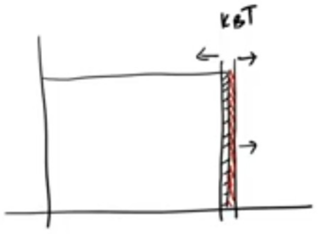
So, you know at a finite temperature if you want to calculate quantities say for example, we show these say N how do you calculate N that is 2 of F epsilon G epsilon D epsilon. And this F of epsilon is nothing but the step distribution, which is written as exponential beta epsilon minus mu plus 1. So that is the distribution. This distribution is important and has to be distinguished from the Bose distribution, where μ had to be negative and would only approach 0 in the quantum limit—the extreme quantum limit. Whereas mu is positive here, it's on the right side, mu resides on the right side of the diagram and that denotes the largest occupied state and you have this at T equal to 0.

which is 1 over kT, all states which are epsilon greater than mu are unoccupied, and all states which are less than mu are occupied. So this mu at T equal to 0 is nothing but epsilon F. So you can see this easily if you write this down as (epsilon minus mu) by kT. And plus 1. So if you put T equal to 0, this would like to be infinity, but only infinity when epsilon is greater than mu. And in that case, this is equal to 0 because 1 by infinity is 0.

The plus 1 doesn't matter. But if T is 0 and still epsilon is less than mu, then it is exponential minus infinity, which tells you that the occupancy is equal to 1. And that is why we see that step structure that you see here. So, I will just write that once again for epsilon greater than mu f of epsilon equal to 0 for reasons that I just told you. For epsilon less than mu, f of epsilon equals 1.

And it has a sharp discontinuity at T equal to 0. And this is very important. And that discontinuity, if you take a derivative of the Fermi function, there it looks like a delta function. Okay, so this is elementary discussion about free Fermi gas distribution and so on. One more important thing or rather like a hand waving thing we can talk about and we just draw this distribution once more and suppose we want to know that what is the specific heat due to electrons at t equal to 0 or at t equal to you know 300 Kelvin as I said it does not matter.

$$N = 2\int f(\varepsilon)\, g(\varepsilon)\, d\varepsilon .$$



$$f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)}+1} .$$

at $T=0$, ε_F .

$$= \frac{1}{e^{(\varepsilon-\mu)/k_BT}+1}$$

for $\varepsilon > \mu$, $f(\varepsilon) = 0$.

$\varepsilon < \mu$, $f(\varepsilon) = 1$.

$$E \sim \left(\frac{k_BT}{k_BT_F}\right) k_BT$$

$$\sim T^2$$

$$\boxed{C \sim T}$$

So, you have to understand that if you specifically it is the heat required to you know raise the temperature of the body by a degree which means that the electrons from this region would be excited and they would go to the higher energy region. So, this at a temperature T, this is width is like is of the order of KT. So, they would come here and they would like go to this thing. Let me write it with or they had make the hatches with a different sign.

So, the black ones are where the electrons come from. If you increase the temperature just a little bit, the red region is the one that they go to or they occupy at that temperature. So this is of the order of KT and so this is the you know the so what is the fraction of these particles that are excited its KT by KTF okay but TF is the Fermi temperature. So, each of these has an energy which is like kT, and that is why this whole thing—so the energy goes as T squared, and so the specific heat would go as T, which is the relation that we all are aware of, okay. In some sense that if you have studied forming gas before, then you know the specific heat due to electrons is that the variation with temperature is linear in T.

So, for a material if you are doing an experiment and you want to find the specific heat, the specific heat actually goes as alpha T plus gamma T cube and this is the electronic contribution and this is the phonon contribution that we have seen that is the device theory and this alpha T is the linear in temperature which we just shown in a hand waving way and can be derived. Okay, so once you are familiar with this, this is all taught in the first course of solid-state physics while you learn about metals. Let me go back to the statistical mechanics of the free Fermi gas. And the way we have done it earlier, we will do it in the same way. So, log of ZG is our starting point, as you have seen earlier for the Bose case.

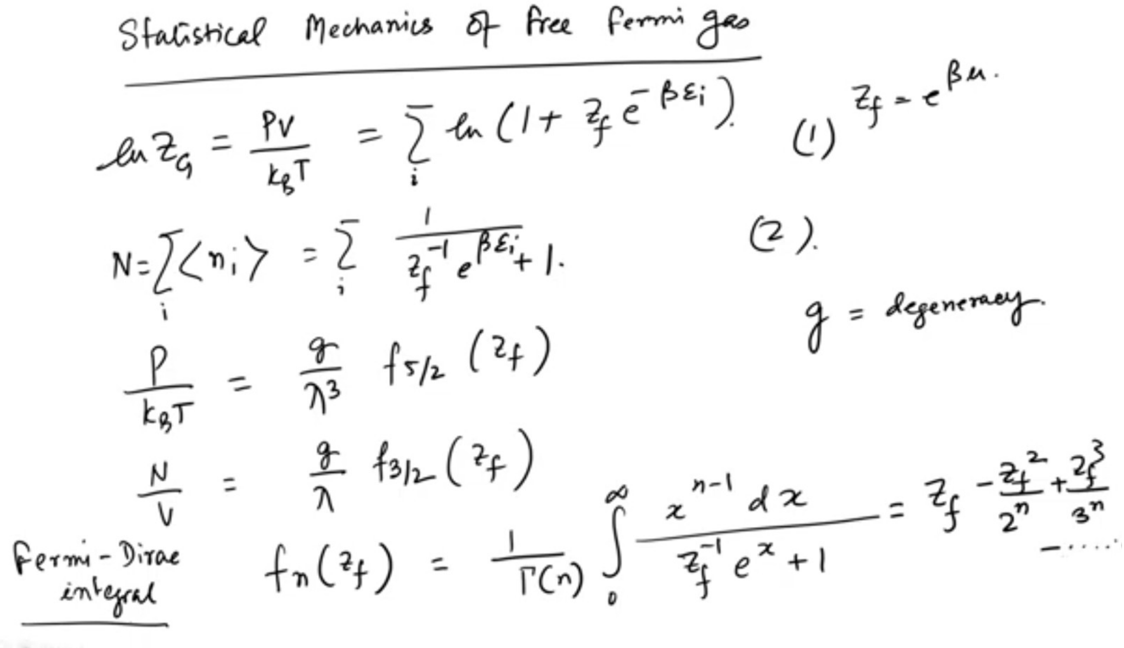
This is equal to sum over i log of 1 plus ZF exponential beta epsilon i. Epsilon i are the, you know, the single particle states. ZF is still the fugacity, which is exponential beta mu. And we can write this equation as equation 1. So we can write down the number equation as this and there is a sum over i and this is like a sum over i and the distribution. We are not really writing the spin degeneracy here but you can write a factor of 2 if you wish.

So, this is say equation 2. So, we convert this summations into integrals and we can write down P over KT equal to this is you can write a G over lambda cube. I will tell you what G is usually it is written with the G and G is that the degeneracy and in particular you may want to talk about the spin degeneracy. So, it is G over lambda cube F 5 by 2 you can remove G also if you wish if you are not talking about spinless if you talk about spinless particles.

So, N over V is again G over lambda cube f of 3 by 2 z f, we are writing Fermi functions, Fermi integrals with f and Bose integrals were written with a g, okay. And f of n z f, that is a Fermi Dirac integral, it is called as a Fermi Dirac integral. So this is equal to 1 by gamma n, there is a 0 to infinity x to the power n minus 1 dx divided by ZF inverse e to the power x plus 1. Of course, the connection has to be said that x equal to beta epsilon and so on so forth.

Now, for small ZF, it has this expansion as 2 to the power n plus ZF cube 3 to the power n minus and so on. So, it alternates plus minus plus minus. So, these are the expansions for small ZF, which of course correspond to the classical limit, but then we know that In

Fermi systems, the classical limit or all accessible temperature regimes actually fall in the quantum limit. So, what is the internal energy of this Fermi gas?



Statistical Mechanics of Free Fermi gas

$$\ln Z_G = \frac{PV}{k_B T} = \sum_i \ln\left(1 + z_f e^{-\beta \varepsilon_i}\right) \quad (1) \qquad z_f = e^{\beta\mu}.$$

$$N = \sum_i \langle n_i \rangle = \sum_i \frac{1}{z_f^{-1} e^{\beta \varepsilon_i} + 1}. \quad (2).$$

g = degeneracy.

$$\frac{P}{k_B T} = \frac{g}{\lambda^3} f_{5/2}(z_f)$$

$$\frac{N}{V} = \frac{g}{\lambda} f_{3/2}(z_f)$$

Fermi-Dirac integral

$$f_n(z_f) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}\, dx}{z_f^{-1} e^{x} + 1} = z_f - \frac{z_f^2}{2^n} + \frac{z_f^3}{3^n} - \cdots$$

You know, after we have given an introduction to the Fermi-Dirac statistics and so on, we are getting into this formal mode of obtaining the physical quantities from the log ZG. That is the statistical tool we have developed over these last so many weeks, and that is what we want to stick to. Otherwise, every solid state physics book has a derivation in terms of for the specific heat or for say internal energy and ground state energy and pressure and this and that and so on. I think there was just a go back, and I believe there is an N missing here. So, I think that is missing here.

So, that N is the density. So actually P0 goes as n to the power, so this 2 by 3, so 2 by 3 plus 1, so this is like n to the power 5 by 3, okay? So, that's the ground state pressure. So, I missed that n, which is the density, okay? So, the internal energy—how do we get it?

We get it using our very familiar form, which is log of ZG, this del del beta log of ZG and ZF and V being constant, and a little bit of simplification can be achieved. When you do a del del t of log of ZG with a KT square there and ZFV. So, this can be written as 3 by 2 KT, this GV over lambda cube. f of 5 by 2 z f. That is the Fermi integral that we have just defined in the previous slide. So, this Fermi integral with, you know, z f inside and n is equal to

3 by 2 or 5 by 2 depending on whether you're talking about pressure or internal energy and the number of particles so this is the expression for internal energy let's call it as equation so we have equation 1 2 3 4 and we have say for example 5 and we can we can use P equal to two-third U by V and this is universally true for all non-relativistic gases and the factor is simply replaced by the coefficient is replaced by 1 over 3 for the this one non-relativistic and 1 over 3 is for relativistic okay. So if you differentiate 5 with respect to, you know, this, let's call it a 6. So differentiate 5. With respect to T and use this formula.

Internal energy

$$U = -\frac{\partial}{\partial \beta} \ln Z_g \Big|_{z_f, V} = k_B T^2 \frac{\partial}{\partial T} \ln Z_f \Big|_{z_f, V} .$$

$$= \frac{3}{2} k_B T \frac{gV}{\lambda^3} f_{5/2}(z_f) \qquad (5).$$

$$P = \frac{2}{3} \frac{U}{V} \qquad (6)$$

Differentiating (5) wrt T and use:

$$\frac{1}{z_f}\left(\frac{\partial z_f}{\partial T}\right)_v = -\frac{3}{2T} \frac{f_{3/2}(z_f)}{f_{1/2}(z_f)} .$$

Again, this formula has been, or something very similar has been used for the Bose gas as well, is del Z del T at a constant V. So, this is equal to minus 3 over 2 T f of 3 by 2 ZF. Divided by F of half ZF, okay. Now, the Fermi integrals replace the Bose integrals. If you use this and take a derivative of U with respect to T, then what you get is the following. So, you get a CV equal to NK.

and f of 5 by 2 z f divided by f of 3 by 2 z f and minus 9 by 4 f of 3 by 2 z f f of half z f, okay. So, this is equation 7. I hope I have written 6. Okay, 6 is there. This is 7.

This is the expression for the specific heat at any temperature of a Fermi gas. You just have to use these Fermi integrals to know their temperature, exact temperature dependence. And the free energy, of course, you can write down the free energy here.

They're not very important, but sometimes for some particular reason that you may need this. So free energy F is equal to mu N minus PV, and that's equal to NKT.

log of Zg, Zf and F of 5 by 2 Zf and F of 3 by 2 Zf, okay. That is your free energy and We are not doing anything more with this, just getting the expressions in terms of the Fermi distribution function or the Fermi Dirac integral. Now, to get the exact temperature dependence at all temperature, you have to expand this. And for ZF to be small, it's easy because all these things will become just simply ZF.

So ZF will kind of cancel. And you can see that it becomes 1 minus 9 by 4 ZF. There are these things that will sort of, you know, go to zero—rather, this F half ZF. All these will have to be calculated. I mean, they are general expressions, you know, as a function of temperature.

And that's why. So this is the most general expression that you get at all temperatures. Okay. Low, high, very high, very low, and all that. Okay.

And similarly, the entropy can be written as. So there's specific heat. And entropy is written as S equal to U minus F divided by T. You got U and you got F both, and then divided by T, which is simply nothing but NK. And then we have a 5 by 2 entropy. f of 5 by 2 zf, f of 3 by 2 zf, and so on, okay, minus log of zf.

Let us call this 9, and so these are some of the expressions that you have. Now, they are, of course, sort of, you know, the Fermi gas or rather we are interested in in most of the cases as low temperature gases. Properties of these things, and in order to get the low-temperature properties, we need these forms of the Fermi-Dirac integrals, and the expression of the Fermi-Dirac integrals, as you know, at finite but low temperature. At T equal to 0, it is easy, but at finite and low temperature, one has to do a Sommerfeld expansion of these Fermi functions. I mean, I just write these Fermi functions, or you call them the Fermi-Dirac integrals, which are really the integrals that we have introduced, are needed at low but finite temperature.

Specific heat

$$C_V = Nk_B\left[\frac{f_{5/2}(z_f)}{f_{3/2}(z_f)} - \frac{9}{4}\,\frac{f_{3/2}(z_f)}{f_{1/2}(z_f)}\right] \qquad (7)$$

free energy

$$F = \mu N - PV = Nk_BT\left\{\ln z_f - \frac{f_{5/2}(z_f)}{f_{3/2}(z_f)}\right\}. \qquad (8)$$

Entropy

$$S = \frac{U-F}{T} = Nk_B\left[\frac{5}{2}\,\frac{f_{5/2}(z_f)}{f_{3/2}(z_f)} - \ln z_f\right] \qquad (9)$$

Sommerfeld expansion of the FD integral are needed
at low but finite temperature.

And this is the limit that is of interest to us because the properties of these systems are very important at low temperature, as the quantum effects dominate at low temperature. At very high temperature, of course, classical limits would dominate. Even if we understand that even room temperature is the same as zero temperature for electrons, But what about other properties that are intertwined with this? They are mixed with this.

So we do these experiments. Any experiments in solid-state physics are done at a reasonably low temperature to see the quantum behavior to be perceived. So, these things can be written as this F of 5 by 2 ZF. This is equal to 8 over 15 root pi. These are the Sommerfeld expansions.

We do not get into the details of that. These are mathematical properties and they are quite important for many usage and we will sort of see, but we will keep only the leading term, one term after the leading term, that is important. because this one will give you exactly the behavior at t equal to 0. Anything more than that you want you have to go to one at least one order and F 3 by 2 ZF is equal to 4 by 3 root pi and log of ZF These are only given information.

They are not derived. If you really want to get into this, you can look at this Sommerfeld expansion of the Fermi Dirac integrals. But they are only used, the results are only useful to us. And that is why I simply write these expressions. you would only need them for getting results at small but finite temperatures.

So F half, these are the three that are involved, and this is equal to 2 by root over pi, and there's a log of ZF to the power half, and 1 minus pi squared over 24, log of ZF to the power minus 2, and plus all that. other terms. So if you want the expression for the density of particles, simply take the F 3 by 2 and put it in this N by V equation which is 8 pi over 3, 2 pi over H square, H square 3 by 2, KT log ZF, hold to the power 3 by 2. and 1 plus pi square by 8 log of ZF to the power minus 2. And there are other terms which you may not want to write.

So, in the 0th approximation, that is at 0 temperature, we have said that So, in the 0th approximation, what do we have? We have the KT log Z, ZF is equal to mu, which is equal to 3N by 8 pi whole to the power 2 by 3 and 8 square over 2M. And this is nothing but epsilon F. Okay, so at T equal to 0, mu is equal to epsilon F, which is what we have said earlier, that they are not distinct, and this is equal to mu.

$$f_{5/2}(z_f) = \frac{8}{15\sqrt{\pi}}(\ln z_f)^{5/2}\left[1 + \underbrace{\frac{5\pi^2}{8}(\ln z_f)^{-2}} + \cdots\right].$$

$$f_{3/2}(z_f) = \frac{4}{3\sqrt{\pi}}(\ln z_f)^{3/2}\left[1 + \frac{\pi^2}{8}(\ln z_f)^{-2} + \cdots\right].$$

$$f_{1/2}(z_f) = \frac{2}{\sqrt{\pi}}(\ln z_f)^{1/2}\left[1 - \frac{\pi^2}{24}(\ln z_f)^{-2} + \cdots\right].$$

$$\frac{N}{V} = \frac{8\pi}{3}\left(\frac{2m}{h^2}\right)^{3/2}(k_B T \ln z_f)^{3/2}\left[1 + \frac{\pi^2}{8}(\ln z_f)^{-2} + \cdots\right]$$

In the zeroth approximation:

$$k_B T \ln z_f = \mu = \left(\frac{3n}{8\pi}\right)^{2/3}\frac{h^2}{2m} = \varepsilon_F.$$

$$\boxed{\begin{array}{l}\text{at } T = 0 \\ \mu = \varepsilon_F\end{array}}$$

At the first order,

$$\mu = \varepsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \cdots\right]$$

And so, if you neglect all other terms, you get this one term, and that gives you this KT log Z. There is a T that is missing here. So that is your epsilon F. And at the first order, that is, at a temperature that slightly differs from T equal to 0, okay? So taking the first order correction to this, we can get this as, so we have this mu which is equal to epsilon f that is leaving the first term We have 1 minus pi square by 12 kT whole square kT by epsilon f whole square and then there are other terms there, okay.

So, at T equal to 0, so put the second term equal to 0, mu equal to epsilon f which is what we have claimed. at t not equal to 0, mu deviates from epsilon f. We have said that, but how it deviates is shown here, that the first order correction to mu compared to epsilon f is of the order of t square, okay. And this can be, you know, carried on to the internal energy as well. So, we can write down the internal energy expression u over n is equal to 3 over 5 epsilon f 1 plus 5 pi square by 12 kT by epsilon f whole square and plus all that.

So what I mean to say is that suppose this is the ground state energy and this ground state energy per particle is 3 by 5 epsilon F. Now this is easy to get because you have this E0 equal to epsilon G epsilon D epsilon and from all the way from to epsilon F. At t equal to 0, okay? No, you get the first term. What is the correction? So, the correction to the ground state energy—or rather, the total ground state energy at t not equal to 0—you will have to add this Fermi function here.

And this is the function which is 1 divided by exponential beta epsilon minus mu plus 1. Now, epsilon is simply, you know, epsilon and g epsilon is epsilon to the power half and so on. And if you try to do this integral, it's going to be very difficult. An exact solution is probably available because that's what people have done for a long time—dealing with these integrals and trying to find a closed form for that. And so if you wish to know that was the first order correction to this E0 compared to the t equal to 0 value, which is 3 by 5 epsilon f, this goes as 5 pi square by 12 and a correction which is of the order of t square.

So, if t is not too small—that is, if it cannot be neglected—then include it. And if you are really at very low temperature, then, of course, you can. Suppose you are at 50 Kelvin and you know the classical limit is at 10 to the power 5, 10 to the power 6 Kelvin. And I mean classical limit with the epsilon F, which is what you compare it with. So if KT is like say 100, 150 Kelvin and epsilon F is like, you know, 50,000 or 1 lakh Kelvin or, you know, 0.1 million Kelvin.

Internal energy

$$\frac{U}{N} = \frac{3}{5}\varepsilon_F \left[1 + \underbrace{\frac{5\pi^2}{12}\left(\frac{k_B T}{\varepsilon_F}\right)^2}_{} + \cdots \right]$$

$$E_0 = \int_0^{\varepsilon_F} \varepsilon\, g(\varepsilon)\, d\varepsilon \quad \text{at } T=0 .$$

$$E_0 = \int_0^{\varepsilon_F} \varepsilon\, g(\varepsilon) \underset{\begin{array}{c}\downarrow \\ \frac{1}{e^{\beta(\varepsilon-\mu)}+1}\end{array}}{f(\varepsilon)}\, d\varepsilon \cdot$$

then, of course, it can be neglected. But if you are still at, say, 1,000 Kelvin or, you know, 5,000 Kelvin safe for that kind of temperature, then that thing can be included. But, of course, there is nothing called 5,000 Kelvin in the experimental realm. Okay, so we can... In that case, we can neglect it, but at least it gives you the first order correction to the T equal to 0 result.

And similarly for the pressure, so this P is equal to two-third U by V. and this is equal to 2 5th n epsilon f, a result that we have already seen, and it is 1 by 5 pi square by 12, and k T square by epsilon f whole square, and you have other terms which can safely be neglected, okay. So the leading term always gives you t equal to 0 result in all the cases that we have seen. And so the CV finally can be written as it's nk pi squared over 2. and kT over epsilon f. So this is a result that we are interested in, and Cv for a fermionic system or an electronic system at low temperature, it goes as Tq, okay.

Pressure

$$P = \frac{2}{3}\frac{U}{V} = \frac{2}{5} n \varepsilon_F \left[1 + \frac{5\pi^2}{12}\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \cdots \right]$$

$$\boxed{C_V = N k_B \frac{\pi^2}{2} \frac{k_B T}{\varepsilon_F} .} \qquad C_V \sim T^3$$

So, this is by and large, you know, the story of thermodynamics for a free Fermi gas, okay. And it's very important to realize that even at room temperature, we are really talking about the Fermi gas as if it's highly degenerate and at very low temperature—that is, at T equal to zero. If you want any correction due to this really T not equal to 0 and T equal to 300 Kelvin or at least 150 to 100 Kelvin, you can consider these corrections to these expressions of pressure, internal energy, number of particles, etc. That we have just given. So we shall stop here and continue with some more applications of this Fermi-Dirac statistics and as I said one of them would be the paramagnetic or the Pauli susceptibility of due to the free electrons and we will also talk about the white dwarf stars.

We will stop here. Amen.