ELEMENTS OF MODERN PHYSICS

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Week-01

Lec 2: Lagrangian Mechanics

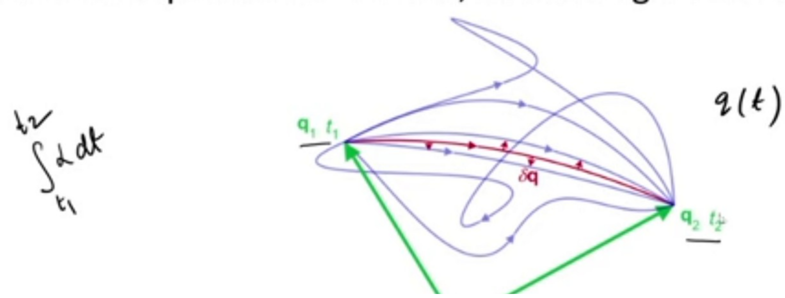
So we shall start with Lagrangian mechanics. We have done a little bit of Newtonian mechanics, where we have introduced the free body diagram and how to solve the equations of motion, how to get trajectories, etc. Now we will shift over to Lagrangian mechanics, and the reason that Lagrangian mechanics is done is that it is more powerful in some sense, as it not only predicts the trajectory of the particle, but it also identifies the trajectory that the particle takes, which Newton's laws are unable to predict—exact trajectories in going from some x1, t1, some space-time point x1, t1, to some final space-time point x2, t2, and thus this is more elegant. and more informative, but it also involves mathematical complexities like calculus of variations, etc.

That is why it is taught at the undergraduate and postgraduate levels, whereas Newtonian mechanics you have been exposed to since your school days. So we'll study why we study Lagrangian mechanics. We'll talk about that. We'll talk about D'Alembert's principle, virtual displacement, and the principle of virtual work. We'll talk about generalized coordinates and different types of constraints. And these constraints are going to affect the equations of motion, and we need coordinates which are free of these constraints, and they are called generalized coordinates. I will talk about generalized force, we will talk about the Lagrangian, and hence Lagrange's equation of motion, and of course, we will back it up with some applications of this equation of motion, Lagrange's equation of motion. So the question is, why do we study Lagrangian mechanics? Because Newtonian mechanics has been developed earlier, and it is able to predict, under the application of a force, what is the space-time point as it moves from one space-time point, here shown as q1, t1. to q2, t2, and the reason that we are not talking about x1, x2, but q1 and q2, will be sort of clear later. So, Lagrangian mechanics incorporates the principle of least action, which is a variational principle that, when applied to the action of a mechanical system, will tell what the action is.

Basically, the action is the Lagrangian density integrated over time from some t1 to t2. We will see that in a while. So, the action of a system, of a mechanical system, can be used to obtain the equations of motion for that system. So, it is historically called 'least' because it requires finding the path of motion that has the least value. So, the action has a least value, and the principle can be used to derive the Newtonian equations, which we are already familiar with, Lagrangian and Hamiltonian equations of motion, and even those of general relativity. So, if you look at these two space-time points, which are given by q1, t1 and q2, t2, there are many, many ways, an infinite number of ways that the particle can take in going from these two points, space-time points, and Newton's law would only say that there are all these possibilities and it will just tell you that, starting from a q1, t1 and under the application of a certain force, It has gone to a q2, t2, but will not be able to tell, distinguish between the one that is actually taken by the particle among this plethora of paths that we are showing here. And so, when the system evolves, it traces a path through the configuration space. Okay.

Why study Lagrangian Mechanics?

The principle of least action is a variational principle that, when applied to the action of a mechanical system, can be used to obtain the equations of motion for that system. It is historically called "least" because it requires finding the path of motion in space that has the least value. The principle can be used to derive Newtonian, Lagrangian and Hamiltonian equations of motion, and even general relativity.



So, this is a configuration space that you see here, which is all the q points as a function of t. So, that is the configuration space, this space. And there are two points being shown in that space, Okay. So, the path taken by the system has to have a stationary action, that is, the action has to be extremized or it has to be minimized under small configuration changes in the configuration of the system. So, if you go from q to q plus dq, then the requirement is that this action, which we call S, the elementary change or infinitesimal change in action, has to vanish. So, this is called the principle of least action, or the path taken by the particle will be in conformity with the least action principle. So, this action of a physical system is defined as the integral of the Lagrangian or the Lagrangian density between two instants of time, t1 and t2.

As the system evolves, it traces a path through the configuration space. The path taken by the system, has a stationary action (δS = 0) under small changes in the configuration of the system.

The *action* (S), of a physical system defined as the integral of the Lagrangian L between two instants of time t_1 and t_2 – a functional of the N generalized coordinates $\boldsymbol{q} = (q_1, q_2, \dots, q_N)$ and is given by,

$$S[q, t_1, t_2] = \int t_1 t_2 L(q(t), \dot{q}(t), t)dt$$

where the dot denotes the time derivative, and t is time.

Mathematically the principle is written as:
$\delta S = 0$, where δS means a *small* change in the action.

So, where this L is actually a functional in generalized coordinates, we will come to what generalized coordinates are in more detail later. Just we are trying to emphasize why Lagrange's equation of motion or why this Lagrangian mechanics has come into being and how they are, you know, important for us to learn. So, this is how the action is written. It's a function of these generalized coordinates, which we call q, and these two times that we are showing here. So, this is really, you know, this integral is over. So, this t1 and t2 are really here. So, these are like t1 and t2. And this L dt is called an action. And mathematically, the principle that's the principle of least action is written as delta S is equal to 0, where delta S means a small change in action.

So, the path that is taken, say for example, is the one that is shown in red, and all the other ones that are shown in blue are not the path that the particle has taken in coming from or rather in traveling from Q1, T1 to Q2, T2, because say for example, along this red path, the action is minimum. So, let us see what virtual displacement is. So, a virtual displacement, let us call it delta ri. ri is a vector, and we are sort of writing it in bold letters, not putting the vector sign, but it means the same thing. So, delta ri refers to an arbitrary infinitesimal change in the configuration of the system. And by I, we really mean the ith particle. Okay, so and it's true for all particles that are present in the system. It's called virtual to distinguish it from the actual displacement during which the forces and the constraints or the constraint conditions may be changing. If it does, then these delta ri s are not really the actual displacement that has taken place by the particle or other that has, you know, affected by the that has come into force for the particle. But so this is really a virtual displacement.

Virtual displacement:

A virtual displacement $\delta \boldsymbol{r}_i$ refers to an arbitrary, infinitesimal change in configuration of the system.

It is called *virtual* to distinguish it from *actual* displacement during which the forces and the constraints may be changing.

If a system is in equilibrium, the total force vanishes.

$$\sum_i \boldsymbol{F}_i = 0$$

Taking a dot product with virtual displacement also vanishes.

$$\sum_i \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i = 0$$

And for the system to be in equilibrium, we know that the total force on all the particles would vanish. So, which means that the sum of F I should be equal to zero. And if you take it with the dot product with the virtual displacement, so this is really the virtual work. The virtual work also vanishes because the sum over F i is equal to 0. So, F i delta R i also has to vanish. And so this is the that work done is equal to 0. Now, what we do is that we split the force acting on the particle indexed by i by F i superscript a, where the a actually means the applied force, and F i are the force due to the constraints. Now, what kind of constraints are we talking about? Say, for example, in a rigid body. So, this is a body which remains rigid throughout the course of its motion. So, as the body moves, it does not change its shape. It does not get elongated. It does not contract or shrink.

Nothing happens to the shape of the body, which means that the distance between two points remains what they are. So, this is a constraint condition that is always constant. Suppose a ball is hit, it is not hit very hard, and then of course the ball would continue to move and would retain its shape. Say for example, another example, let us talk about a huge sphere and we are talking about a very small pebble on that. So, this is constrained to move on the surface of the sphere, and say the sphere is enormously large, and then it sort of continues to move forever, say for example, even if it, you know, falls down, we are not considering that. So, this is always at a distance r, and that is the constraint condition where r is the radius of the sphere, Okay. So, these are some of the constraints that we are talking about. We'll see more constraints in terms of mathematics, in terms of, you know, sort of concepts of such constraints. We'll see that.

So if we split that into an applied force, the force on the particle indexed by i, by an applied force and the force due to constraint, and then we write this F i as F i a plus F i and then dotted with delta r i, that's equal to 0. And it is not unusual to assume that the forces of constraints do not do any work, which means that if it is really because of the interaction potential between the two atoms or between the two objects in a rigid body, they would not do any work and this force would actually balance or, you know, go to zero pairwise and so on and it will not do any work on the on the system because it is an internal property of the system, okay. So, if this is equal to 0, the one that we are left with is the applied force dotted with the virtual displacement equal to 0 and this is called the principle of virtual work, okay. So, F i a that is applied dot delta r i and sum over i is equal to 0 is called the principle of virtual work, okay. There is a very subtle point that requires our attention here.

Suppose:

$\boldsymbol{F}_i = \boldsymbol{F}_i^a + \boldsymbol{f}_i$ with $\boldsymbol{F}_i^a$: applied force and $\boldsymbol{f}_i$: force of constraints

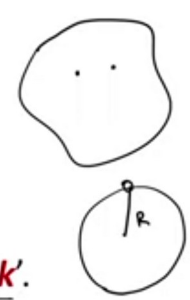
$$\sum_i (\boldsymbol{F}_i^a + \boldsymbol{f}_i) \cdot \delta \boldsymbol{r}_i = 0$$

Suppose the force of constraints don't do any work, then

$$\sum_i \boldsymbol{f}_i \cdot \delta \boldsymbol{r}_i = 0$$

Hence,

$$\sum_i \boldsymbol{F}_i^a \cdot \delta \boldsymbol{r}_i = 0$$



The above equations is called '***Principle of Virtual Work***'.

Delta r i, one cannot say that the coefficients of delta r i are equal to 0. That is because that which means that F i a cannot be put to zero, even if the sum over i F i a dot delta r i equal to zero, because the delta r i s are not independent of each other. And there are certain constraint conditions which are making them dependent. So all the r i s, so r i and r j actually can have a relation between them or r i and r k can have a relationship between them where ijk s are all, you know, indices of particles that we are talking about, okay. So, with the forces of constraints not contributing to the work done, if we wish to look into dynamics, what would we do? We will write down F i equal to dpi by dt, which we write as a p i dot. So, which means F i minus p i dot, dot delta r i equal to 0.

A subtle point:

The coefficients of $\delta \boldsymbol{r_i}$ can't be put to zero, $\boldsymbol{F_i^a} \neq \boldsymbol{0}$, since $\delta \boldsymbol{r_i}$ s are not independent of each other and are connected by **constraints**.

With the forces of constraints not contributing to the work done, let us look for the dynamics.

Writing $\boldsymbol{F_i} = \dot{\boldsymbol{p}}_i$, $\sum_i (\boldsymbol{F_i} - \dot{\boldsymbol{p}}_i) \cdot \delta \boldsymbol{r_i} = 0$ $\qquad \vec{F}_i = \vec{F}_i^a + \vec{f}_i$

Again assuming the work done via the force of constraints is zero,

$$\sum_i (\boldsymbol{F_i^a} - \dot{\boldsymbol{p}}_i) \cdot \delta \boldsymbol{r_i} = 0$$

- *D'Alembert's Principle*

So if you again assume in this case, so if you again split this as now I am writing it as a vector because I cannot show the bold figure. So this is a and plus F i. So if you again take this work done due to F i equal to 0, which are the forces of constraint. So we get this equation as F i a minus p i dot. delta r i equal to 0 and this called D'Alembert's principle. So, this the dynamics of the particle is embedded into it previously we were simply talking about the statics of the particle. So this is called D'Alembert's principle. And we are now want to, you know, talk about these generalized coordinates. So want to go into from these r i to q j. If you note that we have written r with a vector sign and q without a vector sign. And this is important because q j may not be the all these the variables of the coordinate system that you are, you know familiar with okay. So, we have you know different type of constraints that are there in talking about bodies objects and so on the motion of them and they are broadly categorized into.

So, we are talking about constraints in order to go from normal coordinates that we were familiar with in the Newton's laws to the generalized coordinates. So, what are the constraints? The constraints are the holonomic constraints and the two would be a non-holonomic constraints. And just to tell you a priori, we will not talk about non-holonomic constraints. In fact, they are microscopic in nature and they may not be essential for us to talk about the macroscopic motion of objects. So what are holonomic constraints? So holonomic constraints are like the ones that we can express it by using some equalities, which are like all these things. And so this is an equality. And what's sort of realization or what's an example of such an equation is that for a particle which is in order for rigid bodies, this is equal to r i minus r j equal to some c ij square.

So, that is saying that the distance between two objects always remains the same, and c ij is simply a constant. It could be dependent on ij or it may not depend upon ij at all, okay. Well, I mean, ij means these are a pair of particles. So their indices are of a pair of particles. So they would remain, you know, constant. So r i minus r j becomes equal to a constant. So this is an example of a holonomic constraint. And what is a non-holonomic constraint? Say, for example, you have a small sphere, not a large one, a small sphere, and you have another one which is smaller than that, and it is rolling on the surface of this, which means that it will roll a while and then fall off when the normal reaction occurs. Between the small pebble or the small sphere and the large sphere, that becomes 0.

So this small sphere will lose contact with the large sphere and will fall off, and we can then write as r square minus this is a radius, so this is r square minus a square, that has to be greater than 0. So, as long as it is in contact with the sphere, the equality sign holds, but then as it loses contact, it does not hold, then it becomes greater than 0. And this is, the equality is not there, so it is called a non-holonomic constraint. So these are examples, and as I said, I will do or rather deal with holonomic constraints only, and well, there are, you know, further classifications of these things, and they are, you know, called scleronomous and rheonomous constraints. So these constraints, they are independent of time, and real numbers, they are explicitly dependent on time.

So it is not, yeah, independent of time and dependent on time, that is fine, okay. So, these are different constraints that we look at, as I said earlier, we will look at holonomic constraints, and they are mostly independent of time, which means that they do not have any time dependence, all these constraints are time-independent constraints. So, suppose you ordinarily have 3 n coordinates in a particular problem, there are n particles, and each having three coordinates in three spatial directions. So, there are three n coordinates, and there are k constraints, okay, k number of constraints. Okay, such that one constraint we have shown here, that the distance between the particles, the square of the distance, they would remain constant. So, say there are k of them, and then we need to talk about 3 n minus k independent coordinates or independent degrees of freedom, okay, independent degrees of freedom.

Generalized Coordinates $(\boldsymbol{r}_i \to q_j)$:

Constraints

(1) Holonomic (2) Non-holonomic

$f(\vec{r}_1, \vec{r}_2, \vec{r}_3 \ldots \vec{r}_N) = 0$

$(r_i - r_j)^2 = c_{ij}^2 \longrightarrow$ Holonomic Constraints.

$r^2 - a^2 \geqslant 0 \longrightarrow$ Non-holonomic Constraints.

Scleronomous — independent of time.
rheonomous → dependent on time.

And this is called DOF, degree of freedom, and that is equal to the number of generalized coordinates. We will see that. So we introduce now the generalized coordinates. So we go from r i to q i as said earlier. So we have these R1, which is a function of these generalized coordinates, which are q1, q2. And how many of them? They are 3 n minus k. And of course, time and all the way r n, that is also a function of r n, and they are functions of this q1, q2, and q3 and minus k and time. So, these are the qs, the generalized coordinates. How they are different from the normal coordinates that we have dealt with. These coordinates, they include the constraint conditions that are there. Now, if you remember, we have said that there is a subtle point here, which tells you that the coefficients of delta Ri cannot be put to 0.

But now, if you use q's or the qi's, then if there is a situation, you can put the coefficients of delta qi equal to 0, because they are independent of each other. So, q's are the generalized coordinates that we have. So, what are the properties of the generalized coordinates? So now, the key properties are, okay, so one is that, unlike the Cartesian coordinates, they will not be associated with three mutually perpendicular directions. Like, for example, in some system, they just theta and phi could be the generalized coordinates. And in fact, even without constraints, they are useful. And of course, the process of defining the holonomic constraints in formulating the generalized coordinates is useful.

$3N$ Coordinates.
K constraints.
$3N-K$ independent degrees of freedom (DOF).

$\vec{r}_i \rightarrow q_i$ $\qquad$ q's are the generalized coordinates.

$$\vec{r}_1 = \vec{r}_1(q_1, q_2 \cdots q_{3N-K}, t).$$
$$\vdots$$
$$\vec{r}_N = \vec{r}_N(q_1, q_2 \cdots q_{3N-K}, t)$$

So, we will talk about the holonomic constraints only. And so, what is the final outcome or what is the biggest incentive of this is the laws of motion become independent of constraints. Okay, so this is the main advantage of using these coordinates, these generalized coordinates. And let us see how we put them into use and derive what is called Lagrange's equation of motion. Okay, that is the main idea that we have. And we start from this equation that we have talked about earlier. So, r i is actually a function of all these generalized coordinates. So, we had, while formulating Newton's laws, we have used these r i's which had some constraint conditions embedded into them. And from there, we have migrated to a set of coordinates which have no constraints or rather they take into account the constraints and they are independent coordinates. So, these are like q1, q2, and so on, and let me write it as q_n where n could be 3n minus k and time, okay. So, there are n independent coordinates. All right, so how do we write the velocity? The velocity is written as dri by dt. We know that as the velocity of the ith particle. And if you write this in terms of the generalized coordinates, then it is the sum over k and a del r i del q k and a q k dot, which will be called as a generalized velocity. And if there is this r i has some explicit time dependence, so this is called as a generalized velocity.

So, that's your v i. And if we, you know, sort of, we can write this as equation 1, say, for example, or we can actually go and number our equations from earlier. So, we have, so let's call this as equation 1, and then equation 2 would be just this, say, equation 2 is this, and let me just do a numbering of equations which will help us later, and so this applied one, so let us call this as equation 3 and this as equation 4, and maybe now we continue with these equations as equation 5, okay. So, that is equation 5, that is the velocity and written in terms of the generalized velocity, and so an arbitrary displacement can be written as that's delta r i, this is equal to sum over j, delta r i, del q j, delta q j, and so on.

<u>Properties of generalized Coordinates</u>

1. They will not be associated with three mutually $\perp$r directions.
2. Even without Constraints, they are useful.
3. Laws of motion become independent of constraints.

$$\vec{r}_i = \vec{r}_i(q_1, q_2 \dots q_n, t).$$ n: independent Coordinates.

$$\vec{v}_i = \frac{d\vec{r}_i}{dt} = \sum_k \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \qquad (5)$$

$\dot{q}_k$: generalized velocity.

An arbitrary displacement $\delta \vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$

And so, in terms of the generalized coordinates, the virtual work or the principle of virtual work becomes work in terms of the generalized coordinates, we write it as q i. So, that is equal to sum over i F i dot delta r i equal to i j. i j and F i dotted with delta r i del q j delta q j, okay, and let us call this as this entire thing as virtual force. So, this is of q j delta q j. So, this is the same thing stated, this quantity is written in terms of the virtual force. So, q j is called as a virtual force, or rather, not the virtual force, I am sorry, I have to correct myself, this is called as a generalized force. This principle of virtual work, not so, this is called as a generalized force which takes into account all these constraints, which is equal to F i delta r i and del q j, and I hope you understand that this del r i del q j because there is a relationship between r i and q j as far as this, you know, this equation that you see here, okay.

Alright, so this is called a generalized force, and it is important to understand that q j do not have the dimension of length. And in that same spirit, q j do not have the dimension of force, okay. So, they can have any dimension. So, let us look at the other term that you see in equation 4. So, this is equation 4. So, we have dealt with this F i a and knowing that the force of constraints do not do any work. So, let us look at the second term pi dot delta r i, okay. So, what does it give you? So, it gives you this pi dot delta r i, okay, is equal to mi r i, so this pi dot delta r i, so m i r i double dot, because that is momentum time, derivative of momentum, which is nothing but the force, which is like ma, so this is into delta r i, and this is equal to I j, so this is i, so this is i j m i r i double dot, and delta r i del q j and delta q j, okay. So, this is how the second term is written and see the last term that is how this term is written in a modified fashion. So, mi r i double dot delta r i del q j okay. Let me just pause for a while and say that what we are trying to do we have of course gone from the real coordinates that are used in Newton's laws to the generalized coordinates which take into account these constrained conditions okay. And from these constrained conditions we have constructed these generalized coordinates which finally yielded a notion of generalized force.

Principle of virtual work in terms of q_i

$$\sum_i \vec{F}_i \cdot \delta\vec{r}_i = \sum_{ij} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\,\delta q_j = \sum_j Q_j\,\delta q_j$$

Q_j : Generalized force. $= \sum_{ij} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$

q_js don't have dimension of length.
" " " force.
Q_j. " "

2nd term in Eq. 4 $\quad \sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i = \sum_i m_i \ddot{\vec{r}}_i \cdot \delta\vec{r}_i = \sum_{ij} m_i \ddot{\vec{r}}_i\, \partial r.$

We know that small q j and capital Q j which are respectively the generalized coordinates and the generalized force, they do not have the dimensions of neither length nor say in

Newton and so on that is the dimension of force. But they are important for us very important for us because they take into account the constraint conditions and each of them are independent of each other.

That is, each of the q j are independent of each other, whereas each of those ri that you talked about earlier, they are not independent. They are constrained by certain conditions. And these conditions are some of these things we have said that the internal conditions These internally a rigid body will have the distance between the particles would remain constant as the motion progresses. And now we are trying to get the dynamics in terms of these constraints taken into account condition, okay. So, we write this down equal to, so this is equal to sum over I. And we have a d dt of m i r i dot delta r i del q j you can see that this is fine and then you have to take out.

mi r i dot d dt of del r i del q j, okay. So, let us call this as equation number 6 and the last term that is this term can be written as. So, the last term of 6 can be written as d dt of del r i del q j is equal to del r i dot del q j, this is equal to k del 2 r i dot del q j. del q j del q k qk dot plus del 2 ri del qj del t okay. So, this is what it comes where we have you know change the order of the differentiation okay and this is nothing but equal to del vi del qj, okay.

$$\sum m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) - \underbrace{m_i \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j} \right)} \right] \quad (6).$$

Last term of (6).

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \frac{\partial \dot{\vec{r}}_i}{\partial q_j} = \sum_k \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_k} \dot{q}_k + \frac{\partial^2 r_i}{\partial q_j \partial t}.$$

$$= \frac{\partial \vec{v}_i}{\partial q_j}$$

From the velocity expression, that is the first term of Eq. 6.

$$\frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}$$

So, that is del v i del q j because r i dot is nothing but v i. So, that from the velocity expression, so that is in the first term of 6, okay. So, then we have a del v i del q j dot and this is equal to a del r i del q j. So, 6 becomes equation 6 becomes sum over i m i r i double dot this is double dot del r i del q j this is equal to sum over i d dt of m i v i dot del v i del q j dot minus m i v i dot del v i del q j and this let us call this as equation 7.

If you do it carefully you will get it. There is just an algebra that we have done and so the second term the full second term in equation 4 which is including the two terms that you see there that is written as sum over j d dt of del q j dot and sum over i half m i v i square and minus del del q j and again this sum over i half m i v i square basically this is a square and then you have to take this as v square or we can simply write it because it is a square so v i dot v i so it is v i square and minus q j that is equal to 0, okay. and dotted with delta q j that is equal to 0. Now this is what this equation 4 gives finally by taking into account all these terms. Now you see that you have this bracket the second bracket that you have can be put to be equal to 0 because these delta q j's are all independent of each other. So the coefficient can be put to be 0 and you can see that the kinetic energy is what is given by sum over i m i v i square. So, this equation is by putting the coefficient to be equal to 0.

So, this is d dt of del T del q j minus del T del q j There is a dot there, del q j and this is equal to the generalized force and this equation will do a bit of more simplification. So, you have this generalized force appearing there and we have this kinetic energy. So, it is a del T del q j dot. And del T del q j and that is minus del T del q j will be equal to q j. So there is a d dt of del T del q j dot minus del T del q j equal to q j is the equation that we get.

Eq.(6) becomes

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} \right) - m_i \vec{v}_i \cdot \frac{\partial v_i}{\partial q_j} \right] \qquad (7)$$

2nd term in Eq(4)

$$\sum_j \underbrace{\left\{ \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right] - \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m_i v_i^2 \right) - Q_j \right\}}_{=0} \cdot \delta q_j = 0.$$

$$T = \sum_i \frac{1}{2} m_i v_i^2 .$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j} \longrightarrow \text{Generalized force.}$$

From equation 4, okay? And now, if the forces are obtained from the gradient, negative gradient of the scalar potential, which are, you know, this negative gradient, so we can write the generalized force as sum over, so this i, index is i and there is a f i and then there is a del r i del q j which is what we have said earlier, see the expression for this generalized force that is equal to f i del r i del q j. So, we write that and this gives equal to minus negative gradient of these V and del r i del q j and this is nothing but equal to minus del V del q j, okay. That is using the definition of gradient and so there is a dot here. So, this is minus del V del q j. So, this equation becomes d dt of del T del q j dot. So, remember this dot or rather take a note of this dot that we write here. So, these are the derivatives of the generalized coordinates with respect to time.

So, this is T minus V del q i that is equal to 0. Now, we have taken this generalized force into account. And if you define the Lagrangian to be equal to T minus V, so this is called as L, and L is called the Lagrangian. Then we have d dt, and we can write down, instead of T, we can write down L, which is del del q j dot. Assuming that the potential that you see here is often, or rather most generally, I mean, usually they are not a function of q j dot. So, instead of T, we introduce T minus V so that the first and the second term have the same numerator. So, it is del T. T minus V del q i is equal to 0. So, this tells you that it is d dt of del L del q j dot minus del L del q j. So, there is this j here and there is a j here, okay, and del L del q j that is equal to 0, okay.

$$\vec{f}_i = -\vec{\nabla}_i V$$

$$Q_j = \sum_i \vec{f}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \nabla_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0 .$$

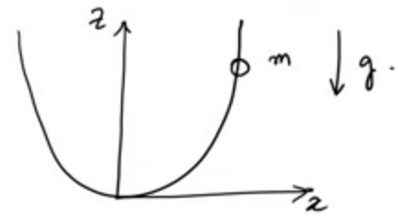
$$L = T - V \qquad L: \text{Lagrangian.}$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}(T-V)\right) - \frac{\partial (T-V)}{\partial q_j} = 0$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0} \rightarrow \text{Lagrange's equation of motion.}$$

So, this is Lagrange's equation of motion, which we have derived from the first principle, derived from the principle of virtual work, okay, and D'Alembert's principle. So, L is known as the Lagrangian, which is kinetic energy minus potential energy, and let us see how to apply them in order to learn more about it, do some examples. So, let me do a simple example where you have example 1. So, you have a parabola, and this parabola lies in the, say, for example, in the x z plane, okay. And there is a bead that's like sliding along the parabola, okay. So, that's your z and x, and the bead is of mass m, and it slides, okay, and the gravity, of course, acts like this. Okay, so this parabola has an equation which is given by z is equal to some, say, a x square, okay, a is a constant. And what is the constraint here? Of course, there are two degrees of freedom as it looks like, but which are x and z. Of course, this is at y equal to zero.

Example#1



Parabola $z = ax^2$ a : Constant.

DOF $= 1$. Generalized Coordinate x.

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{z}^2\right) \qquad z = ax^2 \Rightarrow \dot{z} = 2ax\dot{x}$$

$$= \frac{1}{2} m \left(\dot{x}^2 + 4a^2x^2\dot{x}^2\right) = \frac{1}{2} m\dot{x}^2\left(1 + 4a^2x^2\right)$$

$$V = -mgz$$

So, it's in the y equal to zero plane, say, for example, and there are two degrees of freedom or two coordinates that should be needed to describe it, which are x and z. But then x and z are related. So, the degree of freedom, which we call it as DOF, which is equal to one. And so, we choose the generalized coordinate to be equal to x, okay. So, T is equal to half m x dot square plus z dot square, but because z is equal to a x square, so z dot is equal to 2 a x x dot, okay. So, this becomes equal to half m x dot square plus, you know, 4 a square, x square, x dot square, that is, so you can take the x dot square out, and one can write it as 1 plus 4 a square, x square, and a half m x dot square. So, that is the kinetic energy, and then potential energy is easy, which is minus m g z, which it is, you know, measured such that it has a negative sign.

So, our L, which is T minus V, that is equal to, so the Lagrangian is equal to T minus V, which is equal to half m x dot square 1 plus 4 a square x square and minus m g z, which is nothing but a x square, okay. So, that we have simply L as a function of x. So, now we want to calculate the equation of motion. So, how do we calculate the equation of motion? We have to calculate del L del q j dot. Now, q j is just there is just one coordinate, which is x. So, it is del L del x dot, which is equal to you take the derivative with respect to x. So, x dot square becomes 2 x dot, and this 2 will cancel and then it becomes m x dot and 1 plus 4 a square x square, okay. So that is one term and then we have a del L del x term where both these terms contribute x square and there is another x square there so we have this as 4 m a square x x dot square minus 2 m a g x. This a is a constant that comes with the parabola, the constant of the parabola z equal to a x square.

So, what is the equation of motion? So, the equation of motion is d dt of del L del x dot minus del L del x equals 0. So, you have these d/dt of that. So, this d dt of this is given by, you know, you have to do this: take a derivative with respect to time of the top expression, which is del L del x dot, which gives that m into 1 plus 4a square x square and there is an x double dot plus 8ma square x x dot square minus 4ma square x x dot square plus 2 m a g x equals 0. So, if you sort of arrange them, then it becomes $\ddot{x}$ plus 4a square x square plus 4a square x x dot square plus 2agx equals 0, where we have cancelled m because m is not equal to 0. So, this is the equation of motion. So, we can now solve it using standard methods of solving differential equations. The main important emphasis is getting the equation of motion, which ultimately, if you do not do this Lagrangian mechanics and do Newtonian mechanics, you would get the same equation of motion.

Lagrangian $L = T - V$
$$= \frac{1}{2} m\dot{x}^2 (1 + 4a^2x^2) - mgax^2$$

Calculate EOM

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}(1 + 4a^2x^2).$$

$$\frac{\partial L}{\partial x} = 4ma^2x\dot{x}^2 - 2magx$$

EOM $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0.$

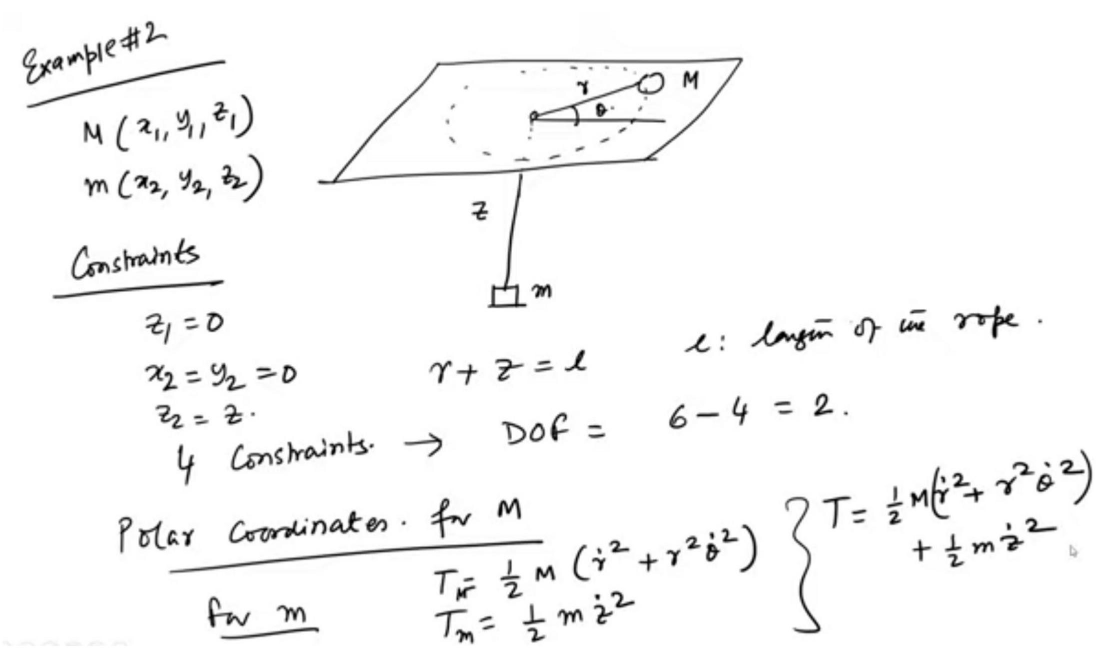
$$m(1 + 4a^2x^2)\ddot{x} + 8ma^2x\dot{x}^2 - 4ma^2x\dot{x}^2 + 2magx = 0.$$

$$\boxed{\ddot{x}(1 + 4a^2x^2) + 4a^2x\dot{x}^2 + 2agx = 0}$$ EOM.

Let me do another problem. This is again the problem that we have done on the previous day, which involves, I mean, involving Newton's laws and wrote down this equation of motion and so on. So, that same problem: there is a table, and there is a hole in the table, and there is a mass. Let us call this mass as capital M, and say there is a mass there, which let us call it as small m. We might have called them by different names, M_1 and M_2, but let me just call them as this capital M and small m. And this, let me draw it a little better. This is angle θ, and say this is r, and you know that this is constrained to move in a circle on the plane, and this small m, the mass that's hanging, will either move up or down, and so there are two masses instead of one mass as in the previous problem. Okay, so let's call this; let's say that this is so m has these coordinates should be x1, y1, and z1. Of course, some of them can be equal to 0 because this is on a plane, and small m can have x2, y2, and z2. So, what are the constraints or constraint conditions? Constraints are z1 equals 0 because it is on a plane.

There are also constraints on small m, which tells you that x2 equals y2, that is equal to 0, because this is only it sort of moves only in the z-direction or in the z-plane. So, out of 6 coordinates, we have been able to come down to 3 because of these constraint equations. And there's one more constraint equation, which tells you that this is a rope. So, once again, I remind you of the problem. So, there's a small hole in a table. It's a horizontal table, and there's a mass that is capital M, which is tied to a rope, and the rope passes through the hole and which supports another mass small m. And this capital M is constrained to move in a circle, and small m is constrained to move in the z-direction. So, we also have another constraint, which is r plus z equals l, l being the length of the rope, which does not change, which means that it is kind of a rigid rope, that is, it does not change over this course of motion. So, we have four constraints. So, the DOF becomes, you know, 6 minus 4, it equals 2. So, we need two coordinates. And let us see what two coordinates we need. So, that is the DOF. And z2, of course, we call it by z. That is the coordinate for that.

And let us use polar coordinates for capital M. For the kinetic energy, because it is moving in a circle, and we want to exploit the symmetry of this motion, because it moves on the circumference of a circle, we use polar coordinates. So, polar coordinates for M that give you the kinetic energy equals half M. We have told this earlier that is r square r dot square plus r square theta sqaure, that is the velocity squared, and also for m, small m, T equals, so this is say T M and this is T m, which is half m z dot sqaure. Okay. So, if you put together, then T equals half m r dot sqaure plus r square theta square) with the bracket here plus half m z dot sqaure. So, that is the total kinetic energy of the system, and the potential energy V. This equals, so if you sort of put it from the, or rather measure it from the table, so this is minus m g z, but z equals l minus r. Okay.



z is equal to l minus r, so this tells you that the potential energy is equal to minus m g. l minus r, okay, and not only that, we have a z dot which is equal to minus r dot and that gives the kinetic energy equal to half m, sorry, this is a capital M, half M r dot square plus r square theta dot square and we have a half m and we have this as the small m r dot square as well because z dot is equal to minus r dot, l being the constant. So that's this. So the Lagrangian is writing it once more. It's a r dot square plus r square theta dot square plus half small m r dot square plus mg, which is T minus V is l minus r. And that's the Lagrangian of the system. Okay, so what do we need to find? We need to find a few things such as del l del r dot r, and del L del r, del L del theta dot and del L del theta. These are the four quantities that we want to calculate.

So, this is equal to M plus m r dot and del L del r is simple is just this. This is just one term which is half capital M r square theta dot square. And of course, there is another one which is minus M g r. So, this is equal to M r theta dot square minus m g. So, these are the two things that are important and we have a del L del theta dot. This is M r square theta dot and del L del theta equal to 0, which tells you that because there is no term involving theta. So, that tells you that theta is a cyclic coordinate, and for a cyclic coordinate, the corresponding momentum is conserved, so p theta is a conserved quantity, okay.

$$V = -mgz$$
$$z = l - r$$
$$V = -mg(l-r).$$
$$\dot{z} = -\dot{r}$$
$$T = \frac{1}{2} M(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2} m\dot{r}^2$$
$$L = \frac{1}{2} M(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2} m\dot{r}^2 + mg(l-r).$$
$$\frac{\partial L}{\partial \dot{r}} = (M+m)\dot{r} \; ; \quad \frac{\partial L}{\partial r} = M r\dot{\theta}^2 - mg.$$
$$\frac{\partial L}{\partial \dot{\theta}} = M r^2\dot{\theta} \; ; \quad \frac{\partial L}{\partial \theta} = 0.$$

θ : a Cyclic Coordinate.

p_θ : a Conserved Quantity.

So, p theta is conserved which means that p theta which is nothing but del L del theta. This is equal to M r square theta dot which is equal to a constant, okay. So, this is so p theta is conserved so this p theta so p theta is conserved this is a constant so it is a constant of motion or it is conserved and so the net angular momentum is equal to r cross M v. So, this is equal to m r r cap that is r vector crossed with the v which is r dot r cap plus r theta dot theta cap. So, there is only one this capital M has only angular momentum, the other one small m has a linear momentum and then you have a r cap cross r cap that is equal to 0 and then you have a r cap cross theta cap which is equal to k cap.

So, this gives you a m r square theta dot k cap this is what we said. So, this conserved momentum is nothing but the angular momentum and this is what we were saying that the generalized momentum is p theta which actually is the angular momentum of the system. So, the Lagrange's equation of motion which is what we want to find, is nothing but M plus m r double dot minus M r theta dot square plus m g. This is equal to 0. So, that tells us that r double dot equal to M r theta dot square and mg divided by M plus m. Now, you see this is the radial acceleration for this or rather the r double dot, which is concerning this capital M. And of course, there is, you know, it is also related to the kinetic energy of the, I mean, r dot is related to the kinetic energy. So, r double dot is the acceleration that you have.

Now, this of course has two components: r double dot and r theta dot square, and so on. So, this is r double dot and r square theta dot equals a constant, which is what we have seen earlier. Now, if, for example, at t equals 0, that is, the whole thing is in a static condition at t equals 0. So, where small r equals capital R, so going back to the figure, this small r is equal to capital R at t equals 0. And the angular speed is given as omega equals omega 0, so that is the initial condition, which we will now use in order to fix or rather write down the equation of motion. So, this equals omega equals omega 0. So, that gives you theta dot equals R square omega 0 divided by r square.

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = M r^2 \dot\theta = \text{const} \qquad (p_\theta \text{ is Conserved}).$$

$$\begin{aligned} \text{Net angular momentum} &= \vec r \times M \vec v \\ &= M(r\hat r \times (\dot r \hat r + r\dot\theta \hat\theta)) \\ &= M r^2 \dot\theta\, \hat k \end{aligned}$$

Lagranges' EOM

$$(M+m)\ddot r - M r \dot\theta^2 + mg = 0 \quad \Rightarrow \quad \ddot r = \frac{M r \dot\theta^2 - mg}{(M+m)}$$

$$r^2 \dot\theta = \text{Constant}.$$

If at $t=0$, $r = R$, $\omega = \omega_0$.. $\boxed{\dot\theta = \frac{R^2 \omega_0}{r^2}}$

So, this constant is calculated as r square omega naught divided by r square. So, that is the. So, this is R double dot we get and R theta dot we get, and from the relations that we have written for the radial and the tangential acceleration, now we can find everything that we have obtained earlier. Okay. So, this is I purposefully did this because this problem was solved using Newton's laws, and now you have solved it using Lagrange's equation of motion. So, establishing a one-to-one correspondence between the two would be easy. We have simply written down the Lagrangian, first identified the degree of freedom, and then used the constraint conditions to write down the kinetic energy and the potential energies, and then wrote down the Lagrangian. There are two generalized coordinates here, two degrees of freedom which are written as this r and theta, and of course, the z was converted, which is the coordinate for small m, was converted into that into r, and then we have written down the equation of motion for both and found out that there is a conserved quantity or there is a constant of motion, which is nothing but the angular momentum, and then we have calculated r double dot and theta dot. So, that is the power of all these Lagrange's equations.

We have made this statement right at the beginning that they can be used to derive Newton's laws of motion and many other things. So, these principles of least action, virtual work, generalized coordinates, and constraint conditions, and then arriving at the equation of motion, applying the equation of motion to this particular two particular cases, one of them is very simple, having just one degree of freedom, which is a bead sliding on a parabola. And then we wrote down the equation of motion, and then we have considered the problem that we have already done, that is, the two masses, one moves on the horizontal plane on a table, and the other is connected to it by a rope and moves in a vertical plane, and finding out their equations of motion. So, we will stop here with Lagrangian mechanics and we will carry on with the rest of classical mechanics that has been mandated for the course.