ELEMENTS OF MODERN PHYSICS

Prof. Saurabh Basu
Department of Physics
IIT Guwahati

Lec 18: Laplaces Equation, Magnetostatics, Maxwell's Equations

Welcome to this last module of electromagnetic theory. We will be discussing a solution of Laplace's equation as has been told earlier. So, in particular we will talk about dielectric sphere in an electric field and this you know would be very similar to a magnetized sphere in a magnetic field which will not do, but the solutions etcetera are very similar for the scalar potential, the magnetic scalar potential. Here we are talking about the electrostatic scalar potential which we just call it as electrostatic potential. Then, we'll give a brief introduction to magnetostatics.

And we'll talk about, you know, magnetic lines of force, B and H fields. And we'll talk about the Lorentz force. And then we'll talk about Ampere's law and then talk about magnetic materials, in particular magnetization and susceptibility. And finally, we'll be talking about time varying electric and magnetic fields. And in that connection, we will talk about electromagnetic induction, Faraday's law and in particular, there is a sign that is very important, which is called as a Lenz's law.

And finally, we will talk about Maxwell's equations. So, let us get ahead with this Laplace's equation. So, we will do the solution of the Laplace's equation. We have formally seen the solution. But we'll see it for a given case.

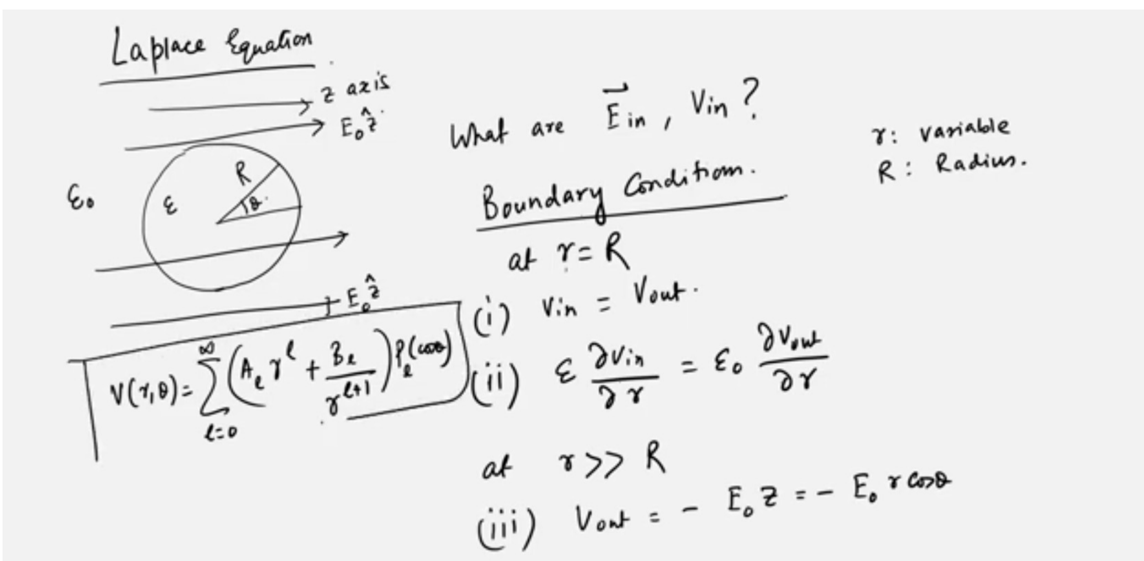
So we have a dielectric sphere. OK. Of, say, for example, radius R. And this is the Z axis. And there's an electric field along the Z axis. So there's a E0 electric field along the Z axis and which actually pierces the material.

Let's call that as E0. And because it's in the z direction, so this is E0 z cap. and so on and say this is you know the angle theta that we have and we have this dielectric sphere of dielectric constant or this permittivity to be equal to epsilon and you can take it the outside to be vacuum and take it as epsilon 0 but it really doesn't matter you can take epsilon 1 and epsilon 2 to be the dielectric constants or the permittivities of inside and outside region. And this electric field outside is constant.

That's a uniform electric field. And what we need to find is that what are E internal electric field and internal potential. Internal means inside the sphere. So that's the question that we have to answer. And what are the boundary conditions?

The boundary conditions are. V, so at r equal to capital R, that is, this r is the, small r is the variable and capital R is the radius of the sphere, okay. So, when at small r equal to capital R, we have V in, that is inside the sphere. The potential electrostatic potential has to be continuous V in equal to V out. And the second one is about the fact that there is no charge density there.

So, epsilon del V in del R. This is equal to epsilon 0 del. epsilon 0 del V out del R, that is the gradient part, but since we are only talking about V, which only has variation with respect to the radial variable, this is the gradient term that is written. Also, at large distances away from the sphere, the V out this can be thought of as a third boundary condition. The V out is nothing but minus E0 Z, where Z is a variable, and we can express it as E0 R cos theta. So, minus E0 R cos theta, that is the value of the potential far away from the sphere.



The sphere will have no influence on very large distances away from the sphere, and it will simply be due to the external field that we have. And we have written down the solutions for this, and the solutions are V of R theta is equal to, we might have written it with M, but here we are writing it with L. So, L equal to 0 to infinity, and we have a AL RL plus a BL R to the power L plus 1 and a PL cos theta. That's the solution that we have written down earlier by solving Laplace's equation in spherical polar coordinates. Now, you see, in this equation, you have two terms, okay? Now, for V in, that is the potential inside, this term will only survive because this term will go to infinity.

The second term will go to infinity because we can come very close to r equal to 0, in which case this second term inside the bracket will blow up, which will make it an unacceptable solution. So, V in will be dominated by the first term, not dominated, it will be represented by the first term and V out on the other hand, if R goes to infinity, the first term will blow up. So, that is not going to be the acceptable solution and we will have to fall back on the second term. So, what I am saying is that V in of R theta is there is no phi variation.

This is equal to sum over L equal to 0 to infinity and we have A L r L P L cos theta And for V out, we have this term, which is L equal to 0 to infinity, B L divided by r L plus 1 P L cos theta. But then there is so a term which is minus E 0 r cos theta, because that's due to the external field, which is always going to stay. So this is this for r smaller than R. and well and this is equal to r greater than R.

So that's the solution that we have. So we have to equate this solution. So let me write it as inside and outside. So this we have already written. This is for inside and this for outside.

And what we are going to do is that we are going to match the solutions at small r equal to capital R. So V in small r equal to capital R dropping the theta dependence because theta is going to cancel out from both sides. it is equal to V out R equal to R. So, that tells us that L from 0 to infinity AL R to the power L PL cos theta, well you have to write that and minus E0 r cos theta. And there is also this l equal to 0 to infinity B l divided by capital R plus l plus 1 P l cosine theta. Now, you see, this is a sort of sum over l, which means that there could be, you know, all values of l possible. And this infinite, in fact, number of values of l are possible.

But you see, the right hand side contains a cosine theta in the first term of the right hand side, which earlier I had written it towards the end. Now, remember that P1 cos theta is equal to cos theta. So, that is the property of the or that is the Legendre polynomial for l equal to 1. Which means that since there is a cosine theta here and which is like a P1 cos theta and there is one term in the right hand side, which is like a P1 cos theta.

So, what we have is that only l equal to 1 will survive. So, we have A1. R, R to the power 1. So, that is your—and this is like a—so there would be an R here, not small r. So, this is like a minus E0 R and plus a B1 by R square. So, this is by dropping all other terms except l equal to 1.

$$V_{in}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \qquad (\text{inside})$$

$$V_{out}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) - E_0 r\cos\theta \,. \qquad (\text{outside}).$$

$$V_{in}(r=R) = V_{out}(r=R).$$

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = -E_0 R\cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta).$$

$$\boxed{P_1(\cos\theta) = \cos\theta}$$

$$A_1 R = -E_0 R + \frac{B_1}{R^2}. \qquad (l=1).$$

$$A_l R^l = \frac{B_l}{R^{l+1}} \qquad (l \neq 1)$$

And so, this is L equal to 1 would survive and then of course, you have also for the other ones then you have this A l R to the power l is equal to B l R to the power l plus 1 that will not include this term that is there. So, this is for l not equal to 1. So now the second condition, which is this epsilon del V in del R equal to epsilon zero del V out del R. That condition gives that I write it as epsilon R. Now, this epsilon R is nothing but epsilon by epsilon zero. So this epsilon R is equal to there is a sum over L equal to zero to infinity. l A l R to the power l minus 1.

So, this is the second boundary condition and PL of cos theta Now, this is a minus E0 cos theta because I am not changing anything. I mean, this R goes because I take a del del R. So, and then a plus of l plus 1 B l divided by R to the power l plus 2 and a P l cosine theta. So, that is the equation coming from the second boundary condition. Where, as we have said, we have used epsilon R equal to epsilon over epsilon 0 because on one side, that is outside, you will get an epsilon 0, and on the left-hand side, you will get an epsilon.

So, I have taken a ratio of that and written it as a relative permittivity. So, again, if you split it into L and L equal to 1 and L not equal to 1, then we have this epsilon R l A1. This is for l equal to 1. So, it is minus E_0 minus 2 B_1 by R cube. So, this is for L equal to 1 and epsilon 0 E_0.

l a l r to the power l minus 1 is equal to minus l plus 1 divided by r to the power l plus 2 and a b l. So, this is for l not equal to 1. So, if you call these as equation 1, this as equation 2, these two can be called as equation 3 and equation 4. And that sort of yields us that from equation 2 and 4, Al over Bl that is equal to 1 by R to the power L plus 2 because we need to calculate or we need to obtain the unknown coefficients in order to arrive at a unique solution, particular solution for the problem. And Al by Bl, this is equal to 1 over epsilon r and this is like minus l plus 1 divided by l, 1 divided by r to the power 2l plus 1. Now, these two equations, you can call them as equation 5 and this is equation 6.

They would be inconsistent if all your a l and b l are not equal to 0 for l not equal to 1. Okay, so you see that there is we got the ratio A l by B l and these A l by BL have two different forms and they do not seem to agree unless we put l equal to 1. So, we have the only possible or rather the only solution that is allowed here is l equal to 1. So, we are left with only l equal to 1 solutions allowed. Alright, so that tells us that we have only two coefficients that we need to determine and that is a1 is minus 3 divided by epsilon r plus 2 e0 and b1 is minus epsilon r minus 1 divided by epsilon r plus 2 into r cubed e0.

2nd boundary condition.

$$\epsilon_r \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = -E_0\cos\theta + \sum \frac{(l+1)B_l}{R^{l+2}} P_l(\cos\theta) \qquad \epsilon_r = \frac{\epsilon}{\epsilon_0}$$

$$\epsilon_r l A_1 = -E_0 - \frac{2B_1}{R^3} \quad (l=1). \qquad (3)$$

$$\epsilon_r l A_l R^{l-1} = -\frac{(l+1)B_l}{R^{l+2}} \quad (l\neq 1) \qquad (4)$$

From Eqs. (2) and (4).

$$\frac{A_l}{B_l} = \frac{1}{R^{l+2}} \quad (5), \qquad \frac{A_l}{B_l} = \frac{1}{\epsilon_r}\,\frac{-l+1}{l}\,\frac{1}{R^{2l+1}}. \quad (6).$$

$$A_l = B_l = 0 \text{ for } l\neq 1.$$

Only $l=1$ soln. allowed.

$$A_1 = -\frac{3}{\epsilon_r+2}E_0 \qquad (7)$$

$$B_1 = -\frac{\epsilon_r-1}{\epsilon_r+2}R^3 E_0. \qquad (8)$$

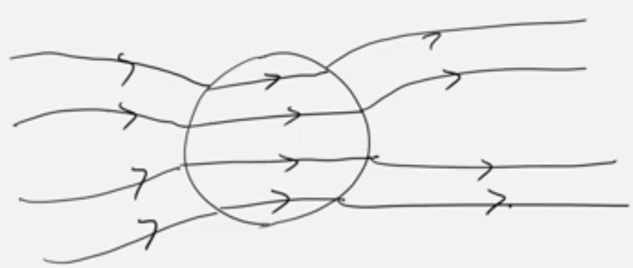
r cubed E0. So these are equations 7 and 8 and now we have obtained the full solution which is V in r theta that's equal to minus 3 E0 divided by epsilon r plus 2 r cosine theta for r to be you know less than capital R. So this tells you that this is actually less than E0, which means that the potential inside is less than these. I mean, is reduced from these this external contribution, which is E0 r cos theta. So this is E0 r cos theta and which is reduced by this factor 3 divided by epsilon r.

plus 2 and this is also in a direction which is the potential has a negative sign okay not direction but it has a negative sign similarly we have a v out r theta that's equal to minus e 0 r cosine theta plus epsilon r minus 1 epsilon r plus 2 and we have this r cube by r square e 0 cos theta, okay? Now, this term will go to 0 for r to be going to infinity and we will be left with only this term, which is what is expected, okay? And how do we calculate this, okay? The E in and E out, so E in is equal to, that is inside is equal to minus del V in del r, r cap and so on.

$$V_{in}(r,\theta) = -\frac{3E_0}{\varepsilon_r+2}\, r\cos\theta \cdot \qquad < (E_0\, r\cos\theta \cdot).$$

$$V_{out}(r,\theta) = -\underbrace{E_0\, r\cos\theta} + \underbrace{\left(\frac{\varepsilon_r-1}{\varepsilon_r+2}\right)\frac{R^3}{r^2}E_0\cos\theta \cdot}_{\to 0 \text{ for } r\to\infty}$$

$$\vec{E}_{in} = -\frac{\partial V_{in}}{\partial r}\hat{r} \cdot = \frac{3}{\varepsilon_r+2}\vec{E}_0 \qquad |\vec{E}_{in}| < |\vec{E}_0|.$$



And this gives you that this is equal to 3 epsilon R plus 2. E 0 vector. So, this is like E 0 r cos theta. So, what happens is that so, in this sphere you have this and if you can you can calculate E out by taking a derivative with respect to r for this equation. So, at very large distances of course, that is the external field and then it looks like this and this is for the outside region. So, it only bends a little in the vicinity of the sphere and otherwise it becomes you know it takes this value. Now, for inside it is slightly is slightly reduced, so that is the field lines, the electric field lines for this particular case and it is reduced because epsilon r is greater than, so this is less than E0, I mean mod of E0 if you want, so that is saying that this E in magnitude is less than E0 magnitude. So, that is the solution of this in the spherical polar coordinate system.

So, this completes one of our electrostatics parts, and let us now get into magnetostatics. And we will do that very quickly because a lot of things are common, and they look exactly like electrostatics. And we will start with, you know, magnetic field lines, okay. And so, this magnetic field lines, you know, they, if you take a bar magnet with, you know, a north pole and the south pole showed like this. Then, the magnetic field lines will go like this from the north pole to the south pole in the vicinity, and then, of course, they will diverge like this and diverge like this at, you know, large distances, okay.

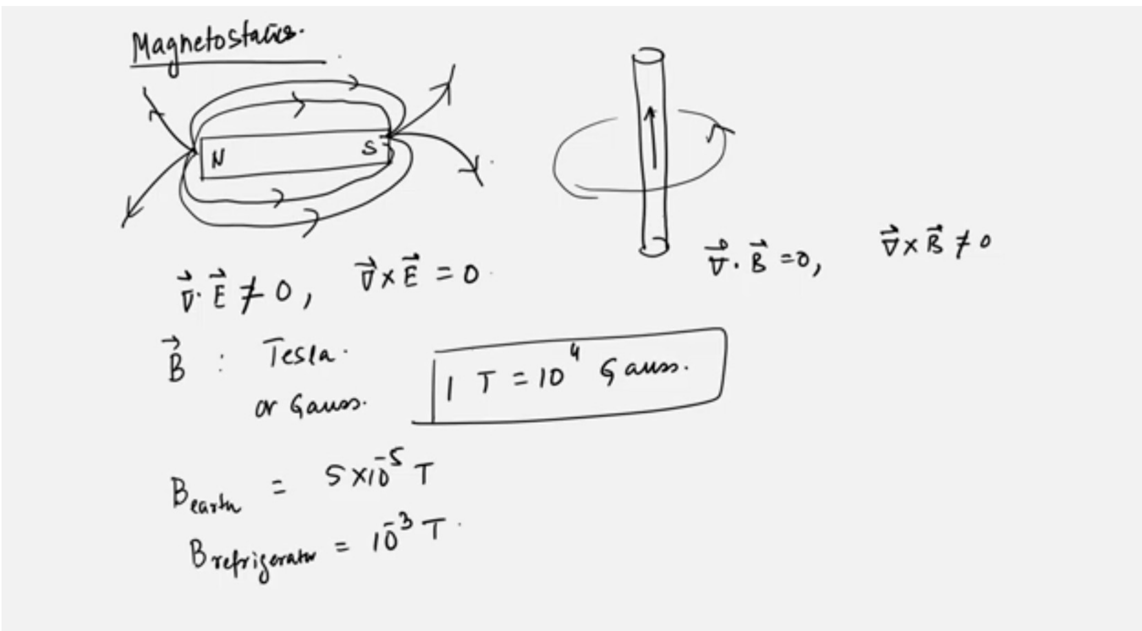
So, that is there are of course lot of field lines that are there and each one will have these originating from the North Pole and the South Pole. And if you talk about a current-carrying wire, which we are all familiar with, and say it is carrying a current in this direction, then, you know, the magnetic field will curl around like this. So, this is the direction of the magnetic field. Which is really the theta-cap direction. So if you notice my hand, so this is the direction.

So, this is a current-carrying wire. So, this pen, and the magnetic field is curling around. So it goes in the in the theta direction. And it's basically given by this right hand rule where you can the thumb actually points in the direction of the The current and the magnetic field is the rest of the four fingers that will point in the direction of this magnetic field.

So, that is the direction of magnetic field for static poles. And they are very distinct from the electric field because the electric field is characterized by divergence of E not equal to 0. and the curl of E equal to 0. We are of course talking about static case, but on the other hand for the magnetic case, we have divergence of B equal to 0 and curl of B is not equal to 0 and it is equal to what we will see in just a while. So, it has really properties very distinct from the electric field, but the important thing is that they both are valid vectors because we know what are the divergences and the curls, basically the divergence and curl for each of these fields.

So, it is, you know, the magnetic field is usually expressed by this B. And it's also called as a magnetic induction. And there's also another quantity called as H, which is often called as a magnetic field or the magnetic field intensity. So, there is a bit of notations that you would see and most of the time this B is called as a magnetic field or we loosely call it as a magnetic field, but strictly that is the name B is used for magnetic induction vector and H is used for magnetic field or the magnetic field intensity. The unit is Tesla.

So, that is the SI unit, and it is also expressed in Gauss, and the relationship between Tesla and Gauss is that 1 Tesla is equal to 10 to the power of 4 Gauss. And if you want to know that what kind of magnetic fields we have around Earth is that the Earth has a magnetic field. So B of Earth, this is about 5 into 10 to the power minus 5 Tesla. And the home refrigerator that we use for kitchen that has a B that is, you know, nearly 10 to the power minus 3 Tesla. So that is the kind of magnetic field that we have.



So Tesla is a unit where things look small. But if you express it in Gauss, they do not look small. But that is, you know, the conversion between them. And now, if we are talking about current and the magnetic field associated with it, how do we know that there is a magnetic field associated with the current? If you keep an ammeter in the vicinity of a current-carrying wire, it will show deflections.

And this deflection shows that there is a magnetic field. So if you switch off the current, the deflection will go to zero, which means that in the vicinity of a wire or maybe inside a solenoid. And what a solenoid is that it's like a cylindrical object which has several turning of wires around it. And these wires carry current. And there is a constant magnetic field, very nearly constant magnetic field, excepting probably at the edges if it's a small solenoid.

But let us ignore that. So, there is a very uniform magnetic field that you can get inside a solenoid and that is available in any lab. Let us now talk about the charge, a moving charge field. if it is you know experiences a magnetic field or a region it enters into a region that there is a magnetic field. So, suppose there is a magnetic field lines and by these crosses what we mean is that if you look at an arrow from behind it will look like that there is a cross there.

So, which means that this magnetic field is inside the screen that you are seeing. So, a charged particle with a velocity v and maybe a q, charged q enters this, it has also a mass m and when it enters, it will get deflected and it will follow a circular trajectory and this magnetic force which is given by or it is called as a Lorentz force, this is equal to q into v cross B. where v is the velocity of this and B is the direction of the magnetic field. So, v cross B is a vector that is perpendicular to both v and B and takes place in the direction if you place this part of the hand along the first vector that is v and curl it towards the second vector, the thumb will point in the direction of the force that is acting on the charged particle. So, this will give a circular trajectory and this circular trajectory is maintained because of this magnetic field.

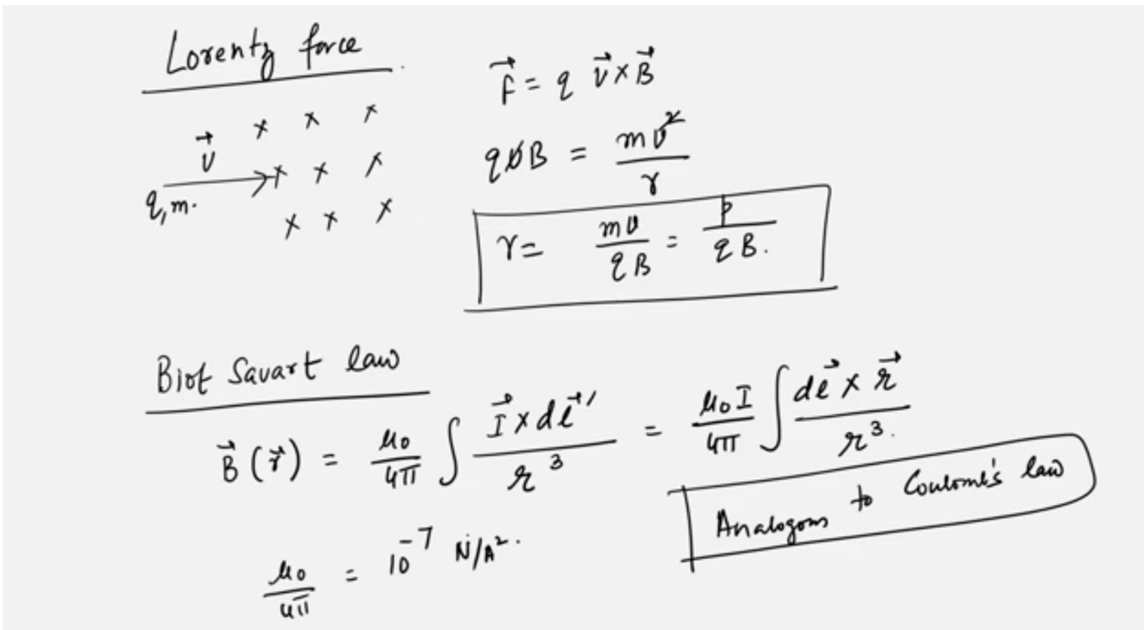
So, if you write it in scalar form or assume that v and B are perpendicular, then we have this q v B and that gives rise to the centripetal force. This is required for you know, executing the circular motion. So, this circular motion will have a radius which is given by 1v will cancel from here and we have this as m v over q B and we have this as a momentum over q B and so on. So, one can actually measure this radius

the radius of the trajectory and which is given nothing but the momentum of the particle divided by its charge multiplied by the magnetic field. So, that is quite well known. I mean, this is taught at the school level. And so, the only thing that is important here is to, you know, acknowledge that this is a direction is V cross B. So, always V and B are not perpendicular to each other. There could be a component and so on.

So, you have to Really treat it as a vectorial equation or a vector equation and then solve for this m x double dot m y double dot and so on so forth. All right. So let us now write down the Biot-Savart law. And what is Biot-Savart law?

Biot-Savart law gives you the magnetic field due to a current distribution. Just like Coulomb's law gives you this electric field due to a charge distribution or it gives you for the electric field due to point charges. Here, of course, we talk about charge distribution as a general case and it will give you the magnetic field for this. OK. And again, we'll use those notations that we have done earlier.

So this is equal to mu 0 over 4 pi. This is I cross dl. DL prime divided by this R cube or it could be written as a mu 0 over 4 pi mu 0 is known as the permeability of free space and this mu 0 divided by 4 pi has a value we will just tell you that in a while. If the current is constant we can write it like this and we can write this as the d l cross R. r divided by the r cube, and this is analogous to Coulomb's law in electrostatics.



In the sense that, you know, Coulomb's law gives electric field and Biot-Savart law gives magnetic field due to this charge distribution. So, this mu 0 over 4 pi is nothing but equal to 10 to the power minus 7 Newton per ampere square and so on. That's called as a permeability of free space. And one can actually calculate these for a given distribution. Either it's a straight wire or maybe there is a wire that is circular in shape and you want to calculate the current at the middle of the wire that is at the center.

at a point which is at the center of the wire and so on and various other you know kind of things that one can calculate just like we have done it for large number of cases or other we probably have not done it, but then we have given a sort of indication that it can be done for a number of cases which are given as examples. So, let me draw a similar triangle that we have done earlier and for the electrostatics and this triangle now would correspond to three quantities which is J which is the current density and then there is A which is the vector potential I will define what vector potential is and then it is B. So, the vector potential is defined as B is equal to curl A and this is true because if divergence of a vector is equal to 0 in particular for this case the magnetic field is divergence less. Then, this vector B can be written as a curl of a potential, which is now no longer a scalar potential, but is called as a vector potential, because divergence of curl is equal to 0. So, B can always be written as the curl of a vector potential and this is the vector potential that we see.

We will talk a little more about it and if you want like if you know J and you want to get A, then you can use this integral relation. So, A is equal to mu 0 over 4 pi and J and divided by R and dV or something that is a volume. We are talking about a volume current distribution. And so, if you know J, you can get A or if you know A, you can take a Laplacian of that and can get a mu 0 J. Now, all these things are completely analogous to these quantities, some of the quantities that we have seen in electrostatics.

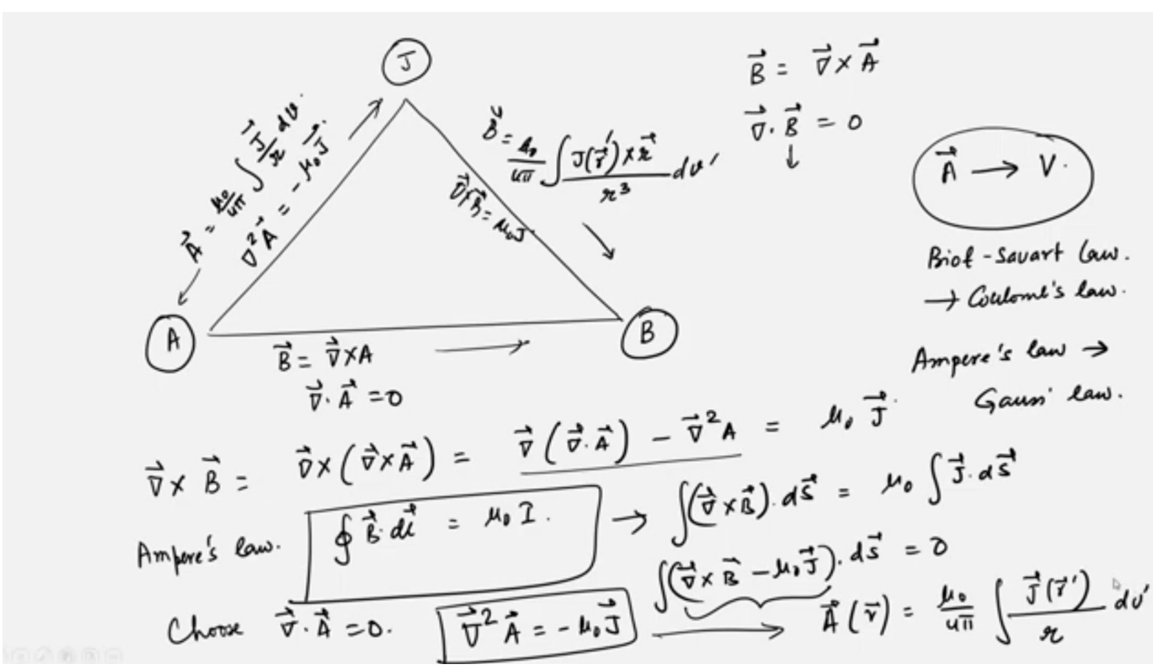
So, A is the vector potential is analogous to the scalar potential, which we have written by V. And this equation that you see here is that this Laplacian of A equal to minus mu 0 J, that is analogous to the Laplace's equation that we have seen there. And if you know, say, for example, A, you can get B to be equal to curl of A, which is what we have just said. OK, so you can get this if you know A, then you can get B. And this subject to the, you know, the condition that a divergence of these thing is called as a gauge condition. And we can always choose a gauge condition such that this divergence of A equal to 0. And similarly, if you know J, then you can get a B by, you know, B is equal to this mu 0 by 4 pi and this J by J r.

cross R. So, this J R prime cross R divided by R cube and dV and so on or dV prime. So, this if you know J you can get B and if you know B then you can get J by using this curl of B is equal to mu 0 J. And of course, divergence of B is equal to 0. So, it is a similar triangle that we have seen earlier. And so, we have this, as I said that these are the analogous quantities.

So, we have also the Biosavart law to be, this is analogous to Coulomb's law. then we have these will just see in a while that Ampere's law is analogous to the Gauss's law of electrostatics. and so on. So, there are these exact similarities that make things much easier and let us say you know convince ourselves about some of these things that we have just seen. So, we have curl of B that is equal to if you want to write it in terms of so, we are just taking curl of B because we know that the curl of B is not equal to 0 and what it is

We need to see which is what we have written here also. So curl of B means curl of curl of A. And curl of curl of A can be written as gradient of divergence of A minus a Laplacian of A. And this is equal to mu 0 J. And so this actually comes from this Ampere's law. So we should also write down Ampere's law. So, it tells you that the B dot D L is equal to a mu 0 I and so this you know I mean this I is magnitude.

So, this is the Ampere's law and then if we use Stokes theorem which will make it more curl of B dS and I can be converted as J dS and so on. So, that is a surface current and so this you can take this curl of B minus mu 0 J. So, let me just do it. So, this is a curl of B dot ds this is equal to a mu 0 j dot ds because this ds is arbitrary one can take it inside and so curl of B minus mu 0 j dotted with some arbitrary ds is equal to 0, which cannot be satisfied unless you have this bracket equal to 0. So, the curl of B equal to mu 0 j and that is what we have written. Now, we wanted to express it in terms of the vector potential and the vector potential gave these two terms. But what you can do is that you can choose without any loss of generality, we can choose divergence of A equal to 0 which means that gives you this Laplace's equation for the vector potential which is minus mu 0 j. So, this is exactly same as the scalar equation that is Laplace's equation for the scalar potential which is Laplacian of this V equal to some minus rho over epsilon 0 and this is minus mu 0 j.



So, this Laplace n of a equals to minus mu 0 j, and the solution of that is precisely what we have written: the a of r is equal to mu 0 over 4 pi j of r prime. Divided by R and dV prime, and this is what we have written in the triangle. So, all these relations that we have written are consistent and let us now look at magnetic materials and these magnetic materials are analogous to the dielectric material. So, let us talk about the magnetic materials. And these magnetic materials, just like the D vector and the P vector and so on —if I remind you of the dielectric—where we have an electric field, a D vector, which we called the displacement vector, and a P vector, which we called the polarization vector.

Here also, we will have B vector, H vector and M vector. So, B is called as the magnetic induction, H is a magnetic field or the magnetic field intensity and M is the magnetization. So, we write down for linear magnetic material, the magnetization of that is written as M is equal to chi M into H, M is the magnetization. And Chi M is called as a magnetic susceptibility. It is often written with just a Chi, but because we have talked about electric susceptibility with a Chi E, we are writing it with a Chi M. So, this is a magnetic susceptibility and H is the magnetic field intensity.

So, we are no longer talking about vacuum, we are talking about magnetic materials which is represented the magnetization is represented by this and where does this magnetization come and if you define the magnetic dipole density then this M equal to N into mu say and where mu is called as the magnetic dipole density. So, we have again just like this potential term, the scalar potential we have expressed it in terms of the bound and the free charge densities. We can express the magnetic vector potential in terms of the bound surface current and bound volume current densities. that can be seen easily we can we ignore the derivation of that but let me write this as a is equal to mu 0 over 4 pi and And so this is J b r prime divided by r minus r prime dV prime.

You would have seen this similar expression with J replaced by rho, everything else. So, mu 0 by 4 pi was 1 by 4 pi epsilon 0. And plus a mu 0 by 4 pi. And then there's a closed integral, which is a Kb r prime. And r, this r minus r prime is nothing but these curly r. So let me write it as curly r here.

And there is curly r here. And then there's a ds prime. So that's absolutely analogous to what we have seen for the scalar potential, the electrostatic scalar potential. So, this Jb is called the bound current density, the bound volume current density.

And KB is the bound surface current density. And they are more familiarly called as JM for the magnetic part, and this is called as KM. The letter B was used to show that these are bound currents, so they are distinct from the free currents. And so these, this JM or the JB is obtained from the magnetization by taking a curl of the magnetization just like it was divergence of P for this the volume charge density and the KM is the M cross N.

Magnetic Materials

$\vec{M} = \chi_m \vec{H}$

$\vec{M}$: Magnetization.

χ_m : magnetic susceptibility.

$\vec{H}$: magnetic field intensity.

$\vec{M} = N \vec{\mu}$

$\vec{\mu}$: magnetic dipole density.

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}_b(\vec{r}')}{\mathscr{r}} dv' + \frac{\mu_0}{4\pi} \oint \frac{K_b(\vec{r}')}{\mathscr{r}} ds'.$$

$\vec{J}_b$: Bound volume current density $= \vec{J}_M$

$\vec{K}_b$: " Surface " " $= \vec{K}_M$

$\boxed{\vec{J}_M = \vec{\nabla} \times \vec{M}}$, $\boxed{\vec{K}_M = \vec{M} \times \hat{n}}$

So, just like the bound volume charge density and the bound surface charge densities were obtained from P, divergence of P and P dot n, here it is curl of m and m cross n. There was a minus sign for this bound volume charge density. Okay, so what are the relationships between, you know, these H and B and so on. So, if you write down this, the differential form of the Ampere's law, which as I said is nothing but analogous to the Gauss's law. So, this is equal to mu 0 j. So, in a medium what we can do is that we can write this as the curl of B is the J actually splits up into two part one is the bound part of the other is a free part.

So, it is a mu 0 J instead of writing it as free we just leave it as that or you can for more clarity you can write it as this and then the bound one which we have decided to write it as this. So, this is equal to mu 0 J free. plus mu 0 curl of M because that is the definition of this volume current. So, this curl of M and so I can combine the two sides and can write this curls as B minus mu 0 M. This is equal to a mu 0 J free. And this is nothing but we can write as curl of H is equal to mu 0 J free.

Okay. And Well, what you can do is that this is we are writing it in material. So, we can remove this 0 because it could be simply you know the permeability of that medium. So, we can write this down as curl of H is equal to mu J free and this is nothing but the Ampere's law in a magnetic medium.

just like we have Gauss's law in dielectric medium, which was just divergence of D equal to rho free. Here it is mu into, well, there will not be any mu actually. I mean, there is a mu there. So, we have defined H to be equal to B over mu 0 minus M. So, we can actually keep this mu 0 if you like.

And so, this becomes, there is no mu here. So, it becomes equal to J free because the mu will get absorbed because of this. H is defined as B by mu into M. So, this is Ampere's law in differential form. So that is the form of Ampere's law and it could be a little misleading and this Ampere's law because curl of H equal to J free gives you an idea that if one is given J free that is one knows. that was the free current density, then H can be obtained.

But that is not true because we will not be able to compute H without an ambiguity because the divergence of H is not equal to 0, unlike B. So, this is a tricky point that one needs to keep in mind that this Ampere's law is fine, but the Ampere's law, if you are given J free, you will not be able to calculate H just by taking or rather by integrating this equation because divergence of H is not equal to 0. So, for most magnetic materials, except very good ferromagnets, we have these linear relations that hold, which we have written earlier. It is chi M into H or H is equal to, you know, B over mu 0 minus M or this is equal to B over mu 0 minus chi M H. Okay, and so we have told what these values of mu 0 etc. are.

Now, so your B is equal to mu. into H which is equal to mu into H. And this mu is called as a magnetic permeability of the medium. And just some values of mu, mu is equal to some 0.99 something is very close to 1 of these mu 0, which is of that free space. This is for a paramagnet, say, for example, bismuth, which is a paramagnet.

So paramagnets of mu almost equal to 1. And mu equal to, you know, it is 1.00002 mu 0. This is for aluminum. And mu is very large, of the order of 10 to the power 5 for, you know, ferromagnets. But, of course, there is no guarantee that this linear relationship between B and H will hold for ferromagnets.

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}.$$
$$\vec{\nabla}\times\vec{B} = \mu\left(\vec{J}_{free} + \vec{J}_M\right) = \mu \vec{J}_{free} + \mu\, \vec{\nabla}\times\vec{M}$$
$$\vec{\nabla}\times\left(\vec{B} - \mu\, \vec{M}\right) = \mu\, \vec{J}_{free} \qquad \left(\vec{H} = \frac{\vec{B}}{\mu} - \vec{M}\right).$$
$$\boxed{\vec{\nabla}\times\vec{H} = J_{free}.} \longrightarrow \text{Ampere's law in differential form.}$$
$$\boxed{\vec{\nabla}\cdot\vec{H} \neq 0 \quad \text{unlike } B.!!}$$
$$\vec{M} = \chi_m \vec{H}.$$
$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} = \frac{\vec{B}}{\mu_0} - \chi_M \vec{H}.$$
$$\vec{B} = \mu_0(1+\chi_m)\vec{H} = \mu\vec{H}..$$

$\mu = 0.99\ \mu_0.$
Bismuth (paramagnetic)

$\mu = 1.00002\ \mu_0$
Aluminium

$\mu = 10^5\ \mu_0.$
for Ferromagnets.

All right, so again, one can solve similar equations, the Laplace's equations. One may not do it for the vector potential, but there is, you can also convert it into a magnetic scalar potential and solve this equation. Okay. All right, so we are, you know, closing in, but let us, you know, do one more quite important thing and which will bring us to the last part of this module and which is time varying electric and magnetic fields. All right.

So, we start by relaxing the condition that we are no longer going to talk about static charges and, you know, just current densities that are there. But we talk about electric fields and magnetic fields that are time-dependent. And the first work was started by Faraday, so stated in the form of Faraday's law. and who saw first time for the first time that the currents are placed in you know time varying. So, the behavior of currents basically placed in time varying magnetic fields and what was the observation?

The observation is that there is a current that is induced. So, current induced. In a circuit, basically, if a steady current flowing in an adjacent circuit is switched on or off, So what it means is that suppose you have one circuit here and then maybe another circuit here. This was carrying a current and this current was suddenly switched off and then there will be a transient current in the loop that is on the first loop here.

So, there was a current flowing in 2 and it is either, so there was probably no current flowing and then you suddenly switch on the current in loop 2, there will be also a current in loop 1 or there is a current already flowing in loop 2 and suddenly it is switched off. Or it can also happen that these adjacent circuit with this loop number 2 is pulled towards this loop number 1, then there will also be a transient current, okay. Or it can also happen that, you know, I mean, so these are different situations that I'm talking about that, say, for example, there is a, that's the most common example. So there is a solenoid and a solenoid, as we have said, that it is this kind of cylinder where there are several turns of wires there. And now you take a magnet and,

And there's a bar magnet. OK. And this bar magnet is it's put in or it's taken out. And then you'll see that there is a current that will flow. So it will show deflection.

So there was no current. There's no source of current; there is just a dead solenoid. That means it's not carrying any current. That means that it's not carrying any current. It's only there.

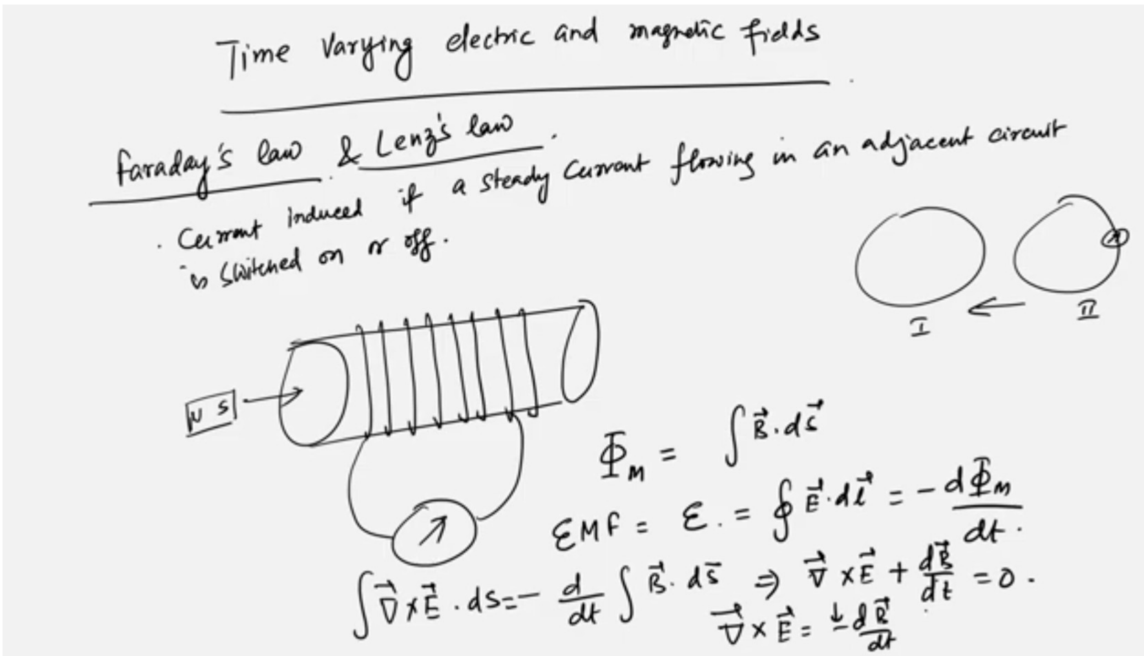
There are a number of turns there and so on. And then you put this bar magnet, you pull it inside and then pull it outside and you will see that there is a current that will flow in the circuit without any external battery being employed here. So, this shows that the time-varying fields are quite distinct as compared to the static fields and in fact in the static case the E and the B fields are completely dealing from each other they have no relationship between them whereas in the time varying case a space varying electric field will produce a time varying magnetic field and a space varying you know magnetic field will also induce a time varying electric field okay. So, one can actually introduce a magnetic flux which is the dot product of this B and some area that is there and there is a quantity called as the EMF.

So, will it is right written with an epsilon and it is called as electromotive force. And this electromotive force is defined as E0 into D, I mean E into dl. This is equal to minus d phi dt. That is what Faraday's law is. And E dot DL can be converted into a curl E dot ds.

integral of that and minus d dt equal to a minus of d dt of b dot ds. And now we can pull this thing inside. See, this integral is over the surface. That's a space coordinate. And the time is taken, you know, operates from the left.

but time and space are completely delinked in non-relativistic mechanics. So, we can do take a derivative and take this derivative of with respect to time to be inside and this will give us that the curl of E minus del B del t that is equal to 0. So, we will why is equal to 0 because we have written of course, d d t and del B del t we can write it as knowing that B is a function of both time and space. So, curl E is equal to minus del B del t.

And sorry, this is there will be a plus sign here. And so curl of E is nothing but a minus del V del T and or dV dt. And this minus sign is important and this minus sign is given by the Lenz's law. which tells you that the direction of the induced current would be such that it resists the change in the magnetic flux through the circuit. So, this is the Faraday's law and Lenz's law basically combined.



which gives the Faraday's law gives you this how this curl or the space varying electric field is linked with the time varying magnetic field. There is a minus sign which is given by the Lenz's law which simply says that there is a direction of this current that flows into the circuit and the current that we just talked about and it, the direction is such that it will resist any change in magnetic flux through the circuit. Alright, so we are closer to, you know, what we wanted to have and so let us collate all these information that we have gotten so far from these electrostatic magnetostatics and then these time varying fields that we see we can of course write down these equations which is a divergence of D this is equal to rho free that is called as the Gauss's law.

and curl of E is equal to minus del B del T okay. So, this is called as a Faraday's law as we just saw now along with the sign that is given by the Lenz's law and divergence of B equal to 0 that is an important equation. It has a meaning that So, this magnetic field lines cannot you know sort of they do not diverge or they do not start from somewhere and end somewhere and it has to start from a north pole and end in a south pole okay and this also means that there is no magnetic monopole being present. This is quite unlike that we have a single charge that is present.

There is no single pole that is present. Even if you break the magnet and keep breaking it until it becomes smaller and smaller, even at the molecular level, it will have a north pole and a south pole, and the divergence of B will always be equal to 0. And the curl of H is equal to J free. You can write this down as, you know, the curl of B is also equal to $\mu_0 J$, but they mean the same thing. So, this is, you know, probably this has no name, but we can write 'no monopole.'

And this is Ampère's law. And one more thing that we should not forget is the equation of continuity. This is called the equation of continuity. Alright, this set of five equations somehow presents an inconsistency that one sees here. Sorry about that.

And this inconsistency is seen in this equation: if you take Ampère's law and take the divergence of that curl of H, that should be equal to 0. But the divergence of J free is not equal to 0 because of the continuity equation. So, there is an inconsistency. The inconsistency is that because the divergence of J, whether it is free or not free—I mean, they are—this divergence of J is not equal to 0, especially when the charge density varies with time. So, how to take care of that?

$\vec{\nabla}\cdot\vec{D} = \rho_{free}$ Gauss's law.

$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$. Faraday's law.

$\vec{\nabla}\cdot\vec{B} = 0$ No monopole.

$\vec{\nabla}\times\vec{H} = \vec{J}_{free}$ Ampere's law.

$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0$. Equation of continuity.

$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H}) = 0$ Inconsistancy !!

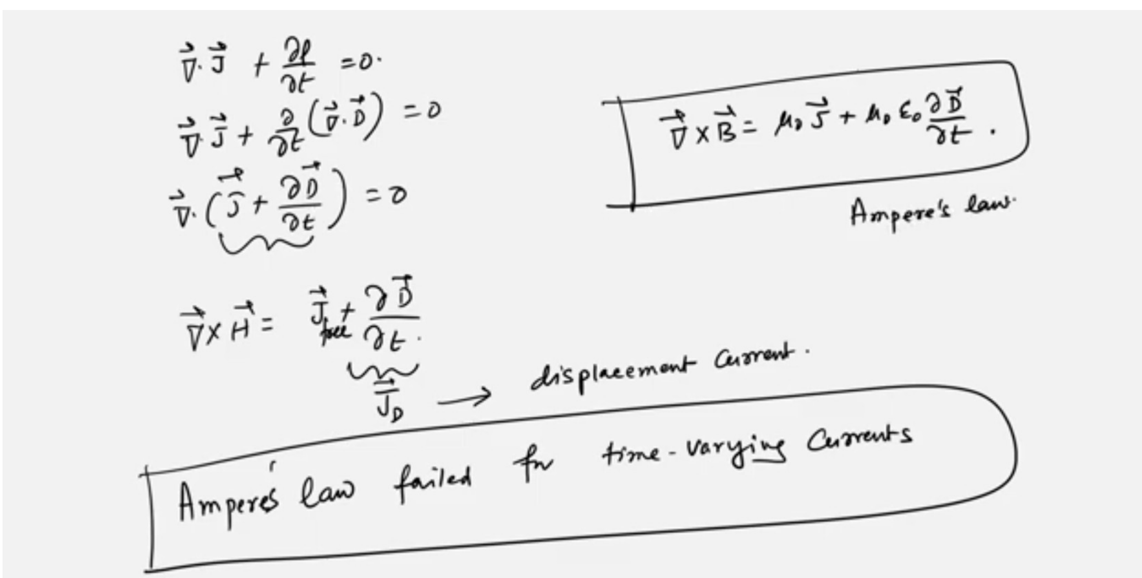
$\vec{\nabla}\cdot\vec{J} \neq 0$.

So if you look at this equation that tells you that these equations are inconsistent or rather the Ampere's law is inconsistent for time varying fields. So we have this divergence of J.

So, divergence of J plus del rho del t that is equal to 0 that is the equation of continuity. So, now rho we know as divergence of D so divergence of J plus del del t of divergence of D. that is assuming that rho is rho free, this is equal to 0.

So, divergence of J plus del D del T, once again we swap this order of derivatives, space and time derivatives, that is equal to 0. So, now, it is really that the J plus del D del T is, the divergence of that has to be 0. In Ampere's law, the J that you saw here in this fourth equation, that has to be replaced by not just J, but J plus del d del T. So, we should have curl of H equal to J plus del d del T, and this is an important discovery called the displacement current. As you see, it is a transient thing—as long as D is a function of t, it will exist, and it does not change anything.

It was, you know, later on realized that this current is important and this current is a transient current and one can actually show that this current actually exists by taking a simple example of a capacitor, okay. So, the main thing is that Ampere's law really failed for non-steady currents or time varying Now if you change this J by you know so this is J free by this del D del T or you know you can write this the same thing as curl of P is equal to mu 0 J plus a mu 0 epsilon 0 del D del T. So, that is the form of Ampere's law now. This is written in terms of B. And these are some of these important things. And this allows us, you know, to write down Maxwell's equations.



And these Maxwell's equations will write it down for both vacuum and in medium separately. So, the equations are for vacuum divergence of E is equal to rho over epsilon

0, divergence of B is equal to 0, curl of E is equal to minus del B del T and curl of B is equal to 0 J plus mu 0 epsilon 0 del E del T which is nothing but this del D del T because D is nothing but epsilon 0 del E del T. And in a linear medium and when I say a medium it means that it has both dielectric properties and magnetic properties. So, divergence of D is equal to rho divergence of B is equal to 0 curl of E is equal to minus del B del T and curl of H is equal to J free plus del D del T.

(i) $\varepsilon < 0, \quad \mu > 0$: plasma

(ii) $\varepsilon > 0, \quad \mu < 0$: artificial material

(iii) $\varepsilon < 0, \quad \mu < 0$: Metamaterials.

Okay, so that is an important part of this. I mean that those are the Maxwell's equation and the Maxwell's equations are Lorentz invariant. So, they, you know, if you do a relativistic transformation, the Lorentz transformation These equations are all invariant. And we also, you know, this C, which is the speed of light, is given by 1 by mu 0 epsilon 0, which comes out as 3 into 10 to the power 8 meters per second, which is what you know.

And this eta is called as this mu 0 by epsilon 0 is called as the impedance of free space has a value which is 377 ohm. Okay. So, one sort of thing one can talk about for these Maxwell's equation, the Maxwell's equation never makes any distinction between positive values of mu 0 or epsilon 0. In fact, it turns out that if you have these I mean, they do not sort of exclude these values and in fact, this epsilon equal to 0 or mu to be, I mean, epsilon less than 0, mu to be greater than 0 and so on in a medium, of course, and we have, you know, so.

This C is equal to the V, or rather, when you talk about V, that is the speed of light in a medium, which is given by C divided by n, where n is the refractive index. n is actually given by the square root of $\mu_r\varepsilon_r$, which are relative permeability and relative permittivity. So, that is the n, the refractive index. This is eta. And this is the refractive index. And as I said, the refractive index can be negative if mu r becomes negative or epsilon r becomes negative or both can become negative.

Maxwell's equations

Vacuum

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}.$$

$$\vec{\nabla} \cdot \vec{B} = 0.$$

$$\vec{\nabla} \times \vec{E} = \frac{-\partial \vec{B}}{\partial t}.$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}.$$

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \ m/s.$$

$$v = \frac{c}{n} \qquad n = \sqrt{\mu_r . \varepsilon_r}.$$

In a linear medium.

$$\vec{\nabla} \cdot \vec{D} = \rho_{free}.$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = \frac{-\partial \vec{B}}{\partial t}.$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_{free} + \frac{\partial \vec{D}}{\partial t}.$$

$$\eta = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377 \, \Omega.$$

eta

n: Refractive index.

In fact, these have certain names and so on. So if epsilon, we can just write down quickly about epsilon less than 0 and mu greater than 0. These are seen in plasma, in which case this... The refractive index is negative. So, this is seen in plasma and 2 epsilon greater than 0 mu less than 0.

This is probably some artificially fabricated material. not really seen in real materials and three one can have both epsilon to be negative and mu to be negative and these are called as a meta materials in optics okay. So, at the end just want to say that these equations are all correct and these equations are Lorentz invariant and we had just a one module which is based on the electrodynamics and though we have done the electrostatics quite elaborately we have spent less time on the magnetic properties and also the time varying fields. But at least it should have given you an idea that there are these equations and these quantities in electrostatics, which are very similar and analogous in magnetostatics, even though the electric and the magnetic fields are they have distinct characteristics. And one more important thing is that the time varying cases have to be dealt separately because the electric field and the magnetic fields in the static case, they were decoupled, they are delinked.

And but in the time varying case, they depend on each other in the sense that the space varying magnetic field gives rise to time varying electric field and vice versa. And these laid down the foundation of mechanics. Maxwell's equations we do not do propagation of wave vector but then we can write down these using the Maxwell's equations we can write down the propagation or these how electric fields propagate in either in vacuum or in different media and so on and that is the electromagnetic fields which is nothing but light and how all I mean the light when it goes from one medium to another, it undergoes reflection, it undergoes refraction and so on and there are various properties of optics. total internal reflection, anomalous properties and so on they all can be explained using the propagation of electromagnetic fields in a medium.

There are you know that strongly decays inside a conductor and as you know that the conductors have electric field inside of conductors have no electric field. And the reason is that that these the electromagnetic fields cannot penetrate and penetrates only up to a very small depth inside a conductor and that is why we have this conductors to be free from electric fields inside. That is why one can actually work in a region where there is large electric field but sitting inside a metal cage or a metal I mean a conductor basically. So we'll stop here for this discussion on the electrodynamics and start with something new for the next module. Thank you.

Thank you.