

# ELEMENTS OF MODERN PHYSICS

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## Lec 21: Identical Particles

Now, I will discuss something very important, which is called the indistinguishability of particles or identical particles. This is very important, and one needs to understand it well. So, identical particles and indistinguishability. Alright, so this is a very important thing, as I said. What I want to tell you right at the outset, a priori, is that the definition of identical particles is really the same in classical and quantum physics. But they have differing implications.

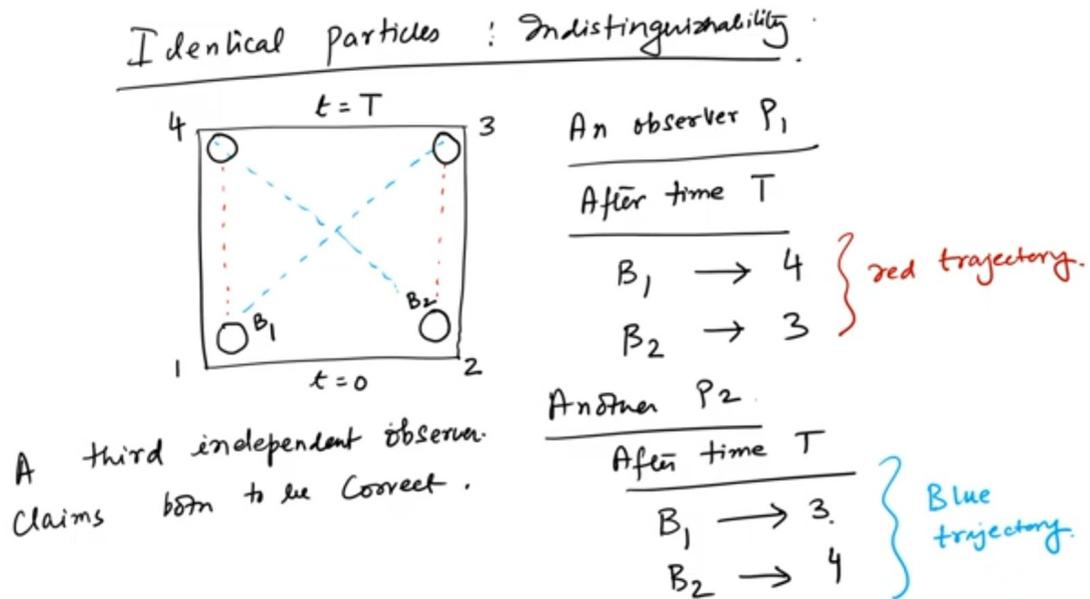
It is quite a sort of powerful and tricky statement. Let me try to explain that, that what kind of different implications could it have for identical particles. And it is not difficult to imagine that if you have a system of electrons or a system of, you know, photons or a system of particles, phonons or any other, you know, quantum object, quantum particles, electrons and so on, baryons, they are intrinsically indistinguishable. You cannot tell one from the other, okay.

So if they are, then what classical mechanics did so far in terms of distinguishability and one-particle description is fine because you know that you are just talking about one particle and nothing else in the universe. But when you have many particles, at least two particles, how would you distinguish one from another? And as I just said, the definition has to be the same in both classical and quantum physics. But they vary or differ by their implications. Let us see how it does.

Let me draw a billiard board. Okay. and this billiard board has say these balls, let us say that these are two balls and even if I know very well that they are identical because we are talking about identical particles, say they are identical, I still have to write for our discussion to proceed as B1 and B2. So, ball 1 and ball 2 are the two sort of balls are the two corners of a billiard board and let us see that this is at a time  $t$  equal to 0.

Now, what will happen is that the billiard ball would be hit and will reach somewhere on the other extremity at after a time  $t$  and let me show these things by two different colors. So, let me show it by this blue color that this actually goes like B1 goes to this end and B2 goes to this end. this end and they are now at  $t$  equal to  $t$ , we are these two balls that

are, that went to this part. So, but there is also another possibility that you see that they can actually go like this and they can go like this. So, say for example, an observer P1 he claims that the blue line is correct and he sort of he takes the initial data. So, his initial data was that the B1 and B2 are at this corners the lower corners at  $t$  equal to 0 and then he takes a data at after time  $t$ . he predicts that B1 let me write it as these edges as 1, 2, 3 and 4. So, he predicts that after time  $t$ , b1 goes to say for example, he says that let us b1 goes to 4, so that is along the red trajectory and b2 goes to 3, so this is the red trajectory. Another observer P2, he also takes data and after you know some player has come and hit the ball to, so he predicts that this goes to 3 and B2 goes to 4.



Okay, and this is the blue trajectory, a blue path. Of course, both can be correct. We know that because forcefully we have named them, but they are identical, so they cannot have any name associated with them. So, an independent observer would actually find that both are correct. So, a third independent observer claims both to be correct. And he is unable to, you know, resolve this conflict that what P1 claims or what P2 claims, which one is correct.

And he is actually very correct in saying that because these are indistinguishable particles, and both seem to be correct. So, what allows us to distinguish one situation from another? That is the situation in which which P1 reports and the situation which P2 reports. The distinction lies, of course. on trajectories, that is whether it is a blue trajectory or a red trajectory, okay, which means that it depends on history.

That is, which path—you can either call it history or trajectory—they mean the same thing in the sense that how they went from one identical particle, you know, from this corner of this thing, from this 1, 2 position to the 3, 4 position, and so on. So, this history is known in classical mechanics, but not in quantum mechanics. There is no definition of the path followed in quantum mechanics. No such history is known.

quantum mechanics. This is a very important statement and very powerful statement as well that we are unable to know which path do they follow and if you have any question that how does classical mechanics know do we really solve Newton's laws and know this exact path. Probably, in a simulation, if you do it, you know, if you sort of discretize this, the forces, if you know the forces acting and the numerical is solved, you probably can generate the path, but any other path is also possible and that answer is very nicely given by the, by your Lagrange's formalism. So, we know that the path that it follows actually for in that along that path, the action has to be minimum or the LDT has to be minimum or extremum.

along that path. So, even if this the Newton's laws do not really specify this, there are many ways of going from one to another that is initial  $x_i$   $t_i$  to  $x_f$   $t_f$ ,  $i$  and  $f$  they stand for initial and final. But the Lagrange's equation of motion that inherently and implicitly assumes that these equations of motion which are identical to that the Newton's laws, they actually take into account the correct path that is followed and the path that actually the particle takes is where the Lagrangian density, this  $L$  is the Lagrangian density and then  $DT$  and so integral of that is minimized which means  $\delta$  of that is equal to 0 and this called as an action. So,  $\delta X$  equal to 0.

The distinction lies of course on Trajectories.

It depends on history!!

History is known in Classical Mechanics.

NO Such history is known in Quantum Mechanics.

$$S = \int L dt \text{ has to be a minimum}$$
$$\delta S = 0.$$

We will not go into details of that, but then there is nothing like this in quantum physics. Quantum physics does not really care or it is possible to construct a path through which the particle goes. We only know about the probability density for a particle to be found at a space time point, nothing more than that. There is no trajectory that could be generated in quantum physics. And this is the inherent difference between classical and quantum physics, where the distinguishability and the indistinguishability become so important and the statistics become important.

And as you have seen that the Gibbs paradox tries to take into account in a sort of hand waving manner by dividing either the canonical partition function or the entropy by some factor of  $N$  or  $N$  factorial and then do some, get some results for the Gibbs paradox. That is, if you allow two identical gases to mix, there should be no change in entropy before mixing and after mixing. So, the entropy of mixing is equal to 0, and no such things are required for quantum mechanics, and that is the power of quantum mechanics. So, we just look at a little about the symmetric and anti-symmetric wave function that these under the exchange that happens to the wave function.

So, if you have a particle A and a particle B, and if you take A to B and B to A, then classical mechanics demands that it is actually a new state, a distinct state that needs to be taken into account or there is a distinct configuration that has to be incorporated. Quantum mechanics says no, they are the same state; they can at most vary by sign, that is, a phase factor. And let us just try to, you know, sort of put it with some symbols and so on. So, we have, say, a wave function consisting of two particles, and we are simply able to write it, we are able to write it with just a ket, say A and B. And the particles, if they are by interchanging them, this becomes BA, and it becomes a distinct state as far as classical mechanics.

So, these are classical mechanics even though you do not see a wave function, but we are simply talking about, say, in the  $p$ - $q$  space or in the phase space, this is just a configuration that we are talking about. So, this is classical mechanics. would assert that  $\Psi$  and  $\Psi$  prime are distinct. But in quantum mechanics, This is not distinct.

In fact, this is equal to  $\psi$  equal to AB equal to some  $\alpha$   $\psi$  prime and this is equal to some BA. And you know that in quantum mechanics, if you multiply a state or a ket by a number or by a factor, constant factor, then the state does not change. Because for getting a new state, that state has to be orthogonal with respect to the earlier state. And they have to be linearly independent. And clearly, this is not linearly independent.

It is just there is a phase factor that multiplies this one state to another. So, these are not, so  $\psi$  and  $\psi'$  are not distinct states and related to each other. Okay, so that seems fine. So from here on, we have written it as  $\alpha$ , which means that  $\alpha$  is some number that we have not committed ourselves. And here we go and can define there are two kinds of combinations.

One is called as a symmetric combination of states. which tells you that this  $\Psi$  has to be like  $\frac{1}{\sqrt{2}}$  and  $AB$  and  $A$  plus  $BA$ . Why is this symmetric? Because if you change  $A$  to  $B$ , it does not pick up a sign, it just you know the second term becomes first term and the first term becomes second term, it just does not matter which one you write first inside the bracket, so this is like a symmetric combination. and an anti-symmetric combination is also there and it tells you that this, so let us write it as symmetric here and anti-symmetric here and it is just, it just comes with a negative sign.

Exchange statistics.

$$\begin{aligned} |\psi\rangle &= |a\ b\rangle \\ |\psi'\rangle &= |b\ a\rangle \end{aligned} \left. \vphantom{\begin{aligned} |\psi\rangle \\ |\psi'\rangle \end{aligned}} \right\} \begin{array}{l} \text{Classical} \\ \text{Mechanics} \end{array} \quad \psi \ \& \ \psi' \ \text{are distinct.}$$


In quantum Mechanics

$$|\psi\rangle = |ab\rangle = \alpha |\psi'\rangle = |ba\rangle \rightarrow \psi \ \& \ \psi' \ \text{are not distinct states and related to each other.}$$

Symmetric combination  $|\psi\rangle_S = \frac{1}{\sqrt{2}} [ |ab\rangle + |ba\rangle ]$

Antisymmetric combination  $|\psi\rangle_A = \frac{1}{\sqrt{2}} [ |ab\rangle - |ba\rangle ]$

Now, you see that it is anti-symmetric because if you change  $A$  and  $B$ , there is a sign, a negative sign that comes in. Okay, let us try to make it a little more physical or something that you know instead of  $A$  and  $B$ . Let us give an example that we are talking about particles inside a one-dimensional potential box, which means there are infinite walls there and the particle is really trapped within that. And let us think that the two particles are trapped in  $n$  equal to 3 state and  $n$  equal to 4 states. So, two particles are trapped in a 1D potential box from 0 to  $L$ . OK, so we want to write down the symmetric and the anti-symmetric combination.

And let us write down the symmetric combination easily, which we know is  $1/\sqrt{2}$ . And we still continue that notation in terms of this. So instead of A and B, we now write it in terms of these  $N_1$  and  $N_2$  and so on. But they mean the same thing because we use the basis that is most convenient to us. And if you remember the basis for a particle in a box is that this  $E_n$  is really quantified by this  $n^2 \pi^2 \hbar^2 / 2mL^2$  and the  $\psi_n$  is if it is 0 to L box then it is  $1/\sqrt{L} \sin n\pi x/L$ .

Okay? So, of course, if it is a symmetric box that is between  $-L/2$  to  $+L/2$ , there are, you know, there are parity symmetric solutions would come where  $N$  will only be equal to even for this solution and for  $N$  odd, you will have to have a cosine solution. So, this is fine. So, we will write this in the  $n_1, n_2$  basis. So, we will write it as 3, 4 and 4, 3.

Each one of them is like, so each  $n, n_1, n_2$ , it is like your  $1/\sqrt{L^2}$  and it is a direct product of these things. So, it is like  $\sin 3\pi x/L, \sin 4\pi x/L$  and so on so forth. So, that is the product state and we have taken a square of that and so on. So, this is a meaning. So, there is a symmetric combination.

And if you want the anti-symmetric combination, well, we have written this  $S$  to be outside. So, let me have this  $S$  to be inside for the symmetric and  $A$  for inside the ket notation. So,  $\psi_A$  is equal to again  $1/\sqrt{2}$  you can write it in this notation or you can write it in this notation as well. That is, you have a  $\psi_3$ .

Okay, let me write just 1, then that will be clearer. So, any  $n$  state is like  $1/\sqrt{L}$  and it is  $\sin n\pi x/L$ . Okay, that is the Let us not write a product state because this is what we want to show that  $\psi_3(x_1)\psi_4(x_2) - \psi_4(x_1)\psi_3(x_2)$  and so on. So, this is the anti-symmetric combination.

So, we can write this as well as the symmetric combination. It is a  $\psi_3(x_1)\psi_4(x_2) + \psi_4(x_1)\psi_3(x_2)$ . Well, let me just squeeze it in here,  $\psi_4(x_1)\psi_3(x_2)$ . So, this is a symmetric and the anti-symmetric and very nicely this anti-symmetric part can be written as a matrix or as a determinant rather, this is called as the Slater determinant and the  $\psi_4(x_1)$  and the  $\psi_3(x_2)$  and there is a  $\psi_4(x_2)$  and so on. This is called as the Slater determinant.

Example  $n=3, n=4 \rightarrow$  two particles trapped in 1D pot. box.

$$|\Psi_s\rangle = \frac{1}{\sqrt{2}} [ |3,4\rangle + |4,3\rangle ] = \frac{1}{\sqrt{2}} \left[ \begin{matrix} \psi_3(x_1)\psi_4(x_2) + \psi_4(x_1)\psi_3(x_2) \\ E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \end{matrix} \right] \quad |a,b\rangle \rightarrow |n_1, n_2\rangle$$

$$|n\rangle = \left(\frac{\sqrt{2}}{L}\right) \sin \frac{n\pi x}{L} \quad \psi_n = \frac{\sqrt{2}}{L} \sin \frac{n\pi x}{L}$$

$$|\Psi_a\rangle = \frac{1}{\sqrt{2}} [ \psi_3(x_1)\psi_4(x_2) - \psi_4(x_1)\psi_3(x_2) ]$$

$$= \frac{1}{\sqrt{2}} \begin{vmatrix} \psi_3(x_1) & \psi_4(x_1) \\ \psi_3(x_2) & \psi_4(x_2) \end{vmatrix} \leftarrow \text{Slater determinant}$$

So, the anti symmetric wave function can be written as a determinant. So, if you take the determinant of that, you will get the step that is you see above. All right. So, for you know, larger number of particles, it is a little complicated, but nevertheless, it is still doable, say for three particles, it is instructive for you to learn. So, for symmetric, we can write it easily because it is a 1 by 3 factorial

And we will just use this the fog basis as it is called the number basis or rather we just simply use this the same context that we have done. So, it is  $n_1, n_2, n_3$  plus  $n_1, n_3, n_2$  plus  $n_2, n_3, n_1$  all the permutations that are possible.  $n_2, n_1, n_3$  and  $n_3, n_2, n_1$  and  $n_3, n_1, n_2$  and so on, okay. So, these are all the permutation of the pairwise particles like this one is just the same order and then we have kept  $N_1$ , the first particle to be in  $n_1$  and then  $n_3$  and  $n_2$  are swapped and so on so forth. So, one is able to make the six combination and the normalization demands that there is a 1 by  $n$  factorial.

And this one for the anti-symmetric, you just need to understand that for these pairwise exchanges, for one pairwise exchange, you should pick up a negative sign just like this negative sign that you have it here. that let me show it with the color. So, this is the negative sign that we are talking about this is here. So, the pairwise exchanges will have a negative sign, but 2 pairs of such exchanges will bring back another negative sign which means that it should come with a positive sign. And so, this positive and negative, all these things can be written a little carefully.

And if you write them down, it is like  $n_1, n_2, n_3$ , minus  $n_1, n_3, n_2$ , because that there is one exchange. And then there is the two exchanges, which is  $n_2, n_3, n_1$ . So, it is a negative sign and then it is a  $n_2, n_1, n_3$  that has odd exchanges and plus again even

exchanges  $n_3, n_1, n_2$  minus  $n_3, n_1, n_2$  and so on,  $n_3, n_2, n_1$ . Okay, so this will give rise to a sort of, you know, this Slater determinant which gives you say for example,  $x_1, n_2, x_2$  and  $n_3, x_3$  and so on,  $n_1, x_2$ , this is sorry, this is  $n$ , so this  $x_1, x_1, x_2, \psi_{n_2}, x_2$  and  $\psi_{n_3}, x_2$ .

For 3 particles.

$$|\Psi_S\rangle = \frac{1}{\sqrt{3!}} \left[ |n_1 n_2 n_3\rangle + |n_1 n_3 n_2\rangle + |n_2 n_3 n_1\rangle + |n_2 n_1 n_3\rangle + |n_3 n_2 n_1\rangle + |n_3 n_1 n_2\rangle \right]$$

$$|\Psi_A\rangle = \frac{1}{\sqrt{3!}} \left[ |n_1 n_2 n_3\rangle - |n_1 n_3 n_2\rangle + |n_2 n_3 n_1\rangle - |n_2 n_1 n_3\rangle + |n_3 n_2 n_1\rangle - |n_3 n_1 n_2\rangle \right]$$

$$= \frac{1}{\sqrt{3!}} \begin{vmatrix} \psi_{n_1}(x_1) & \psi_{n_2}(x_1) & \psi_{n_3}(x_1) \\ \psi_{n_1}(x_2) & \psi_{n_2}(x_2) & \psi_{n_3}(x_2) \\ \psi_{n_1}(x_3) & \psi_{n_2}(x_3) & \psi_{n_3}(x_3) \end{vmatrix}$$

And this is  $\psi_{n_1} x_3, \psi_{n_2} x_3$  and  $\psi_{n_3} x_3$ . Now, it is written correctly and it is the Slater determinant. So, we should simply draw a vertical line there. So, this for three particles, you can exchange it to, you know, larger number of particles. Again, these two particles, let me take a mod square of that.

So, we want to calculate the probability here. For the symmetric and anti-symmetric cases, okay. So, S and A both if you write it for these two particles which are could be at the locations this. So, we have a S. So, again go back to the two particle because three particle is too complicated. to write this modulus because you have to take many terms.

So, S A  $x_1$  and  $x_2$  mod square, and this is nothing but equal to half of this, and then you have a mod square of  $\psi^3$ . We go back to our original problem of, you know,  $N_3$  and  $N_4$ , so that and then we have this:  $\Psi_4, X_1, \Psi_3, X_2$ , and so on, this and the mod square. Now, the plus sign is for the symmetric, and the negative sign is for the anti-symmetric. And if you open the bracket, this is really, you know,  $\psi^3 x_1$  square,  $\psi^4 x_2$  square, and a plus  $\psi^3 x_2$  square and  $\psi^4 x_1$  square. And plus minus, you have a  $\Psi^3$

star  $x_1$   $\Psi_4$   $x_2$   $\Psi_4$  star  $x_2$  and  $\Psi_3$   $x_1$ , that is one, you know, interference kind of term or mixed term.

And then you have a  $\Psi_4$  star  $X_1$ ,  $\Psi_3$   $x_2$ ,  $\Psi_3$  star  $x_1$ , and a  $\Psi_4$   $x_2$ . So, this is the probability density for each of these states. And this is very interesting. The interesting thing is that the first two states, Are predicted by this third observer who came and joined the game at that capital T time.

So, he could only see this, but however, a quantum mechanical system In addition to the two terms, they need a crucial interference term. In fact, if you remember Young's double-slit experiment, these two terms—the first two terms—are not important, or rather, they are important, but they are not crucial in explaining the interference pattern. The interference purely came from the last two terms.

$$\begin{aligned}
 \text{2 particles} \\
 P_{S/A}(x_1, x_2) &= |\Psi_{S,A}(x_1, x_2)|^2 \\
 &= \frac{1}{2} \left| \left[ \Psi_3(x_1)\Psi_4(x_2) \pm \Psi_4(x_1)\Psi_3(x_2) \right] \right|^2 \\
 &= \underbrace{|\Psi_3(x_1)|^2 |\Psi_4(x_2)|^2 + |\Psi_3(x_2)|^2 |\Psi_4(x_1)|^2}_{P_1 + P_2} \\
 &\quad \pm \left[ \Psi_3^*(x_1)\Psi_4(x_2)\Psi_4^*(x_2)\Psi_3(x_1) \right. \\
 &\quad \left. + \Psi_4^*(x_1)\Psi_3(x_2)\Psi_3^*(x_1)\Psi_4(x_2) \right] \\
 &\hspace{15em} \text{Quantum Theory.}
 \end{aligned}$$

So, the classical observer who joined at small  $t$  equal to capital T that is at a later time, he sees that one this P that is the P1 observer and P2 observer or the vice versa and he sort of said that both are correct and he was correct, but then he missed this all these main ingredients of the quantum theory. Okay so, let me now sort of quickly do this different ensembles and or rather this different distribution and let us take a simple case in which we have two particles and three states and so on, okay. So, basically, an introduction to quantum statistics. So, the first one is Maxwell-Boltzmann statistics. And what it demands is that it sort of you take any particle in any state and there is an overall factor

of  $1/n$  factorial that should come in counting the number of microstates that are possible.

So, your  $\Omega$  would simply become equal to  $1/N$  factorial and then sum over you know  $K$  and then you just sum over this one you know microstates which is between an energy  $E$  and  $E + D$  or it is sort of restricted in energy space or in the number of particle space and so on so forth. So this is how the Maxwell-Boltzmann statistics is reconciled that since we are talking about it in the same bracket as the other two statistics let us just talk about that there are let us not talk about distinguishable and indistinguishable particles for the Maxwell-Boltzmann statistics which of course we know that this is it is valid for distinguishable particles, but if you are talking about it in the same bracket as Fermi-Dirac and Bose-Einstein, let us just say that you treat them as distinguishable, yet you divide it by this  $N$  factorial number in counting the number of microstates, just like what we have shown here, and there is a crucial  $n$  factorial term required, which is called a Gibbs correction.

And it is only true in the limiting sense of the other two statistics, which we are more interested in. So, we will write down this Bose-Einstein as will tell you about, you know, some historical things about Bose-Einstein statistics and so on. So, this Bose-Einstein statistics tells you that any number of particles can occupy a given quantum state or given energy state, and there is no need for this  $n$  factorial. So, this is an important thing; there is no correction that is needed.

Okay, and of course, here in the Maxwell-Boltzmann, we can simply go ahead and calculate the partition function, which would give rise to this exponential minus  $\beta E_k$ , and we have seen how this gives rise to or rather avoids giving the wrong results by the mixing of these gases to identical gases, and this is called the Sackur-Tetrode equation, which we have seen. Now, in Bose-Einstein, any number of particles can occupy any energy level; there is no need to have this thing. And similarly, in the Fermi-Dirac case as well, we do not have any statistics. Again, no  $n$  factorial needed. Here, any number of particles can occupy any state; here, only no two particles can occupy a given state.

## Introduction to Quantum Statistics.

(1) Maxwell Boltzmann Statistics.

$$\frac{1}{N!} \quad \Omega = \frac{1}{N!} \sum_{\mathbf{k}} 1$$

$$Z = \frac{1}{N!} \sum_{\mathbf{k}} e^{-\beta E_{\mathbf{k}}}$$

(2) Bose Einstein Statistics.

NO  $N!$  needed.

Any number of particles can occupy any state.

(3) Fermi-Dirac Statistics.

NO  $N!$  needed.

NO two particles can occupy a given state.

Okay, so bosons are, they love the presence of other bosons, whereas these are completely, they repel any other fermions that come into their vicinity and so on. And this gives you the main difference between the three statistics. And we'll see examples of that. We would, you know, in a simple situation, start with a few particles that can be distributed in a few energy states. and do these calculate this distribution and calculate the partition function maybe the canonical partition function for those and show that what are the differences between these three statistics in very simple situations and then we shall go ahead and talk about many particles and introduce these statistics and from a more formal perspective

And finally, we would go ahead and do exercises on each of those or other applications of each of those statistics in various physical situations. For example, the Bose-Einstein statistics would give us the right specific heat expression for phonons, say, for example, or it would give rise to the radiation pressure due to a black body or due to electromagnetic radiation. The pressure due to that or you have these fermions will talk about metals which are free electrons and we can talk about the Pauli paramagnetism or we can talk about these white dwarf stars. which are taken as or which can be considered as a very extremely degenerate Fermi gas.

And we'll do these calculations or rather these example problems or these applications on these various statistics. So, I hope the idea has been made clear this week that we are really getting into quantum statistics. There are various corrections or fixations that need

to be done for one to actually transition from classical statistics to quantum statistics. So, we will carry on with quantum statistics from next week onwards. Amen.