

ELEMENTS OF MODERN PHYSICS

Prof. Saurabh Basu
Department of Physics
IIT Guwahati

Lec 16: Electrostatics

We shall start a new topic. So far, we have seen classical mechanics. We have seen relativity. We have seen modern physics, and we have seen quantum physics. And now, the new topic that we're going to see is electrodynamics.

And it's broadly categorized into electrostatics, magnetostatics, and Maxwell's equations. So we will mostly see this electrostatics in some details and the magnetostatics will fall almost similar and there is a parallel that we are going to establish which will make understanding of magnetostatics to be easy and then of course we will do Maxwell's equations and if time permits we will do a little bit of wave propagation. So, in electrostatics, we have to deal with Coulomb's law, electric field, Gauss's law, electrostatic potential, and work done in bringing a charge from some far-off point—say, infinity—to a point where other charges exist or where an electric potential exists. We will talk about Laplace's equation and specifically we will use a theorem called as a uniqueness theorem along with certain boundary conditions and apply the method of images to solve these potentials basically which are solutions of the Laplace's equation. So that is the basic plan for today but it is not very clear that how far we can do but if we fall short of this plan we will continue on the next day.

So, let us do the Coulomb's law, which is, it says nothing but the force between, so there are two charges, point charges, say q_1 and q_2 , and they are separated by a distance, say r . This law says that the force between these two charges is proportional to the product of the two charges, q_1 and q_2 . and it's also inversely proportional to the distance between the two charges or the spacing between the two charges which goes as 1 over r square. Now, this kind of 1 over r square potential is there in many places, you know, particularly in gravitational field. So, it's the same kind of potential.

So, it falls off as 1 over r square. So, if the distance between the two charges increases, then the force will vary or rather decrease as 1 over R square. If you combine these two, one can write this as F is proportional to $q_1 q_2$ divided by r square. And if I use a

proportionality constant, let us say K , which is equal to $K q_1 q_2$ divided by R square, okay, where q_1 and q_2 are two point charges.

So, this is known to everyone and so basically this what is this K , K is numerically equal to if there are two charges which are of magnitude 1 coulomb each and they are separated by 1 meter of distance then K is equal to numerically equal to you know equal to the force which is expressed in Newton. K is actually in vacuum, it's 1 over $4\pi\epsilon_0$, where ϵ_0 is the permittivity of free space. We say free space or we use vacuum, they mean the same thing. Okay. And it has a value, this ϵ_0 has a value which is 8.85 into 10 to the power minus 12 .

Coulomb square, Newton inverse and meter to the power minus 2 . And so this in a medium, of course, it will go as, you know, this K in a medium will go as 1 by $4\pi\epsilon$. And this ϵ is the permittivity of the medium. So when we say a medium, we usually talk about a dielectric medium. All right.

So this 1 by $4\pi\epsilon_0$ or this K is often called Coulomb's constant. And it has a value which is 9 into 10 to the power 9 Newton meters square per Coulomb square or Coulomb to the power minus 2 . And combining these, we can write down Coulomb's law, which is equal to $q_1 q_2$ divided by $4\pi\epsilon_0 r$ square. And once again, r is the distance between the two charges. But this is only part of the story because force has to be a vector.

So we have to put a vector sign. And if we do it for the left-hand side, we have to do it for the right-hand side as well. And if you allow me to redraw this line. So this is that distance. And this, the direction of the force is in the r -cap direction, which means it's connecting the two charges.

So this is the r -cap direction. And this is the direction of the force, which is along the line joining the two charges. So this is the story for the Coulomb force. And suppose there are a number of charges—two, three, four, five charges, and so on. And you want to calculate the force between them.

Coulomb's law

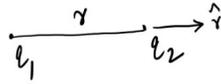
$$F \propto q_1 q_2$$

$$F \propto \frac{1}{r^2}$$

$$K = \frac{1}{4\pi\epsilon_0}$$

$$K = \frac{1}{4\pi\epsilon}$$

$$K = 9 \times 10^9 \text{ Nm}^2 \text{ C}^{-2}$$



$$F \propto \frac{q_1 q_2}{r^2} = \frac{K q_1 q_2}{r^2}$$

ϵ_0 : permittivity of free space/vacuum.
 $= 8.85 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}$

ϵ : permittivity of the medium.

$$\vec{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2} \hat{r}$$

So it will act or rather it will add up vectorially that they'll all have, you know. So there'll be a superposition of these forces between various charges. So the net charge will be one. a vectorial form or a, you know, decomposed in the coordinate system that's convenient to you. So you'll have a F x F Y, and F Z for all those charges which are interacting or rather which are feeling a force with each other.

So let's call this as an electromagnetic force. or it's really the electrostatic force, but, so let's call it electrostatic then, because these two charges are at rest, so this electrostatic force, or it's also called as the electromagnetic force, because when the magnetic field is, you know, coming into picture, and if you categorize the four forces that we have in nature, which are called as the strong interaction forces, And then we have electromagnetic interaction I mean, interaction, what we mean is that we are talking in terms of potential, but they would mean the same thing that there is also a force associated with these particles, which are, you know, interacting either strongly or they're interacting by electromagnetic interaction and so on and so forth. So, and then we have weak interactions.

We have written interaction, but you can simply change this word interaction by forces. They'll not make any difference here and gravitational interaction. And to tell you if this we scale it to 1, that is the highest force or the strongest force, if this is 1, then electromagnetic forces are of the order of 10 to the power minus 2. the weak interaction is 10 to the power minus 10 and the gravitational interactions are usually of the order of

10 to the power minus 38. So how strong like the electromagnetic interactions and the gravitational interactions if we take a ratio of them and try to calculate this ratio that is F gravity divided by F

let's call the electromagnetic by EM and do it for an electron so we have electronic charge equal to 1.6×10^{-19} coulomb mass to be equal to, you know, 9.1×10^{-31} kg. And we have given this $1 / (4 \pi \epsilon_0)$ in the last slide. So the gravitational interaction or rather the gravitational force as a scale, which is 6.6×10^{-11} Newton meter square per kg square. And so the gravitational interaction will have this form, which is $G m_1 m_2 / r^2$, along with, again, if you write it in the vector form, there will be a \hat{r} direction, which is connecting the two masses.

Electrostatic force.

1. Strong interaction (1)
2. Electromagnetic interaction (10^{-2})
3. Weak interaction (10^{-10})
4. Gravitational interaction (10^{-38})

$$\frac{F_{\text{grav}}}{F_{\text{em}}} \sim 10^{-46}$$

$$e = 1.6 \times 10^{-19} \text{ C}$$

$$m = 9.1 \times 10^{-31} \text{ kg}$$

$$G = 6.6 \times 10^{-11} \text{ N m}^2 / \text{kg}^2$$

$$F = \frac{G m_1 m_2}{r^2}$$

So, these are the two masses that we are talking about and now both the masses can be electrons. So, we are trying to calculate what is the gravitational force between two electrons and with respect to the electromagnetic force where it will be like the e^2 and then all these other things k into e^2 by r^2 and if you keep them at the same distance because the distance will cancel out if you take the ratio. And this really comes out as 10^{-46} . So you understand gravity is 10^{-46} times smaller, which is an enormously small number compared to the electromagnetic force. So electromagnetic force is quite strong, at least as compared to the gravitational force.

And as you see in this graph, ordering of the forces, it is only two orders of magnitude lower than the strong interaction. Okay, so now introduce quantity which is quite important in the context that is called as an electric field and this even though the force is not a measurable quantity, it can be you know measured by indirect means by taking a force balance equation with in an appropriate way. But what the electric field is a more sort of physical quantity that's or relatable to real situations.

So, we define this electric field as the force. So, electric field E is equal to the force experienced by a unit positive charge. which means that it's basically, suppose, so this is a force, or rather, if you have a force, then this is a force divided by the charge. We made it equal to 1 such that E becomes equal to F , but it's basically divided by the charge, so that's the electric field due to a charge. So, we can just like what we have done for the force that is $q_1 q_2$ divided by $4\pi\epsilon_0 R^2$.

We can calculate the electric field just by dividing it by one of the forces. So, if you are still talking about q_1 and q_2 , you could ask the question what is the field electric field due to q_1 at a distance r . which means where the charge q_2 is located and that is given by that is equal to the force divided by this q_1 and that will give you the electric field that exists at a distance r where q_2 is located. So, that is the electric field that q_2 is going to feel due to q_1 being present at a distance r , okay.

So, that is what is shown. I have chosen a different charge with a capital Q for a general definition, but this is how the electric field is defined, okay. So there is, you know, I mean, we are so far talking about electric field for discrete charges and Coulomb force for discrete charges. So this will, of course, tell us that this discrete case, we still have a $4\pi\epsilon_0 r^2$. And if you write it in a vector notation, it's still r^2 .

Let us see what happens for continuous charge distributions. So, you do not always have discrete charges that you find in nature. Discrete charges, like a point charge existing in nature, are not possible. So, we usually talk about continuous charge distributions, and the expressions for these continuous charge distributions are the following that you have, you know, for a line charge, or let's write it down. Sort of a general formula for the electric field for a continuous charge distribution at a given point R , which is at a distance from this charge distribution.

So, what I am trying to say is that suppose you have a linear charge distribution. So, there is a wire which has charges, you know, pasted on it, and you want to know what is the electric field due to this charge distribution. This is exactly one of the problems that we

are going to do. And the general formula for this, for that distribution, is that we have, you know, these r cap divided by r square and dq , where dq is a small element of charge. And what does this vector mean?

We have written a sort of unit vector as well as these vectors square, or rather the magnitude of the vector square here. in the denominator. So, this is equal to r minus r prime. Now, this is very important for you to remember because this is really the source point, and this is the field point. What do they mean?

They mean that the source of charge is located at R prime with respect to some coordinate system. Suppose you have a coordinate system here. So that is the origin of the coordinate system. And you are talking about a dq charge here. So this is your r prime.

And you want to calculate this electric field at this point P , which is at a distance r . from the same origin, okay. So, this distance which is the, you know, the distance between these two points or the resultant of this. So, one is r prime and this is r . So, this would be r , this curly r which is equal to r minus r prime. So that's the definition of this curly r and the unit vector associated with this curly r is that r cap vector.

So E_R is equal to 1 by 4 pi epsilon 0 r cap by r square. And if we now do it, this I've given you a line charge example, we can have a charge which is on a surface. So there is a surface charge. So everywhere there is a surface charge density. So usually a line charge density is denoted by λ , which is equal to the total charge divided by the length of the, you know, the wire or length of whichever element that the total length of the element where the charge is distributed.

And this is the surface charge and which is denoted by σ . σ is basically the total charge divided by the total area of this. So, it is called a surface charge density. Or you have a volume, okay, it is a closed volume where you have a volume charge density. This is not a, it is a sort of solid, you know, area where you have a volume charge density.

So, you have charges everywhere. So, λ we define as a linear charge density. Then we talk about σ to be surface charge density. And then we talk about ρ , which is the volume charge density. And if you use these three, that is, if you want to calculate for a linear charge or a surface charge or a volume charge, the electric fields at this point R , R is the field point where you want to calculate the field.

Electric field.

$E =$ Force experienced by a unit positive charge.

$$\vec{E} = \frac{\vec{F}}{q} = \frac{q}{4\pi\epsilon_0 r^2} \hat{r}$$

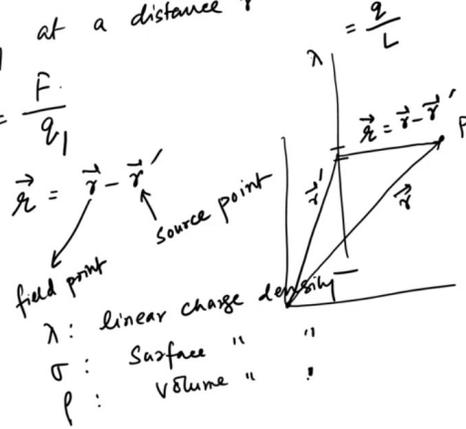
q_1, q_2

q_1 at a distance r

$$E = \frac{F}{q_1}$$

Continuous charge distributions.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda}{r^2} dq$$



So that's called a field point. R prime is the source point. So this is equal to, you know, so 1 over $4\pi\epsilon_0$, which will come out and you have a λ here. And DL prime, you know, that's a prime variable because it's inside the integral. So you integrate over all L . And so this is equal to this R square.

And and it has a direction which is. The R cap direction and that's it. So λdl prime divided by r square into r cap. It's not clear that whether λ is actually a function of, you know, is a constant or a function of maybe r prime. which means that it varies from point to point.

So, you can write down λ with this dependency on this at every source point, it may be varying in some fashion, but nevertheless, this is the form for this electric field due to a linear charge distribution. What is it for the surface charge distribution? The left hand side, of course, remains the same with 1 by $4\pi\epsilon_0$ remaining same as well. This is that σ . Once again, we assume that there is a variation there.

And then there is a ds prime divided by r squared. And this is interestingly, this would be, you know, our cap vector. But this r -cap vector in this case, so the r -cap vector is still with respect to just the way we have shown. Let me just show it once for this thing. So this is the surface charge density with a charge σ .

And this is the point where you want to calculate the electric field. And suppose you have... A coordinate system, some arbitrary coordinate system whose origin is here. Now, you take a small surface area here, which is ds . Because it is inside an integral, we take a ds prime. So, this is your r prime vector, and this is your r vector.

And the difference between the two is this vector, and the direction is in this direction. So, that—and similarly for the volume charge—it is equal to 1 over $4\pi\epsilon_0$, and we have a ρ here. R prime again, and dV prime r^2 r cap. That's the definition for all these different charge distributions. So, one can do a lot of examples with this.

Let me do one example, which is quite well known and is done in all books. So, we take an infinite sheet of charge. Okay, so it's a sheet of charge, which means that it's a surface charge density. And we say that it has uniform density. σ , okay? Which means a uniform surface charge density. So, this is the—you know—it is in the plane, and we want to calculate it at this point. Let us just, you know, draw this coordinate system like this and like this and so on, and then take some small element. Let me show it by some...

Patch here. So we have taken this to be the elemental. So that's the ds containing some charge. And so let's just show this by, you know, this is So that's your point, let's say, P , and this is θ .

And this is from the origin of the coordinate system. This is your R prime, and you want to calculate it here. So this. Is the point that you want to calculate, and let's, you know, call it. So you have this as the y -axis, and this is the x -axis.

So this point has some coordinates, which are given by x and y , and this is your z -axis. Okay, and this angle is ϕ and contains uniform charge density σ , and this is your r . Where you want to calculate this point. So, this vector that you see, let me show the vector. So, this vector is this r , and this vector is the normal r , let me write that.

So, this vector is r and that vector is r prime; this vector is r prime. If it is not clear because we have So, this is our prime and so on. OK, so this is a picture, and this is a schematic diagram of an infinite sheet. So, this looks finite, but it's actually infinite.

So, you can sort of make that distinction. And we want to calculate what the electric field is at this point due to this infinite charge distribution. So, this has cylindrical symmetry, okay? The whole problem has cylindrical symmetry because you have some—say, the board of this, the screen that you're seeing on the board—is the plane in which there is a

uniform charge density, and you're talking about a point which is vertically on top of this surface, okay? So, the coordinate of the field point is,

which is P is 0, 0, z, okay? So, it is purely on the z axis, okay? And you want to calculate it on the z axis, okay? So, E would be purely along the z axis, and this would be equal to this over the entire surface, which is S, like the sigma dx, dy, okay? and a 4 pi epsilon 0, which can stay outside if you wish, and there is an r square, and now because it is on the z axis, this angle is phi.

So, the z direction this will be along the cos phi because for this element it will be in this direction, I mean along that line. Draw it a little better. It is along this line. So, you take the projection along the z-axis. So, that is where the cos phi comes.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(\vec{r}') d\ell'}{r^2} \hat{r}$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\vec{r}') ds'}{r^2} \hat{r}$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}') dV'}{r^2} \hat{r}$$

Infinite sheet of charge with uniform σ

Coordinate of the field point $P = (0, 0, z)$

$$E_z = \frac{1}{4\pi\epsilon_0} \int_S \frac{\sigma dx dy}{r^2} \cos\phi$$

$$\cos\phi = \frac{z}{r}$$

Phi angle is shown. σ is the total uniform charge density, and dx dy is that small area painted in red. And this we want to calculate the electric field at the point P due to the small element of charge that is there. So, if you look at the figure, cos phi is nothing but equal to z over r, this r. So, that is the angle because this angle is making a right angle at the origin and we also have r squared. this is equal to x square plus y square plus z square.

That is when it is resolved in the three coordinate systems. And this x square plus y square is actually equal to rho square in the cylindrical coordinate system. So, we write it

in terms of ρ ϕ z . So, it is $\rho^2 + z^2$. So, your $\cos \phi$ is again z/r . So, E_z which is equal to these expression that we have written, now we have to do an integration over.

So, minus infinity to plus infinity and minus infinity to plus infinity, we have $\int dx dy$ and then z , okay, we can write the z here, and then we have a $1/4\pi\epsilon_0$, and now we have these $x^2 + y^2 + z^2$ whole to the power $3/2$. There is a bit of a problem in doing this integral because it will look like that it is not a very convenient integral to do because these limits are minus infinity to plus infinity and you have a $dx dy$ in the numerator with these x^2 , y^2 and z^2 . But if you transform it into the polar coordinate system, it becomes easy because your $dx dy$ becomes equal to $\rho d\rho d\theta$, right? And because if you do this integral, it will give you this πr^2 , okay?

So, $\rho d\rho$ will give you a $\rho^2/2$ and θ is from 0 to 2π or this is like, you know, ϕ actually. So, this is $\rho d\rho$ and we do not have, I mean, we have of course, written down a θ , but this is actually θ because that is on the plane $\rho d\rho d\theta$. We have shown $d\theta$ here. θ is the angle that you see which these small element makes with the x axis.

So, it is $\rho d\rho d\theta$. So, E_z becomes equal to 0 to 2π and 0 to infinity and $\int \rho d\rho d\theta$ and then you have a z there and then you have a $\rho^2 + z^2$ whole to the power $3/2$ and if you take this $\rho^2 + z^2$ equal to some other variable say that is equal to this ξ^2 , then this becomes equal to a $\pi \int z$ and we have a $4\pi\epsilon_0$ that I have forgotten here. So, it is 4π

ϵ_0 , and we have this integral from z^2 to infinity, and this is z^2 to the power minus $3/2$ $d z$, and this is equal to a constant which is $\sigma/2\epsilon_0$. So, this is an interesting result, and why it is interesting is that if it is an infinite sheet, whether you take it at a distance h or at a distance $2h$ or anywhere you take this point where you want to calculate the field, it is always a constant and has a value which is $\sigma/2\epsilon_0$. And so, this is the one that makes a lot of things easy and will again derive the same result when we get into Gauss's law. So, now let us go to Gauss's law and sort of look at various things that are important.

$$\begin{aligned}
 r^2 &= x^2 + y^2 + z^2 = \rho^2 + z^2. \\
 E_z &= \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sigma dx dy z}{(x^2 + y^2 + z^2)^{3/2}}. \\
 dx dy &= \rho d\rho d\theta. \\
 E_z &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^{\infty} \frac{\sigma \rho d\rho d\theta z}{(\rho^2 + z^2)^{3/2}}. & \rho^2 + z^2 = \xi^2. \\
 &= \frac{\pi \sigma z}{2\epsilon_0} \int_{\xi=z}^{\infty} \xi^{-3/2} d\xi = \frac{\sigma}{2\epsilon_0}.
 \end{aligned}$$

So, what is Gauss's law? So, Gauss's law gives you So, it gives you a handle to calculate the electric field, and it says that the closed surface integral of this quantity is equal to the Q enclosed, whatever Q is enclosed, divided by epsilon 0. So, this is called the flux of the electric field. E is the electric field, E dot ds is the flux.

So, whenever you multiply or rather take a dot product of the vector with the surface and surface has a direction which is unit drawn normal. So, suppose we talk about a sphere and want to calculate the N cap. So, the N cap is in this direction, which is, so there is an element of surface, say ds, and the direction is in the unit drawn normal. For a sphere, this becomes the same as R cap, which is a radial vector that you know connects this.

But say you take about you take a flat sheet and you want to calculate what is ds that is the outward drawn normal. So, ds is the vector which is along this n cap direction which is the unit drawn normal. So, this normal direction so, it is a flux of the electric field and the direction of that you know the surface area. is a unit drawn normal and E is also a vector which has a direction. Somehow, if these two directions are orthogonal, then of course, that there is no flux of the field in that direction.

But at least in this problem with spherical symmetry, the R and the N cap are in the same direction and they will sort of, you know, just the dot product will just behave like a scalar quantity. In any case, whatever the charge is enclosed in this region, maybe there is a charge at the origin or there may be a charge here with q1 and with q2 and with q3. So,

the law says that the flux of the electric field is equal to the total charge enclosed, which in this case is equal to q plus q_1 plus q_2 plus q_3 and all that, divided by ϵ_0 . If one of them is negative, say it is a minus q_2 , we will have a minus q_2 here.

So, that is the net charge or the enclosed charge through this or the flux of the electric field through this area is equal to the charge enclosed divided by the ϵ_0 . So, now there is one relation about the divergence of the electric field that will emerge if you take this equation, the left-hand side of this equation or Gauss's law and use Gauss's divergence theorem. And what is divergence theorem? Divergence theorem says that this elementary vector calculus, you can look it up, it says that

For any arbitrary vector, which is a sort of good vector, meaning it has all the properties of a vector and so on. This is equal to the divergence of A and dV , okay? So, this is like a closed surface integral, and this volume actually defines the surface. So, basically, if you have—if you are talking about, you know, a sort of cylinder, okay?

So, the surface in this case is the curved surface; let us call that S_1 , the curved one, and this is the flat surface DS_2 or S_2 , and this is another surface, S_3 , which is below. In each of the directions, the arrow shows the vector. So, these are the surface and the volume is this entire cylinder that you have. So, this is that volume and the surface area are these three surface areas that you see. For a sphere, it's very easy because that's the surface area, and the volume is basically what the sphere encloses.

So, the surface actually covers the area of the volume that you are talking about. Now, if this is true, so you need to take a divergence and I will not go too much into this divergence gradient and curl. But just to tell you in one line: gradient—let us start with the gradient. Gradient is actually the direction in which a function changes the fastest. If you talk about a hill,

So, at the topmost point, you can actually calculate the gradient to know which direction it is changing the fastest. Divergence is how a vector diverges—whether it has any divergence or it diverges. It does not diverge at all. So, that is given by the divergence. So, if you take a divergence of a vector, arbitrary vector r , which is monotonically, you know, going away from the origin, it is $x \hat{i}$ plus $y \hat{j}$ plus $z \hat{k}$.

This is equal to 3 because it diverges equally in all the x , y , z directions. So, it is how just like this you know the for a point charge if you draw the field lines that just looks like this. So, that is the divergence of the field. So, that is how electric field behaves in the

vicinity of a point charge and that is exactly what \mathbf{r} vector also looks like and we have showed that I mean we have shown that this direction of the electric field is the \mathbf{r} cap direction which is the in the radial direction. And the last thing that is curl, curl is a measure if the vector circulates in space, okay.

If it does not circulate at all, then the curl of it is 0, and this curl of a vector being 0 has other consequences. It is said that if the curl vanishes, then the vector field is conservative. So, the closed line integral of that vector field which means the work done due to that vector field over a closed path is equal to 0. So, the word 'conservative' when curl of a vector field equals 0—say, curl of \mathbf{A} equals 0—means the vector field is conservative. And it means it also means that the closed line integral of the vector is also 0, which means it does not do any work if it goes in a closed path and comes back to the same point.

So definitely, you know, our vector sort of diverges in space and it doesn't curl around. And if you want to, you know, in real life, if you want to understand where this curl arises, if you open a tap and look at the water passing out through the nozzle, you'll see that the water field or the vector associated with the water flow circulates as it goes down through the nozzle and so on to the outlet. So, the curl of \mathbf{r} definitely is equal to 0. So, as soon as you see a vector field which has a direction which is like \mathbf{r} cap or in a given direction in space, then that vector has a curl to be equal to 0. So, by that argument, it is easy to see that all these forces or all these fields are

that we are talking about the field associated with the gravitational interaction or the field associated with the electromagnetic interaction with both of these are, you know, they have a direction which is connecting the two charges or connecting the two masses, the curl of that field is 0. So, curl of these electrostatic fields or, you know, other fields which has a dependence or rather which has a direction which is \mathbf{r} cap, that is equal to 0. So this tells us that if we apply this theorem—coming back to Gauss's law—if we apply Gauss's divergence theorem, then we get that the left-hand side becomes equal to the divergence of \mathbf{E} . $\int \mathbf{E} \cdot d\mathbf{V}$, it is over a volume. So, there is a volume here, and this is equal to Q enclosed over ϵ_0 .

And if we are allowed to write this Q enclosed to be equal to some ρ dV for a you know, continuous charge distribution. Now, you see that the divergence of \mathbf{E} minus ρ over ϵ_0 and over some arbitrary volume equals 0. So, $\int dV$ equals 0. Now, that tells

us that if this integral is equal to 0 for any arbitrary elemental volume, then the integrand has to vanish.

Gauss' law

$$\oint \vec{E} \cdot d\vec{S} = \frac{q_{enc}}{\epsilon_0}$$

flux of electric field

$$= \frac{q + q_1 - q_2 + q_3}{\epsilon_0}$$

Use Gauss' divergence theorem $\oint \vec{A} \cdot d\vec{S} = \int_V (\vec{\nabla} \cdot \vec{A}) dV$

$$\int_V (\vec{\nabla} \cdot \vec{E}) dV = \frac{q_{enc}}{\epsilon_0}$$

$$= \frac{1}{\epsilon_0} \int_V \rho dV$$

$$\int_V (\vec{\nabla} \cdot \vec{E} - \frac{\rho}{\epsilon_0}) dV = 0 \Rightarrow \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$\vec{\nabla} \cdot \vec{r} = 3$
 $\vec{\nabla} \times \vec{r} = 0$

So, that tells us that divergence of E is equal to rho by epsilon 0 and this is another form of Gauss's law. The integral form is given by the first line of this slide. And the differential form is given by the divergence of E, which is del dot E equal to rho over epsilon 0. So, this is in SI unit. If you read an old book, which is written in CGS units, you will see instead of a 1 over epsilon 0, you see a 4 pi.

So, it is a 4 pi rho. OK, so let us, you know, just whatever we have seen, let us see that once more. And that is that infinite sheet of charge. And. So, once again, that infinite sheet of charge, I will just now draw it like this, but it is the same thing.

It is just infinite sheet of charge containing charge sigma. Now, what we do is that, let me write that sigma somewhere and we want to apply Gauss's theorem. So, there is an example. So, what we do is that we construct a pillbox. You know, the boxes which one uses for keeping the medicine, that's called as a pill box.

And this is slightly, you know, going above and, you know, slightly going below. OK. And this is where the interface is. So it's slightly jutting out of the plane on the upper side. And it's also jutting out a little on the lower side.

So this is like that small thing. And the plane goes through in between. OK. So we have this, you know, for the surface. This is the D . This is the end cap.

And for this other surface, this is the direction of this end cap. OK. And the electric field will be like this here in this direction, and the electric field will be like this in this direction. So this is called a pillbox. And this will act as a Gaussian surface for us.

to apply Gauss's law which we have learned for electrostatics. So this is the Gaussian surface and the whole point about this surface is that it is a closed surface such that we can apply this Gauss's law. and that because it's a uniform charge density so that the surface of that thing like here it contains that σ the charge density. So the charge that is enclosed contains that same charge density that is there. So it's $E \cdot ds$

This is equal to Q enclosed over ϵ_0 , which is σ into A , where, you know, A is just the area of the pillbox, the top area of the pillbox. And indeed, the pillbox is actually cylindrical in shape. So it has also a curved surface, but the curved surface is actually along the plane. And because it's along the plane and the electric field is in the perpendicular direction, which is out of the plane, the curved surface does not give any contribution to this flux. OK, so this flux occurs.

gets no contribution from the curved surface. It is only the top surface which is little jutting out from the plane and the bottom one which is slightly coming down from the plane, they contribute and we have, so this is σA by ϵ_0 . So, we now are convinced that for this pillbox the curved surface do not contribute and the two top and the bottom they contribute and so this gives us its equal to you know I mean this $E \cdot ds$ is really equal to twice A of E this is equal to σA by ϵ_0 . So, we do not need to write this, maybe we can just simply write it as A , because we know that A and ds are along the same direction.

So, the dot product will be the same, I mean, just like a scalar. So, E is equal to σ divided by $2 \epsilon_0$, the result that we got. And if you want to write it in the vector form, it is σ divided by $2 \epsilon_0$ \hat{n} cap, the result that we just saw. It is σ divided by $2 \epsilon_0$.

Example

pillbox \equiv Gaussian surface.

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q_{enc}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

$$\oint \vec{E} \cdot d\vec{S} = 2 A E = \frac{\sigma A}{\epsilon_0}$$

$$E = \frac{\sigma}{2\epsilon_0}$$

$$\vec{E} = \frac{\sigma}{2\epsilon_0} \hat{n}$$

We have not put the direction here, but the direction is perpendicular to that. So, instead of doing this integral by, you know, all this change of variables and trying to do an integral, the problem is much simpler here where you simply apply the divergence theorem. and get the result. So, let me take another note of this divergence of E. So, E is equal to 1 divided by 4 pi epsilon 0 And we have this r cap divided by r square rho prime dV prime.

And just once again, to make this notation clear, it is r minus r prime, and this r cap is equal to r minus r prime divided by r minus r prime, okay? So, that is the unit vector, okay? So, the divergence of E is simply the divergence of this quantity. And so, it is the divergence of R cap divided by R square and rho of R prime dV prime. Interestingly, you know, this divergence only acts on the R variable.

So, which means that it only acts on the field variable, not on the space variable. And so we need to only consider this divergence and do not need to worry about the divergence of rho. Which may still exist, okay? But we do not need to talk about that. So, we have this term, which is the divergence of r cap by r squared.

Now, without proof, I will say that this is a definition of the delta function, okay? So, the delta function means that, so delta of r is equal to infinity at r equal to 0, okay? I mean, so at r equal to 0 means x, y, z or r theta phi, whatever you say is 0. But the good thing is that it is an integrable function, which means that delta r d cube r, this is equal to 1. And not only that, if you take any other function and consider this r minus a, I am writing it in terms of scalar notation and let us just write it as a one-dimensional thing.

So, this is equal to, or rather this is f of r, not f of a. So, this is equal to f of a minus infinity to plus infinity minus infinity to plus infinity. So, even if the function diverges, the integral remains finite. So, that is the property of this delta function, and the divergence of r, this unit vector r, curly r by this curly r squared, is equal to a 4 pi delta function. So, the divergence of E now then comes out as divergence of E if you put it.

$$\begin{aligned}
 &\text{Divergence of } \vec{E} \\
 \vec{E} &= \frac{1}{4\pi\epsilon_0} \int \frac{\hat{r}}{r^2} \rho(\vec{r}') dV' & \hat{r} &= \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|} \\
 \vec{\nabla} \cdot \vec{E} &= \frac{1}{4\pi\epsilon_0} \int \vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right) \rho(\vec{r}') dV' & \hat{r} &= \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|} \\
 &\text{Only acts on } \vec{r} \text{ variable} & & \\
 \vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right) &= 4\pi \delta(\vec{r}) & & \\
 \vec{\nabla} \cdot \vec{E} &= \frac{4\pi}{4\pi\epsilon_0} \int \delta(\vec{r}) \rho(\vec{r}') dV' = \frac{\rho(\vec{r})}{\epsilon_0} & & \\
 & & & \left. \begin{aligned} \delta(\vec{r}) &= \infty \text{ at } \vec{r} = 0 \\ \int \delta(\vec{r}) d^3r &= 1 \\ \int_{-\infty}^{\infty} f(r) \delta(r-a) dr &= f(a) \end{aligned} \right\}
 \end{aligned}$$

So, 1 over 4 pi epsilon 0, we have this 4 pi which we can bring there, and we have a delta of r and we have rho of r prime. dV prime using this property that f of r delta of r minus a equals f of a. So, this is equal to rho of r, and this 4 pi will cancel. So, this is equal to rho of r divided by epsilon 0, and this is what we said can also be obtained from Gauss's divergence theorem. So, so much for the electric field and Gauss's divergence theorem. Now, we will talk about the electrostatic potential very briefly.

And what we mean by electrostatic potential is that it's basically the potential due to a single charge or charge distribution that we have. And for any of these forces we have, or any of these fields or even the force, we know that the curl of E is equal to zero because E is in the R cap direction. Okay. So, if this is 0, then we can write E to be equal to minus the gradient of a scalar function. So, the electric field can always be written as minus the gradient of a scalar function if the first line is valid, which means that if the curl of a vector field is 0, then that vector field can always be written as the negative gradient of a scalar potential.

And this tells us that this negative—I mean, the scalar potential here—is the electrostatic potential. And just like the electric field, the electric potential has a form which is 1 over

$4\pi\epsilon_0$. We have this ρ of r prime, and there is a root r , and then there is a dv prime. So do not confuse between V , which is a potential and the DV that we talk about here is the, you know, the volume element that we sort of say. OK, so I think it's clear because.

The gradient of a scalar quantity or a function, it gives you the direction in which the function changes the fastest, right? So, that is the given direction and the curl of that direction equal to 0. So, these two equations are consistent because if you put E equal to $\text{grad } V$, then curl of a gradient should always vanish, okay? So, that is why you can write this.

And it has this for a charge distribution, it has this form, okay? So, what does this physically mean? So, suppose you have a charge, you know, say you have a test charge, Q , which you transfer from a point, say P , okay, we cannot say P and Q because it is already Q . So, transfer from a point M to a point N , okay. So, the work done

In moving this charge or there has to be some work done by the potential or by the field. So, this work done is nothing but equal to this is a minus $F \cdot DL$ and from this point M initial point M to a final point N . And this is equal to a minus Q and from this point M to this point N and this is $q E \cdot dl$. $E \cdot dl$, okay. dl is the path along which it is moved and E is equal to minus $\text{grad } V$. So, the minus sign goes away and we have M to N and a $\text{grad } V \cdot dl$ and that gives you that this is equal to q into $V_N - V_M$ that is difference in potential between the two points. Okay.

Electrostatic potential.

$$\vec{\nabla} \times \vec{E} = 0$$

$$\vec{E} = -\vec{\nabla} V \quad V: \text{Electrostatic potential.}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{r} dv'$$

Test charge Q transfer from $M \rightarrow N$.

$$\text{work done } W = -\int_M^N \vec{F} \cdot d\vec{l} = -Q \int_M^N \vec{E} \cdot d\vec{l} = Q \int_M^N \vec{\nabla} V \cdot d\vec{l}$$

$$= Q (V_N - V_M)$$

Difference in potential between 2 points.

So, this is the physical meaning of this electrostatic potential, which is nothing but the change in the potential energy or difference in potential energy between two points. And how do we understand these two points? You can take a test charge and move this test charge between the two points, from one point to another. The work done will be stored as the electrostatic potential energy, and you divide it by the charge concerned—the test charge that you are talking about—that will give you the electrostatic potential. So, that is not difficult, and you have the general form of this charge distribution for a given charge distribution.

Let us see what form the electrostatic potential takes. So, this potential energy or the work done—they are the same. So, this is equal to half rho into V, V is the potential and this is a dV. So, this is the volume. So, I actually let me make this V, let me make it a little curly V with a small thing so that capital V which is potential and small v, these are sufficiently different.

Okay. And this is also equal to half of this sigma V ds, and so on and so forth. Okay. All right. So, that is the work done.

And what you can do is that you can use this rho to be, I mean, this is not required. We are just talking about the volume charge. So, rho is equal to epsilon 0 divergence of E. And if you put this W equal to, which is the potential energy of the work done, this is equal to—so there will be an epsilon 0 here and there will be a divergence of E into, in V, divergence of E, V. and dV.

And if you need to integrate this, then you do integration by parts. So, by parts, if you integrate, then this becomes equal to—so this W is equal to epsilon 0 by 2, And this E dot del V dV plus there is another term which is V E dot dS, that is the surface term and so on. So this is there. This is the general definition of that.

$$\begin{aligned}
 W &= \frac{1}{2} \int \rho V \, d\alpha \\
 \rho &= \epsilon_0 \nabla \cdot \vec{E} \\
 W &= \frac{\epsilon_0}{2} \int (\nabla \cdot \vec{E}) V \, d\alpha \\
 \text{Doing by parts.} \\
 W &= \frac{\epsilon_0}{2} \left[\int \vec{E} \cdot \nabla V \, dV + \oint V \vec{E} \cdot d\vec{S} \right] \\
 &= \frac{\epsilon_0}{2} \int \vec{E}^2 \, dV
 \end{aligned}$$

$\frac{1}{r}$ $\frac{1}{r^2}$ $r \rightarrow \infty$
 $\frac{1}{r} \rightarrow 0$

But then, you know, this term, the surface term, actually does not contribute because this goes as 1 over r and this goes as 1 over r squared, and this grows as r squared. So the whole dependence is still as 1 over r. Now, this term can go to 0 if r goes to infinity, which means that even if this work done expression or the electrostatic potential energy expression involves a volume integral and a surface integral, we can make the surface integral go to 0 by taking that surface to be at infinite distances, and we are left with only this volume integral, and this is nothing but epsilon 0 by 2 e squared dV. So, this E squared dV is the expression for the electrostatic potential energy. All right, so let us—so we have now learned about force, we have learned about electric field, we have learned about Gauss's law and how that gives flux of the electric field, we have now learned the electrostatic potential, which is nothing but the work done, and finally, at least till whatever point we can, you know, do it today, we will show that the Laplace's equation

which gives you a solution for the potential. Okay, so we have this divergence of E that's equal to rho over epsilon 0. We've also learned that curl of E is equal to 0. If you combine them, then E becomes equal to minus gradient of V, where V is the potential.

So, if you put divergence of minus gradient of V, this is equal to rho by epsilon 0. This gives that this del square V, this is equal to minus rho by epsilon 0. This is an equation called as the Poisson's equation. And this operator is called as a Laplacian. And that is why it is called as a Laplace's equation.

So, this $\nabla^2 V$ equal to minus rho by epsilon 0 is a second order differential equation for V and this has to be solved for either a given, you know, charge density and it could also be true that there is no charge density in space. So, which means that there is no charge density that exists in which case it will give you this Laplacian of V is equal to 0 and this is really called as a Laplace's equation. And just like the Poisson's equation, you have a right hand side, which is non-zero. Here, you have a right hand side, which is equal to zero. And now, whenever there is a differential equation that you need to solve, you need to have the boundary conditions.

And so we need boundary conditions. And because there's a second order differential equation, we need two boundary conditions. To solve for V. And what are the two boundary conditions? One boundary condition is that, remember, the electric field is a vector. So the electric field has components.

So we are not talking about two media separated by an interface. So there is one medium, and there is another medium. So there is a medium 1 and there is a medium 2. And this is actually not a line, but a surface. So it's like just separating two media in three dimensions.

And so the surface, the interface, is actually a surface which is two-dimensional. So we have these two boundary conditions that we have for the electric field. So it's a perpendicular component. So one component of the electric field would be along the plane, which we call the parallel component. And we'll have a component which is.

Laplace's equation : solution for the electrostatic potential.

$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
 $\vec{E} = -\vec{\nabla} V$
 $\vec{\nabla} \cdot (-\vec{\nabla} V) = \frac{\rho}{\epsilon_0} \Rightarrow \nabla^2 V = -\frac{\rho}{\epsilon_0}$ Poisson's equation.

$\vec{\nabla} \times \vec{E} = 0$

$\nabla^2 V = 0 \rightarrow$ Laplace's equation.

We need two boundary conditions

E_{\perp}

So that is a parallel component and we also have a component which is perpendicular to the field to the plane and this is called as a E_{perp} or written as E_{\perp} this this one. So that is the perpendicular component and we have this parallel component. So what are the boundary conditions? And the boundary conditions say that $E_{\text{above}} - E_{\text{below}}$, that is for the perpendicular component, this is equal to $\frac{\sigma}{\epsilon_0}$ and it is coming from this curl of E equal to 0 and divergence of E equal to, so this coming from really the divergence of E is equal to ρ by ϵ_0 .

So that's the, you know, the form for this. And divergence is the perpendicular component because when you take a divergence, you, so this is like a dot product. So we are really talking about the perpendicular component perpendicular to the plane. If, of course, σ is 0, there is no charge density on the surface, then, of course, this is continuous, which means that it will be $E_{\text{above}} = E_{\text{below}}$. okay and similarly we have you know so the V_{above} is always equal to V_{below} it does not matter whether we have these discharge at the interface or not and so $\frac{\partial V}{\partial n}$ above

minus $\frac{\partial V}{\partial n}$, n is a unit drawn normal, which is nothing but the field. This is equal to $\frac{\partial V}{\partial n}$, which is $\frac{\sigma}{\epsilon_0} n_{\text{cap}}$, which is the same as equation 1. So, this is equation 2 and this is same as equation 1, just stated in this form of, so this is really that n_{cap} direction. So, either 1 or 3, let us call this as 3 and 2. So, 1 and 2 or 3 and 2, these are the boundary conditions that we have.

And this is basically for the electric field in the parallel direction. So, $E_{\text{parallel above}}$ is equal to $E_{\text{parallel below}}$. And this is simply coming from the curl of E equal to 0. And if that is not satisfactory, then you can write it as $E \cdot dl = 0$ as well, because curl D is equal to 0, and applying Stokes' theorem, we have $E \cdot dl = 0$. So, let me just briefly state Stokes' theorem.

So, it says that for any vector field, the line integral of the vector field is the curl of the surface integral. And the relation between the surface and the line integral is that. So, there is a surface, and so there is S and this is that L or DL . So, this is 1 and this is 1. And this is 1, and we get 3 because the electric field is a vector.

Boundary conditions

$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

$\vec{\nabla} \times \vec{E} = 0$

$\oint \vec{E} \cdot d\vec{l} = 0$

(1) $E_{\perp}^{above} - E_{\perp}^{below} = \frac{\sigma}{\epsilon_0}$ if $\sigma = 0, E_{\perp}^{above} = E_{\perp}^{below}$

(2) $V^{above} = V^{below}$

(3) $\left(\frac{\partial V^{above}}{\partial n} - \frac{\partial V^{below}}{\partial n} \right) \hat{n} = \frac{\sigma}{\epsilon_0} \hat{n} \rightarrow$ Same as Eq 1.

(4) $E_{\parallel}^{above} = E_{\parallel}^{below}$

Stokes's Theorem

$\int \vec{A} \cdot d\vec{l} = \int (\vec{\nabla} \times \vec{E}) \cdot d\vec{S}$



It is important to note that, you know, I mean, you have a relation E equal to minus grad V , V is this potential, and it can be questioned how a scalar quantity—potential is a scalar quantity—how a scalar quantity can give information about the vector quantity. And the reason that it can give is that because the curl of E is equal to 0, which means that all these relations, or rather, there are additional relations between the x , y , and z components of the electric field coming from the curl equation. So it is not only E equal to minus grad V , it is also curl equal to 0. That enables us to get a complete description from a single scalar function, this information about E . Okay, so let me sort of get into this solution of Laplace's equation by the method of images, and this is really a trick.

And why is it a trick? Because it uses one very important concept: that if you can somehow find out for this solution of this V equal to 0 or equal to ρ by ϵ_0 , if you can find a solution for V , which satisfies all the boundary conditions, then that is the only function, and there is no other function. So, you have found the solution. Somehow, if you can find this V which satisfies the boundary conditions, then that is it.

I mean, there is nothing required, and this is called the uniqueness theorem. So, that uniqueness theorem guarantees that this trick that you may have applied is the right trick. Okay. So, it comes from the uniqueness theorem.

And let us take a, you know, sort of a point. So, there is a conducting plane. as the first part one does. So, there is a conducting plane, and then there is a point charge, and you

want to find the potential everywhere. Let us say the conducting plane is infinite so that you do not see what is there below. So, the q is at some distance d or A above this plane.

So, this is at a distance A there. So, you want to find everywhere here, okay, all these places that you have, you want to find the potential everywhere. Now, it could be, you could say that it is Q by $4\pi\epsilon_0 R^2$, which is what we have, or $4\pi\epsilon_0 R$, because this thing has, let me go and check whether, Yeah, this is R . So its potential is 1 over R and electric field is 1 over R^2 . But that's not going to work because this charge is going to induce charges, negative charges on the surface, assuming that Q is positive.

And there will be also potential due to these induced charges, but there is no clear-cut method to calculate these charges or these induced charges. So, what you do is that you completely change this problem to a problem where you do not have anything, you do not have a conducting plane, but you have a charge at a distance a , and you also have a charge at a distance minus a , sorry, I mean, this is, you have a charge which is minus q , again at a distance a , but in the other direction, so it is a minus q . So, this is imaginary, that is, there is no line, it just simply is a reference point, because this is where the earlier this thing was situated, the conducting plane was situated, okay. And what are the boundary conditions that we have? We have V equal to 0 at z equal to 0 .

So, we will sort of set our, so this is z equal to 0 . And we have these, so z equal to 0 is the plane, say for example. And we have this Q having, you know, X , Y , Z , etc. And minus Q has some X , Y , Z in the other direction and the plane is located at Z equal to 0 and V equal to 0 . Why?

Because it is a conducting plane. And also there is one boundary condition. We do not need the electric field boundary condition here, but we have so V equal to 0 far away from the plane. which means that far away means $x^2 + y^2 + z^2$ is much greater than a^2 . So, that is the two conditions that we have.

And if you can write down a potential function which satisfy both these conditions, then that would be the solution. And let us write down these things as $V(x, y, z)$, which is 1 by $4\pi\epsilon_0$. And we have this Q divided by root over $x^2 + y^2 + z^2$ minus a whole squared. And this is for the top charge. And then there is I don't know what these q is, but we can you can write down a q' .

root over x square plus y square plus z plus a whole square and this if you put the boundary condition that is v at x, y, z equal to 0 and for Z equal to 0 and it is equal to infinity, it is equal to infinity for X square plus Y square plus Z square greater than A square that basically the two boundary conditions that you just saw, then you will see that your q prime is equal to minus q. So, we get q prime is equal to minus q. And that is why it is called an image charge. If you look at q and minus q, we did not know initially that it was minus q. But suppose we have a q prime when we wrote it as q prime and apply the boundary conditions and you will have a q prime equal to minus q. So, now if you want to understand that what is the charge density that is induced on the surface that is minus del V del N and this N is nothing but Z here.

Method of Images. (Trick!).

↳ Uniqueness Theorem. $\nabla^2 V = 0$

1) $V = 0$ at $z = 0$.

2) $V = 0$ far away; $x^2 + y^2 + z^2 \gg a^2$.

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{x^2 + y^2 + (z-a)^2}} + \frac{q'}{\sqrt{x^2 + y^2 + (z+a)^2}} \right]$$

$V(x, y, z) = 0$ for $z = 0$
 $= \infty$ for $x^2 + y^2 + z^2 \gg a^2$

And where is it induced? It is induced at z equal to 0. So, we will change this n to be equal to z and if you do a del v del z, you get 1 over 4 pi epsilon 0. And we have these minus q z minus a and x squared plus y squared plus z minus a whole square whole to the power 3 by 2.

and plus QZ plus A and so X square plus Y square plus Z plus A whole square. So this, so Z plus A whole square and we have a 3 by 2. And now if you put del V del Z and calculate at Z equal to 0, You will get it at, so this is sigma, which is a function of xy, which is minus qA divided by 2 pi x squared plus y squared plus z squared. and 3 by 2 and so on.

So, this is a negative charge. So, the induced charge is negative because of the negative sign and so on. And if you want to calculate, so if you want to do a sigma ds, that is a total induced charges, you will still get all these things. If you put and transform it into exactly the same way that we have done it for these cylindrical coordinate system, you will get it as minus q. So, the upshot of this problem is that this potential that you have written down purely by changing the problem from a q in front of a point q in front of a plane conducting plane.

$$\begin{aligned}
 q' &= -q \\
 \sigma &= -\epsilon_0 \left. \frac{\partial V}{\partial z} \right|_{z=0} \quad n \equiv z \\
 \frac{\partial V}{\partial z} &= \frac{1}{4\pi\epsilon_0} \left[\frac{-2(z-a)}{[x^2+y^2+(z-a)^2]^{3/2}} + \frac{2(z+a)}{[x^2+y^2+(z+a)^2]^{3/2}} \right] \\
 \left. \frac{\partial V}{\partial z} \right|_{z=0} &= \sigma(x,y) = \frac{2a}{2\pi(x^2+y^2+z^2)^{3/2}} \\
 q_{\text{ind}} &= \int \sigma \, dS
 \end{aligned}$$

by removing the plane completely and putting two charges say q and q prime later on q prime came out to be minus q and wrote down these potential functions purely from this you know by this trick applying this trick it satisfies the boundary conditions that you need that is because it is a conducting plane, you have V equal to 0 on the plane itself, and V at very far away should also be 0 because there will be no effect of the potential in the far region. So, that is the only solution one has. As a last problem, you can do it for a sphere, and this sphere is also simple. I will just show you the

So, there is a sphere conducting sphere say of radius r and you have a charge which is again put at a distance you know maybe a from the center of the sphere and now again it is a sort of say earth sphere where potential is equal to 0. And this 0 here. So what you do is replace this problem and write only this—the sphere. And there is this charge here, which is the original charge. And then there is a charge somewhere inside.

which let us call it as a q prime. And these distances are important. And we want to calculate the potential at any arbitrary point. And these arbitrary points are these things. So, we have these—let us call this distance B .

And this distance is r , and this angle is θ . This is r prime, this is r , and I remove the sphere completely. The sphere was drawn as a reference for me to know where this q prime is located. See, the q prime cannot be located outside the sphere because that would alter the potential. So, the Q prime will actually be inside the sphere. Now, there is no sphere, but we have to understand this—or rather, visualize this sphere to still be there—so that Q prime is inside.

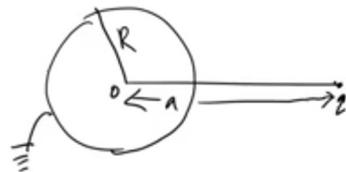
And this distance is, of course, A . And if you again write down V of q , sorry, V at some r , which is $\frac{1}{4\pi\epsilon_0}$. And this is the Q over r plus a q prime over r prime. And then this, in this spherical polar coordinate system, this is $\frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{r^2 + a^2 - 2ar\cos\theta}}$ and so this is and then there is a q prime which is equal to $r^2 + b^2$ the distance b is shown that is a small distance.

$-2Br\cos\theta$, and then we have this. So, that is the potential, and we have to see whether this potential satisfies the boundary condition. We have two unknowns: Q prime and B are unknowns. And how can we calculate this? We take two useful points, and we can take any θ —that is, any θ on the surface of the sphere. But we take two θ s which are useful.

And these θ s which are useful are θ equal to 0 and θ equal to π , in which the cosine takes values which are simple, either it is 1 or -1 . So, if you take these two values, then q divided by a minus r , plus q prime divided by r minus b , this is equal to 0 . So, this is for θ equal to 0 . And we also have q divided by a plus r plus q prime divided by r minus b , this is equal to 0 as well.

So, this is for θ equal to π . Now, you see this equation 1, this satisfies the boundary condition because at small r equal to capital R , it will go to 0 . And, of course, at very large points or very far-off points, it is still equal to 0 . So, if you solve these two equations, what you get is that you get a Q prime equal to $-\frac{r}{a}$ into q and b equal to $\frac{r^2}{a}$. OK.

Sphere



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{r} + \frac{q'}{r'} \right]$$

$$= \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} + \frac{q'}{\sqrt{r^2 + b^2 - 2br\cos\theta}} \right]$$

q', b are unknowns. (1)

$\theta = 0$ & $\theta = \pi$

$$\frac{q}{a-R} + \frac{q'}{R-b} = 0$$

$$\frac{q}{a+R} + \frac{q'}{R-b} = 0$$

$$\theta = 0$$

$$\theta = \pi$$

So, both our assumptions are satisfied. q' has a negative sign, which means the image charge inside the sphere—that imaginary sphere—is negative or rather has the opposite sign. It has an opposite sign with respect to q . And where is it located? It is located, which is inside, you know, the sphere that distance b is actually less than r . Okay, so because a is greater than r . And this is how the entire problem has been changed from the solving of the Laplace's equation.

$$q' = -\frac{R}{a} q$$

$$b = \frac{R^2}{a}$$

We did not have to do it directly, but what we did was apply a trick. And write down a potential function that satisfies all the boundary conditions. And this is a very nice sort of trick that works because of the uniqueness theorem, which says that is the only potential. If you have found one, that is the only potential. So, we will stop here and we will continue the next day.

Thank you. Thank you.