ELEMENTS OF MODERN PHYSICS

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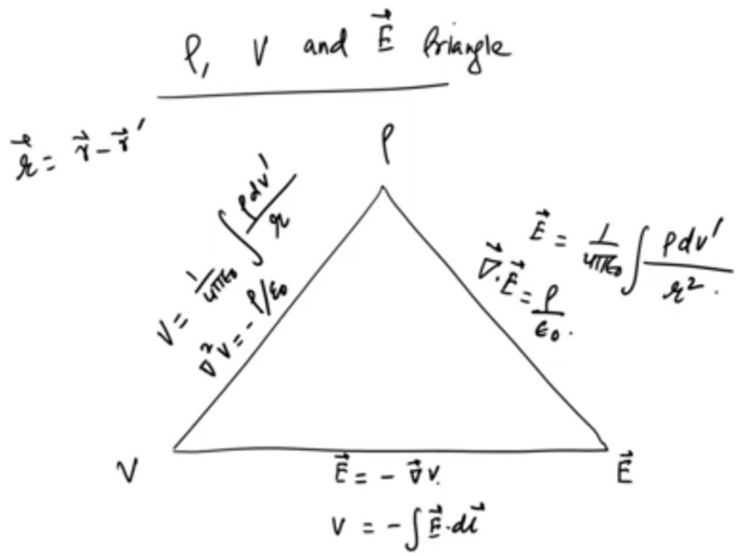
Lec 17: Dielectrics

From what we have gathered from Gauss's law and Coulomb's law, let us summarize them in the form of an interesting triangle where the vertices would be either the charge density or the potential, the electrostatic potential or the electric field. So, let us call this a rho, V, and E triangle. Rho is the charge density, V is the potential, and E is the electric field. So, let us draw a triangle that is big enough so that we have rho here at this vertex and we have the potential here. Remember, potential is often written with a phi scalar potential.

Here, we will write the electrostatic potential as V. So, we have this Laplace's equation which is minus rho by epsilon 0, which means that if you know V, take the Laplacian of that and that will give us this rho or the charge density. or suppose you know the charge density and you want to calculate the potential that is from these integral formula which is say rho dV prime divided by r minus r prime or which we called as r and this r is equal to say r minus r prime okay and this is actually the magnitude of r. Similarly if you know the E then this is the differential form of Gauss's law where you write down this take the divergence of that and that will give you the charge density which is rho over epsilon 0. So, divergence of the electric field is rho over epsilon 0.

So, you know assuming that you know electric field you can take the divergence and get the charge density. Similarly, if you know E or rather if you know rho, then you can get the electric field using this Coulomb's law, which is rho dV prime divided by these R square, okay. And similarly V and E also have some familiar relation E equal to minus gradient of V and V is equal to minus E dot dl or the line integral of that. So if you know V you can take a gradient of that to get the electric field with the directions you know inbuilt into this gradient.



And when you know the electric field, you can take a line integral over some length element, and that will give you the potential. So this triangular relationship between them —when you know one quantity, you can get the other. Let us move on with these discussion of electrostatics and to begin with let us talk about dielectrics and what are dielectrics, what are free and bound charges, what are dipole moment or what is the definition of dipole moment, the polarization, Gauss's law and importantly vector called as a D vector. And hence we will do the solution of Laplace's equation which are by the separation of variables in the Cartesian spherical polar or maybe even cylindrical coordinate system. We will not do a very rigorous analysis of it but we will only do the amount that is necessary for learning and then we will do some examples.

So the question is what are dielectrics, what are their properties, why are they important and why should we study them and in this electrostatics or electrodynamics module of the course. So the dielectrics are really poor conductors of electricity because of the absence of any loosely bound charges to conduct electricity, which means that if you really divide all the materials that we are aware of into conductors and insulators, conductors are characterized by a lot of available electrons for conducting electricity. If you apply a small bias or a small voltage difference, there will be current flowing in the circuit. And in insulators, there are absolutely no free charges.

There's a large band gap in these materials, which makes them insulating. That is, they do not conduct any electricity. Dielectrics are actually closer to insulators. In fact, they have no charges to conduct, but they host large electric fields. They store electrical charges.

They have very high specific resistance and they have high negative coefficient temperature coefficient of resistance. There is an important property. In fact, metals have positive temperature coefficient of resistance, which means that as the temperature is increased, the resistivity or the resistance increases. The reverse happens in the dielectric. So when a dielectric is placed in an external electric field, the positive charges are displaced in the direction of the field.

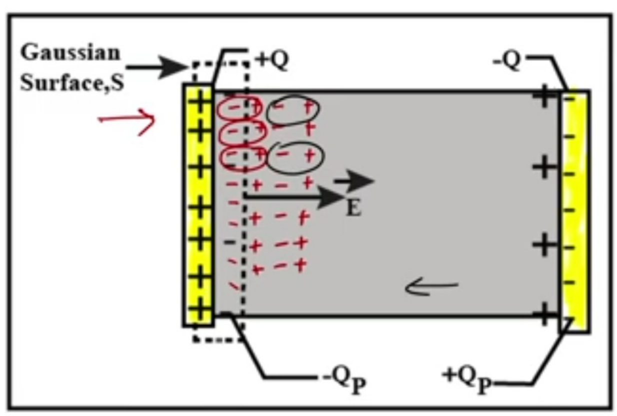
So these are the neutral atoms and these neutral atoms kind of acquire a dipole moment. So we have these atoms which are like this positive in the nucleus, positive charges in the nucleus, and there are negative charges. And when we put them in the electric field, they sort of have this oblong shape. shape where you know the positive charges are along the direction of the field. So, let us say these are the positive charges and these are the negative charges and the atom actually acquires an induced dipole moment and these are important properties of these dielectrics.

This polarization that happens because of this separation of the positive and negative charges that creates a strong internal field and this internal field is in the direction which is opposite to the external field applied field and hence it reduces the overall electric field in the material and that is why They are important and they have, so when you apply an external electric field, the field that is developed inside the material is lesser than that because there is an internal field that develops because of this polarization of charges which is in the opposite direction to the external field. Now, there is a quantity that we will talk about, it is called as a dielectric constant which is a measure of these polarization, the polarization of charges that we are talking about. And substances with low dielectric constant are, say, for example, vacuum or some gases, helium and nitrogen gases, which are there in the atmosphere. And then materials with moderate dielectric constants are ceramics, distilled water, paper, etc.,

Mica, polyethylene, and glass with high dielectric constants are mostly metal oxides, for example, aluminum oxide and so on, which are more like insulators. So, what happens is that let us see a schematic diagram of a dielectric. So, let me use a color. So, these are the negative charges, and then there are positive charges and so on. So, these are negative and positive.

Positive and negative and so on. So these are, you know, the negative charges here, and there are these positive charges here. Sorry, this is a positive charge and negative charge, or positive. So positive, negative, positive and it happens in both the sides and positive, negative, positive, positive, negative, positive and so on. and what happens is that these are they cancel each other okay so these are inside the bulk of the dielectric they cancel each other and what we are left with only that you see are the side ones let us you know kind of put a highlighter here and this is that region where we have some free charges or the unbound charges and these are there on both sides of the system which are not cancelled out and as I said that there is an external field that is there and these external field is say in this direction and then there is a field that is you know produces in this so external field is shown inside and there will be a field inside that will be in the other direction effectively reducing the field and these yellow shaded region contains charges which are free charges and the charges inside are bound charges because they would form you know simply bound they are bound here and so on so there are these bound charges do not take place in conduction and that's why they are more you know the closer to the behavior of insulators and we have only some free charges at the edges of the sample okay And which, of course, do not cancel out. So let us take an example of, you know, so before we do that, let us, you know, sort of see the differences, main differences between dielectrics and insulators, even though we have said that they have some close resemblance. And the dielectrics can develop an internal electric field.

Schematic diagram of a dielectric



We use a color here. The internal electric field, which nullifies the external applied voltage, you know, that the internal fields can be large enough to have these external fields canceled. Dielectrics are easily polarized and can store charges. They contain relatively weak bonds within atoms and mainly used in capacitors, cables, transformers, resonator oscillators, and so on. Insulators do not develop an internal electric field such as the dielectrics.

Insulators actively obstruct electricity; that is, they do not allow the passage of electricity. They are mostly due to strong covalent bonds. However, ionic bonds are mostly present in dielectrics. So used in high voltage line systems, the insulators and electrical appliances, all of us are aware of the fuse that we have is a very good insulator. And so they are, you know, they're used to prevent the users from getting electric shocks.

DIELECTRICS	INSULATORS
Can develop an internal electric field, which nullifies the externally applied voltage.	Do not develop an internal electric field.
Dielectrics are easily polarized and can store charges.	Insulators actively obstruct electricity.
Contain relatively weak bonds within atoms.	Contain strong covalent bonds.
Mainly used in capacitors, power cables, transformers and resonator oscillators.	Used in high-voltage line systems and electrical appliances to prevent users from getting electric shocks.

And it's taken from this stable is taken from this site. So the applications, the potential applications of the dielectric materials are because of the ability to store charges. Dielectrics are most commonly used for energy storage in capacitors. High permittivity dielectric materials are often used to improve properties of semiconductors. In transformers, rheostats, shunt reactors and earth reactors, dielectric materials act as cooling agents and also as insulators.

And dielectrics are also used in liquid crystal displays. Let us get into some more technical details of that. Let us think of an atom which we said that so there is a positive and a negative charge. So, there is this positive charge or say the positive charge is here and the negative charge is here. and we put that in and they are separated by a distance just like there is a model of an atom where we have stripped off those two charges and not showing the atom itself and there is an electric field that is you know is in this direction, okay.

So, this is the electric field direction of the electric field and this charge will have a force acting on it which is QE and And this charge will have a force acting on it, which is minus QE. So, the total force cancels. But however, there is a torque. And this torque is given by basically the force into the magnitude of the distance, which is F into D.

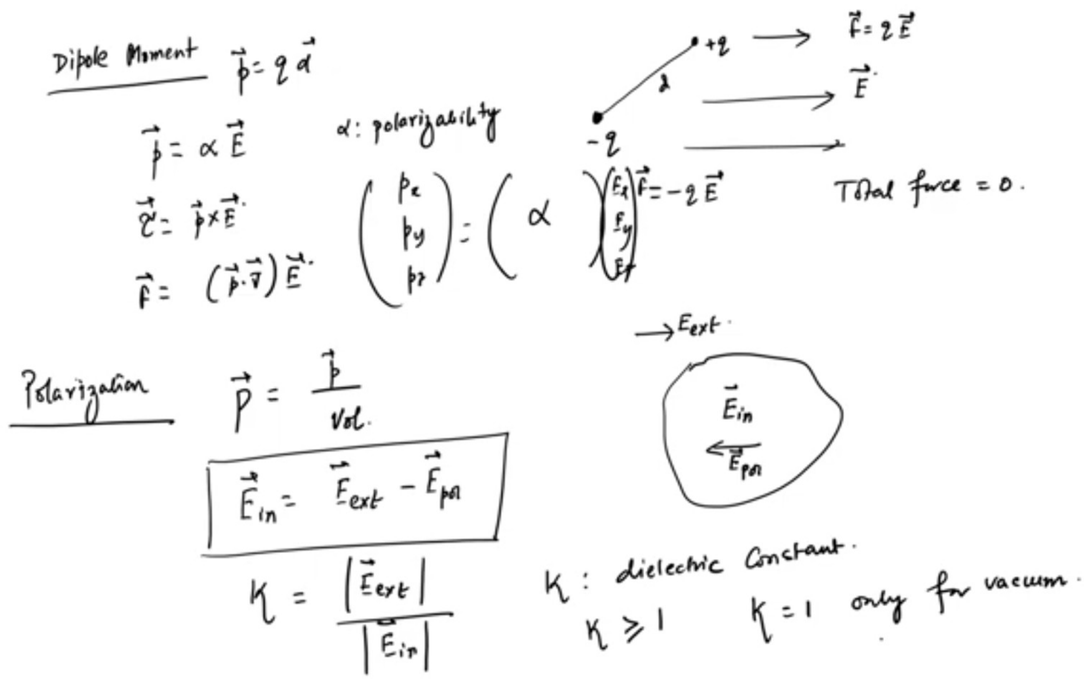
And in terms of these dipole moments, we denote the dipole moment of such a charge configuration, which is, you know, a plus charge and a minus charge equal and opposite, separated by a distance D. So, this P is equal to Q into D, and it, of course, takes place in a direction connecting the two charges. And for linear dielectrics, we shall assume that this polarization is proportional to the electric field that is present and alpha is the polarization constant and alpha is known as the polarizability. So, that tells that how a material is you know susceptible to these formation of a dipole in presence of an electric field okay. And similarly this the torque is given by P cross E and if the for some reason there is a the force that is you know inhomogeneous that will create a force as well. And that is given by P dot del E.

and so this is the definition of dipole moment and what we call as polarization which will you know encounter several times in this discussion of dielectrics. So, that is defined with a capital P vector, which is nothing but this small p, the dipole moment, divided by the volume, okay. So, that is the definition that we use and remember this alpha is a polarizability it may look like a you know, a scalar quantity but it is not because of this vector nature of this equation. So, you may have a px, py and pz which can be, you know, related to this Ex, Ey and Ez and that will demand these polarizability to be a tensor or a matrix of this tensor of rank 2.

So, this is where the polarizability, you know, sits here, which may have different values in different directions and may have off-diagonal elements that is px can be, you know, connected to or they could px can arise from all of Ex, Ey and Ez. So, that way, there will be all kinds of, you know, off-diagonal elements present in this expression. So, that's the dipole moment and the polarizability. And so, what happens is that when you have a dielectric material and you put it in an external electric field, Let us call this as E external and let us call that there is a field developed inside is called E internal and this is this internal field will have something to do with the you know the there will be a polarization that is also going to be generated.

So, your effective field inside is equal to the E external minus the E polarization, and that is an important expression. This is how the electric field inside a dielectric is modified, and this has been told several times that these hosts are or they sort of, you know, have electric fields that are generated. And that's one of the prime differences between insulators and these dielectrics. So, they may host strong electric fields and so on. So, if you define the ratio of the external field, the magnitude of the external field, divided by the field that is generated inside, this is called the dielectric constant.

So, this is the dielectric constant, and from the structure, it is very clear that it is a dimensionless quantity. Okay. And it's, of course, greater than or equal to 1. In a surprising case, it's only equal to 1 when the external E equals the internal E, which will only happen in a vacuum. So, K equals 1 only for a vacuum. which has been always, you know, stated as the worst dielectric or it's not a dielectric at all.



Okay. So, in a vacuum, of course, there are no charges to polarize. So, what will create this internal field to be different from that? So now let's now, you know, come to Gauss's law for the dielectrics because we saw that for conductors or for other charge distribution, the Gauss's law, plays an important role.

So, what is Gauss's law for dielectrics? Okay, so Gauss's law is written as this internal field that we have just written down, and then the ds, where ds is the area element, and this is equal to 1 by epsilon. Now, epsilon is the permittivity of the dielectric medium and it is different than the permittivity of free space. So, 1 by epsilon and we have Q free. That's the free charges.

Now this is important because we have already shown that there are free charges and bound charges in a dielectric. So the Gauss's law only deals with the free charges that form at the edges because they do not find their partners to form a neutral system. So this is coming from the free charges and if you wish to write the same thing in differential notation then this is divergence E in is equal to rho free divided by epsilon and so this permittivity will you know get carried everywhere and in the accounts for the presence of the dielectric and this epsilon will remind us that it is different than the vacuum that we are talking about a different system other than vacuum. So, once we have talked about Gauss's law, let us talk about the free and bound charges.

We have already introduced that there are free charges there, which goes to form the Gauss's law. But let us see that there are also free, I mean the bound charges also come in the discussion of the potential. So, we write down the electrostatic potential inside a dielectric to be equal to 1 over 4 pi epsilon 0 integral of r dot p. This is r prime, that is the form of the potential. So, and this is equal to r square and d cube r prime. So, r prime is the source term, which is inside the dielectric.

This capital P is the polarization, which we have defined. It's a dipole moment per unit volume. r is nothing but this curly r is nothing but r minus r prime, where r is the field point and r prime is the source point. This has all of them have been talked about. And if you remember that we have written down this, that this divergence of 1 over r is

is equal to, it is a vector of course, which is 1 r cap over r squared. So, when you have this r cap by r square here in the equation that is above this, we can write down the potential form as or form of the potential as 1 over 4 pi epsilon 0 and we have a P dot gradient of 1 over r and a d cube r. So what we have just done is that we have, uh, you know, replaced this r over r square by this, uh, gradient of one over r in this, and they've written as P dot of that gradient of r. And, uh, this has to be now solved. And the only way to solve it is that, uh, use an integration by parts, take this as a first function and, uh, the other one as the second function.

Gauss' law for dielectrics

$$\oint \vec{E}_{in} \cdot d\vec{s} = \frac{1}{\epsilon} q_{free}.$$

$$\vec{\nabla} \cdot \vec{E}_{in} = \frac{\rho_{free}}{\epsilon}.$$

Free and bound charges.

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{\mathscr{r}} \cdot \vec{P}(\vec{r}')}{\mathscr{r}^2} d^3r'$$

$$\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \overset{u \text{ (first function)}}{\vec{P}} \cdot \underbrace{\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right)}_{v \text{ (second function)}} d^3r'$$

So, this V is the standard notation that is used. So, the first function and the second function are given as this. And if you do integration by parts, what you get is the following. So your V of r, that's the potential electrostatic potential inside a dielectric is 1 over 4 pi epsilon 0 and integral of the divergence of P over r. d cube r prime minus 1 over r. So, this comes from the second part of the integral, which is the divergence of P d cube r.

prime, okay. So that is the two terms that comes out from this integration by parts and then this can be written as the so one can use Gauss's law here, Gauss's divergence theorem rather, Gauss's divergence theorem. and convert this volume integral into a surface integral. So, this is like 1 over 4 pi epsilon 0, and you have 1 over, so this is like P over r dot ds prime. Again, prime variables refer to the source variables where the charges are present.

And this is 1 over r divergence of P and d cubed r prime. So, this term is due to the you know the surface contribution to the potential and this is to the volume contribution to the potential and we will call this by a sigma B which is a surface bound surface charge density and bound volume charge density which are respectively written as sigma B and rho B and where sigma B is equal to P dot n cap and rho B is equal to divergence of P with a negative sign. The negative sign is coming because you write. So, this contribution is written as 1 by, you know, 4 pi epsilon 0.

Then you have this sigma B dot, you know, n cap ds, n cap dot ds, and plus this volume, which is rho B divided by r d cube r prime, and that is divergence of P. So, these are the bound charge density, surface charge density, and volume charge density. So, we have talked about this volume, the bound charge, and the surface charge from this, you know, this volume. schematic figure the bound charges are in red or black color and this yellow shaded region contains the free charges and enumeration of them are done respectively by this Gauss's law which contains the free charge densities either volume or surface I mean when you talk about Gauss's law they of course talk about the volume charge density free volume charge density

And where do we get the bound charges? The bound charges are obtained from this, from this potential. The potential is a combination of two terms which are like this. So, this is sigma B by r. Sorry, I forgot r here. There was an r there.

So, it is a sigma b by r n cap dot ds prime. So, this is actually prime. It is a prime variable. So, that is why it is prime. All right.

So, your total charge density, rho total, which is often just written as rho, is equal to rho B plus rho free. So, the bound and the free charge density. So, one can actually write down another vector called as a D vector, which is important in this context for a reason that I will just tell you in a while. So, rewriting Gauss's law: epsilon 0 divergence of E is equal to rho, where rho equals rho B plus rho F. That is a bound plus the free and rho B is minus divergence of P plus rho F and this is equal to divergence of epsilon 0 E. Well, the epsilon 0 has to be written, you know, in some consistent form.

So, you can write it like this. and plus a P because you can combine both the divergences and can write this as a rho F. So, let us call this vector as some vector called D. So, that is called as a displacement vector. So, divergence of D equals rho F. This is Gauss's law for D. Even though it has Gauss's law, but there will be no Coulomb's law for D. So, that is a little misleading. Even though it looks like D contains the effect of both E and P, which it does, and that is why it is important, there is no Coulomb's law.

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\int \underbrace{\vec{\nabla}\cdot\left(\frac{\vec{P}}{r}\right)}_{\text{Gauss' divergence theorem}} d^3r' - \int \frac{1}{r}(\vec{\nabla}\cdot\vec{P})\, d^3r'\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\oint \underbrace{\frac{\vec{P}}{r}}_{\sigma_b}\cdot d\vec{s}' - \int \frac{1}{r}\underbrace{(\vec{\nabla}\cdot\vec{P})}_{\rho_b} d^3r'\right] = \frac{1}{4\pi\epsilon_0}\left[\oint \frac{\sigma_b}{r}\cdot\hat{n}\, d\vec{s}' + \int \frac{\rho_b}{r} d^3r'\right]$$

$$\boxed{\begin{array}{l}\sigma_b = \vec{P}\cdot\hat{n} \\ \rho_b = -\vec{\nabla}\cdot\vec{P}\end{array}} \qquad \rho_{total} = \rho = \rho_b + \rho_{free}.$$

$$\epsilon_0 \vec{\nabla}\cdot\vec{E} = \rho = \rho_b + \rho_f. = -\vec{\nabla}\cdot\vec{P} + \rho_f.$$

$$\vec{\nabla}\cdot\underbrace{(\epsilon_0\vec{E} + \vec{P})}_{} = \rho_f \Rightarrow \boxed{\vec{\nabla}\cdot\vec{D} = \rho_f.} \leftarrow \text{Gauss' law for } \vec{D}.$$

Let me first clarify the first statement that I made that D contains both E and P. And why is it that both E and P were known to us? Why did we have to introduce another vector in this context? And the reason is that, you know, take a dielectric. And now talk about it is in an external electric field.

So the field will polarize the dielectric and the polarized charges or polarization charges will produce an internal field and this internal field will in turn polarize the dielectric. and will again create a polarization and so on. So there is an infinite loop that goes on like the external field, it polarizes the system, polarization creates this field, that field will again polarize and will create a polarization, again will have a field due to that. So these two break away from this infinite sequence of electric fields that produce and one actually can talk about D which contains both E and P okay.

So, going a step further, we can write down this electric field, which is equal to, say, sigma free minus P divided by Now, if you write it in vector form, then there is a so sigma is really a number but you can think that it is on a surface and the surface has a direction which is perpendicular to the surface and this is divided by this epsilon 0. If you are uncomfortable with that, we can put a magnitude here everywhere. Okay, so P is equal to this P can be written as these epsilon 0, epsilon 0 chi E and E and so the polarization is proportional to field and this proportionality constant would make it a linear dielectric and this chi E is called as the electric susceptibility. And so this E can be really written as, you know, the sigma free over epsilon zero one divided by one plus chi E. So it's clear to, I mean, it's very apparent that the electric field inside a dielectric is reduced by this factor.

So this factor is the field in Okay, so one can also, you know, write this, this epsilon r is equal to some epsilon by epsilon 0, that is also written as 1 by chi e, and one can replace this 1 over chi e by this epsilon r, which is called as a relative permittivity and so on. So, your d has this form that it's epsilon 0 e plus p and the p is epsilon 0 e plus this epsilon 0 chi e e and this is equal to epsilon 0 1 plus chi e E and so on.

And so, this we can write this as epsilon E. So, D is related to E by this epsilon term where epsilon is equal to epsilon is equal to epsilon 0 1 plus chi E. So, chi is of course 0 when you have epsilon equal to epsilon 0 and that happens in vacuum. So, epsilon r or the relative permittivity can also be used. So, we come to the second note that we have talked about that in spite of there being Gauss's law for this electric field or electric displacement vector. So, if I have not written it, so d is equal to the electric displacement vector.

$$|\vec{E}| = \frac{\sigma_{free} - |\vec{P}|}{\epsilon_0}.$$

χ_e : electric susceptibility.

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

$$|\vec{E}| = \frac{\sigma_{free}}{\epsilon_0} \frac{1}{1+\chi_e}.$$ ← factor reduces the field inside dielectric.

$$\epsilon_r = \frac{\epsilon}{\epsilon_0} = 1 + \chi_e.$$

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E} = \epsilon_0 (1 + \chi_e) \vec{E}.$$

$$= \epsilon \vec{E} \quad \Rightarrow \quad \epsilon = \epsilon_0 (1 + \chi_e)$$

even though Gauss's law exists, this is incorrect, that there is a 1 over 4pi epsilon and then there is a, you know, there is a r cap by r square and then there is a rho free and rho free which is a function of r prime and d cube r prime, this is incorrect. There is no such

Coulomb's law that exists for d. And so, going back to the Gauss's law, we have this divergence of d equal to rho free. Remember, if you write this equation for d, then do not bring in an epsilon in the denominator or epsilon 0. Alright, so we have this rho B which is equal to a divergence of P that gives you a minus divergence of epsilon 0 by epsilon chi E into D.

$\vec{D}$: Electric displacement vector.

$$\vec{D} = \frac{1}{4\pi\epsilon}\int \frac{\hat{r}}{r^2}\rho_{free}(\vec{r}')d^3r'$$

$$\vec{\nabla}\cdot\vec{D} = \rho_{free}.$$

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot\left(\frac{\epsilon_0}{\epsilon}\chi_e\,\vec{D}\right) = \frac{\chi_e}{1+\chi_e}\rho_{free}.$$

And this is equal to a chi E divided by 1 plus chi E and rho free. So this is how the bound charge and the free charge densities are related by this electric susceptibility. All right. So let me now get back to the solution of Laplace's equation. And one could have actually argued that since we are talking about the solution for the electrostatic potential, we have applied one trick called the method of images.

why did we digress from there on to come and talk about this dielectrics and various properties of dielectrics. And the reason is that the solution of Laplace's equation will require us to apply to some system and the dielectrics seem to be one nice system where this thing can be applied or this The concepts of the solution that we get at the end of solving this Laplace's equation can be applied to a dielectric in presence of an external field. And that is the reason that we have done it. So we go back to the solution of Laplace's equation and this has a special name called the separation of variables.

We will demonstrate the separation of variables in the Cartesian coordinate system, but we will be more interested in the spherical polar coordinate system—say, a dielectric which is spherical in shape is put in an electric field and so on. So, let us just start with 1D in Cartesian coordinate system. We will not, you know, do anything more than simply writing it down. So, it is 1D. Laplacian of V equal to 0 and then in 1D we just have one variable so it is equal to 0 and that tells us that dV dx is equal to constant say for example a constant say C and hence V of X is equal to C1x or Cx plus D.

So, that is the solution in 1D, and C and D are, you know, constants that need to be fixed based on the conditions that are given. That may be that potential at x equal to say 1 centimeter is something and potential at the origin is equal to 0, which will set D equal to 0. We will not go into the specifics of this; this is too simple, and let us just leave it at that. Let me do it in 2D, which will be a little more involved and closer to what we know of. So, let us just draw the coordinate axis.

So, this is x-axis and this y-axis and maybe this is z-axis. and we have so along this so we have this plate the two parallel plates let me use a color here and this is that plate And you know, so that is the plane and let us say these are kept at V equal to 0 and well, instead of this, let me use V here. So, this is at some x equal to a and say that here the potential varies as v0 y. It has some variation v0 can be constant as well. So this is where this you know the potential these are two plates which are kept on these plane the xy plane and these two sides are kept at this side and this side are kept at V equal to 0 and then we are planning to solve this problem.

So, this is a problem in 2D where we have the d2V dx2. So, let us write it with a partial derivative. So, del 2v del x2 plus del 2v del y2. This is equal to 0, and the boundary conditions—these are important to set the boundary conditions. In the last problem, we ignored it because the boundary condition is too simple.

Solution of Laplace's equation : separation of variable

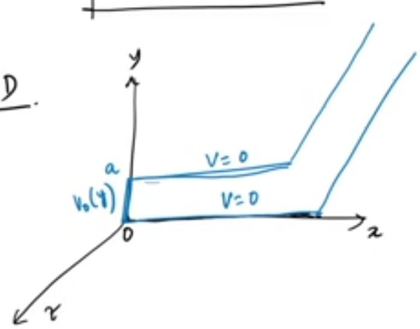
1D

$\nabla^2 V = 0$

$\frac{d^2V}{dx^2} = 0 \Rightarrow \frac{dV}{dx} = C$

$\boxed{V(x) = Cx + D}$

2D.



$$\boxed{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0}$$

Boundary conditions

1. $V=0$ at $y=0$ (1)
2. $V=0$ at $y=a$ (2)
3. $V=V_0(y)$ at $x=0$ (3)
4. $V=0$ at $x=\infty$ (4)

So, one boundary condition here is that V equals 0 at Y equals 0 and V equals 0 at Y equals A. 3 v equal to v0 y can be a constant at x equal to 0. And finally, the trivial condition is that v is equal to 0 at x equal to infinity. So, far away from this arrangement so there are two plates conducting plates and where they are kept at different potential which are shown here and the question is that you obtain the electrostatic potential everywhere by solving this equation the Laplace's equation. So, what we do is that we write this V xy, this is equal to say some function which is solely a function of x, let us call it x of x and then y of y. If you put it in the equation, we get d 2 x d x 2 plus 1 by y, well I write it as capital y because that is a function.

d2 y dy2. Note that, you know, this x and y, x of x means that x is the function and x is basically a function of that. So, x is a variable small x, capital X is a function. So, capital X is a function of small x, capital Y is a function of small y and so on. So, this is a very nice improvement of this because if you look at this equation, the equation that we started with this one, this equation is a partial differential equation and which has been converted into so one PDE or the partial differential equation is converted into two ordinary differential equation and this is you know easier to solve.

And one option is that each one of them is a sum of two terms. One term is a function of x only, the other term is a function of y only. If the sum of them have to be zero, then one trivial solution is that each one of them is equal to zero individually. But that will not give us anything. So what we need is that to get a non-trivial solution.

And in non-trivial solution, what we do is that we write down the first term. as some constant C1. So, 1 by x d2 x dx2 that is equal to c1 and this 1 by y d2 y dy2 So, make sure that you write the numerator I mean in the numerator the function x or function y will appear and also you know at the front, but when you take a derivative with respect to this one that is here in the denominator they are variables okay. And this is equal to say minus, you know, C1 because then only and C's are non-zero constants.

So, what you can do is that you can take c1 equal to k square and c2 equal to minus k square. In this particular case, you cannot take the other choice that is c1 equal to minus k square and c2 equal to k square for reasons that will become clear later. So, that this gives d2x dx2, this is equal to k square x and d2y dy2. this becomes equal to minus k squared y. Okay, so this the first one has a solution that is x has a solution a e to the power kx plus b e to the power minus kx and y has a solution which is equal to c sine ky plus d cos ky.

So, since the solution actually is the product of X and XYFY, so my potential can be written as a product of this which gives AE to the power KX plus BE to the power minus KX multiplied by these solutions. sine and cosine solutions c sine ky plus d cosine ky.

So, now it is important that we find this a, b, c, d because this really does not specify a solution a, b, c and d are unknown constants. So, this gives a family of solutions. what we do is that let us introduce the fourth boundary condition that is V equal to 0 at X equal to infinity.

$$V(x,y) = \underline{X(x)\,Y(y)}$$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = C_1 \quad \& \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = -C_1$$

$$C_1 = k^2, \quad C_2 = -k^2.$$

$$\left.\begin{array}{l} \dfrac{d^2X}{dx^2} = k^2X, \quad \dfrac{d^2Y}{dy^2} = -k^2Y \\ X(x) = Ae^{kx} + Be^{-kx}. \\ Y(y) = C\sin ky + D\cos ky. \end{array}\right\} V(x,y) = \left(Ae^{kx} + Be^{-kx}\right)\left(C\sin ky + D\cos ky\right)$$

So, applying fourth boundary condition let me you know sort of name them so that so this 1, 2, 3, 4. So, from 4 and We have at x equal to infinity, v has to be 0, but a, the constant a does not allow that because the constant a blows up, b is fine because b would go to 0 at x equal to infinity. So, we have to drop this a constant. So, that tells you that a is equal to 0.

So, our vxy gets one step simplified, that is, it is b to the power minus kx. c sine ky and plus d cos ky. Now, there are apparently three constants, but we do not need to carry three constants because BC b multiplied by c can be one constant, b multiplied by d can be another constant, and we can simply write it as exponential kx c prime sine ky, where c prime is equal to b into c plus d prime cos ky. I mean, you can forget the primes because there are no longer any other c and d, so we will just remove the primes there.

Now, what we do is that we apply the first boundary condition, which says that your v equal to 0 at y equal to 0, which is given problem. So this is 0. So the potential is actually 0 here at y equal to 0. So that tells us that because the cosine term will not allow it to be 0 when y equal to 0. So we drop that term.

And so this term is dropped and we have a further simplification, which is equal to c to the power minus kx sine ky. So, it has a harmonic variation along the y-direction and a decaying variation along the x-direction. Now, if you interchange x and y axis, then it will of course have sinusoidal variation along x axis and exponentially dying solution along y-axis. And this is why I said that

you have taken this to be k squared and minus k squared. If you take this as minus k squared and k squared, mathematically that is possible, but not relevant for this particular example. Now use second boundary condition and the second boundary condition you can go back and have a look at it is V equal to 0 at Y equal to A. So, that is the upper plate where the potential is at 0 and this can be sort of written as or rather if you put it you get case to have you know values which are n equal to n pi over a, where n equals 1, 2, 3, and so on. You see, we are solving a classical problem, but we get this quantization similar to the quantum mechanics that we have looked at the Schrodinger equation, solution of Schrodinger equation.

And I have mentioned there as well that this quantization has nothing to do with quantum mechanics or classical mechanics. It is a purely mathematical property where solving some boundary conditions along with their initial properties, or solving in keeping with their boundary conditions, you get this quantization. So, finally, what we get is that we put this now from 3, and we get that is V equal to V0Y. So, we get a V0Y, which is equal to a C sine n pi n.

By a into y, okay. So, that is of course at x equal to 0. So, we have put x equal to 0, so that goes away and so on. So, what is my final solution? My final solution is Vxy is equal to sum over n Cn e to the power minus n pi x by a sin of n pi y by a.

from (4). at $x=\infty$ $V=0$ / $A=0$.

$$V(x,y) = Be^{-kx}(C\sin ky + D\cos ky)$$
$$= e^{-kx}(C\sin ky + D\cos ky)$$

from (1) $V(x,y) = C e^{-kx}\sin ky$.

from (2) $K = \frac{n\pi}{a}$ $n = 1, 2, 3 \ldots$

from (3) $V_0(y) = C\sin\left(\frac{n\pi}{a}\right)y$.

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

That is the full solution or the family of solution and n really runs from 1 to infinity. See, n equal to 0 is not allowed because then the solution does not exist. So, n equal to 1 to infinity, C1, C2, etc., everything is allowed and so on. So now the only thing that you need to find is the Cn and you use this boundary condition that you had that is 3 and from 3 what you do is that your V0Y, let us call it some V0 of Y. Okay, this could be independent of y, then it will be a simple constant else there is some y variation, because this is equal to Cn sin n pi y by a. Okay, this is what we have written as well.

Now, in order to calculate Cn, we use a Fourier trick. So, what we do is that we multiply both sides by sine m pi y by a, where m is just like n is an index and then integrate from y equal to 0 to a. And so, if you do that, then of course, you know, one gets this sum over n equal to 1 to infinity, you have C_n 0 to a, that is where, you know, even if we say infinity, the only thing that is of relevance to us is 0 to a, it is a sin m pi y by a. There is no x because x has been put equal to 0 and dy, so this is equal to 0 to a V0 y sin m pi y by a. Now, what you do is that you use this relation that 0 to a sine n pi y by a sine m pi y by a.

dy this is equal to a by 2 delta mn it is a Kronecker delta which means that if m is not equal to n this is equal to 0 this integral does not give any finite value and if m is equal to n then it is a by 2. So this frees these cn which means it makes it free and you get 2 by a cn equal to 2 by a 0 to a v0 y sin n pi y by a dy. And that solves the problem completely.

There was one unknown which was Cn. So put it in this solution and one can actually solve this provided you know V0.

$$V(0,y) = V_0(y) = \sum_n C_n \sin\frac{n\pi y}{a}$$

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin\left(\frac{n\pi z}{a}\right)\sin\left(\frac{m\pi y}{a}\right) dy = \int_0^a V_0(y)\sin\frac{m\pi y}{a}\, dy$$

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi y}{a}\right) dy = \frac{a}{2}\delta_{mn}$$

$$\boxed{C_n = \frac{2}{a}\int_0^a V_0(y)\sin\frac{n\pi y}{a}\, dy.}$$

$$\text{Suppose } V_0(y) = V_0 \Rightarrow C_n = \frac{2V_0}{n\pi}\left(1-\cos n\pi\right)$$

$$= 0 \text{ for } n: \text{even.}$$

$$= \frac{4V_0}{n\pi} \text{ for } n: \text{odd.}$$

Suppose V0 equal to 0. or V0y equal to V0, then that gives us cn equal to 2 V0 by n pi, which is 1 minus cosine n pi, and this is equal to 0 for n even, and this is equal to 4 V0 by n pi for n equal to odd. So you have an even-odd problem and so on. And nevertheless I mean the problem is completely solved. I will simply now write down these solutions for this spherical polar coordinate system and cylindrical coordinate system.

And you can look at the detailed solution either in Griffiths or in some other book of electrodynamics. So spherical polar coordinate system has all these r theta and phi. So your r is equal to r theta phi and so we solve Laplace's equation there. And the solution is you can, you know, this is usually true that V of r, you can think that it is a function of r and theta, so that the phi really does not make too much of a difference. And writing down the Laplacian in spherical polar coordinate system we have a del V this 1 by r squared del del r of r squared del V del r.

Plus 1 over sine theta del del theta of sine theta del V del theta, that is the separation of variables gives. So, this term is completely dependent on R, and this term is completely dependent on theta. There is no phi dependence because the potential is simply equal to, it is a function of R and theta. So, if you now solve each one of them, the r solution gives r of r is equal to a r to the power m plus b divided by r to the power m plus 1. And this theta of theta, which is a solution that is a function of theta.

So what we do is that V of r theta is equal to r of r, just like x of x, and this theta of theta. And so the r solution is like that, and the theta solution is like this. It's equal to P, say, m cos theta. P m are the Legendre polynomials with some properties that is when m is even like P m x is equal to P m of minus x for even m. And this is equal to also this plus minus, so even, and let me write it together, even, odd.

So the lower sign is for odd and the upper sign is for even. And Pm of 1, which is cos theta equal to 1, is equal to 1 for all m. And so on, and one writes down the solution as V of r theta is equal to these r of r theta of theta and one gets the solution as sum over m A rm plus B rm plus 1 and a Pm cos theta. So that is the solution for the spherical polar coordinate system. Once again, one has to solve this A and B for this given case.

Spherical polar coordinate system $\vec{r} = (r, \theta, \phi)$

$V(\vec{r}) = V(r, \theta)$.

$$\underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right)} + \underbrace{\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right)} = 0 \qquad V(r,\theta) = R(r)\,\Theta(\theta).$$

$R(r) = A r^m + \dfrac{B}{r^{m+1}}$.

$\Theta(\theta) = P_m(\cos\theta)$ $\qquad P_m(x) = \pm P_m(-x)$ for even (odd) m.

$P_m(1) = 1 \quad \forall m$.

$$V(r,\theta) = R(r)\,\Theta(\theta) = \sum_m \left(A r^m + \frac{B}{r^{m+1}}\right) P_m(\cos\theta).$$

And we will again only write down the solution for the cylindrical coordinate system. So, it is rho of phi z and let us you know that V is only a function of rho and phi. So, you have or you can write it R phi z. So, this R phi and you have R of r. and a phi of phi and this solution is written as V r phi is equal to a0 log r plus b0. In fact, here m equal to 0 is allowed and then we have sum over m not equal to 0.

A, M, R to the power M plus B, M, R to the power M again, and we have C, M, cos M phi plus D, M, sin M phi and so on. So, that is the solution for this is the r solution and this is the phi solution and this is the total solution that we have. So, once we have learned how to get this solutions by doing separation of variables. So, in this last two problems we have not done it explicitly but what you can do is that you can assume a solution. and put it into this r part and put that equal to 0 and take that to be some constant and then for the theta part also you can do the same thing.

And this will give rise to this solution what we have written later. It is a sum over m a r to the power m b by r to the power m plus 1 p m cos theta. And for the cylindrical, we have this thing which is, so there is m equal to 0 that exists. So, there is m equal to 0 solution and there is m not equal to 0 solution, okay. So, there is m not equal to 0.

Cylindrical Coordinate System $\quad r, \phi, z.$

$$V(r,\phi) = R(r)\,\Phi(\phi).$$

$$V(r,\phi) = \underbrace{A_0 \ln r + B_0}_{m=0} + \underbrace{\left[\sum_{m\neq 0}\left(A_m r^m + \frac{B_m}{r^m}\right)\left(C_m \cos(m\phi) + D_m \sin(m\phi)\right)\right]}_{m\neq 0.}$$

And then one can solve it for a cylindrical geometry. So we'll stop here for today, but we'll solve at least one or two problems in dielectrics using Laplace's equation or the solution of Laplace's equation. And it is, incidentally, almost identical for the magnetostatic problem as well. So, instead of a dielectric if you do a magnetic or a magnetized media in presence of an external magnetic field the solutions etcetera they look formally very very same. So, once we do it, we will do it only once, and then you can generalize it for the magnetic problem as well.

So, we will stop here and carry on with more electrodynamics in the next class. Thank you.