ELEMENTS OF MODERN PHYSICS

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Lec 15: Clebsch-Gordon Coefficients

We shall be doing angular moment addition of angular momentum in particular Clebsch-Gordon coefficients how to calculate them and see one or two examples and then we will see more examples in the tutorials or these assignment problems. So, just to remind you that we have looked at angular momentum which are like this L which is called as the orbital angular momentum. and then we have also looked at you know spin angular momentum and then we also have looked at J which is the total angular momentum and J becomes an important quantity when there are these L couples to S which is called as a spin orbit coupling. And in each one of these angular momentum, these variables, they have different relations like the relations are mostly, you know, expressed as L cross L is equal to IH cross L. That is a commutation relation, which means that any of these LI, LJ, the components of the angular momentum, which has a relation which is say IH cross epsilon IJK LK. In case we have missed an h cross, it really does not matter.

I mean, sometimes h cross is put equal to 1, but nevertheless, it is important because it sets the scale for the angular momentum. And similarly, we have S cross S, that is equal to ih cross S. And similar relationships follow as well that it's Si Sj commutator is i h cross epsilon ijk. On this epsilon ijk, we have explained that before. It's called the Levi-Civita symbol.

If it's clockwise—that is, ijk maintaining an order—then it's equal to 1. If you flip any of the pairs, that is j and j, if you make it j and i, then it picks up a negative sign. And if any of the two indices are made equal to same, then the leverage of it is symbol that is epsilon ijk is equal to 0. And this is all told and so j is equal to L plus S and this is equal to ih cross j. and we have this J i J J which are two different you know components which is again epsilon i J k J k and so on okay.

So these are the commutation relations that we have seen. Now we talk about the addition of angular momentum, and in particular, we will just talk about J. But you know, these addition rules are sort of valid for all these other two angular momenta as well. So, we will talk about addition, and these additions mean that we are talking about J equal to, you know, J1 plus J2. So, these are two particles.

So, 1 and 2 correspond to two particles. So, we have these: the total J is equal to the sum of J1 and J2, and this quantum number corresponding to J is called j, and the quantum number corresponding to J1 is called small J1, and J2 corresponds to J2. Similarly, the Jz corresponds that corresponds to m, J1z corresponds to m1 and J2z corresponds to m2. So, these are the notations that we have, and we will see what these Clebsch-Gordon coefficients mean in the context of this addition of angular momentum.

$\vec{L}$: orbital angular momentum

$\vec{S}$: Spin " "

$\vec{J}$: Total " " " $= \vec{L} + \vec{S}$.

$\vec{L} \times \vec{L} = i\hbar \vec{L}$ $\qquad [L_i, L_j] = i\hbar \, \varepsilon_{ijk} L_k$

$\vec{S} \times \vec{S} = i\hbar \vec{S}$ $\qquad [S_i, S_j] = i\hbar \, \varepsilon_{ijk} S_k$.

$\vec{J} \times \vec{J} = i\hbar \vec{J}$ $\qquad [J_i, J_j] = i\hbar \, \varepsilon_{ijk} J_k$.

Addition of angular Momentum $\qquad$ 1, 2 : 2 particles.

$$\left.\begin{array}{l} \vec{J} = \vec{J}_1 + \vec{J}_2 . \\ \vec{J} \to j , \quad \vec{J}_1 \to j_1 , \quad \vec{J}_2 \to j_2 . \\ J_z \to m , \quad J_{1z} \to m_1 , \quad J_{2z} \to m_2 \end{array}\right\}$$

Now, it is true that you know the vectors, the J vectors, they actually commute corresponding to the two particles, but there is a problem: J square does not commute with either J1z or J2z, nor does it commute with J square and J2z. And this is not equal to 0. So, either of them is not equal to 0 because the J square contains J1 square plus J2 square plus there is a 2 J1 J2, and this 2 J1 J2 will have terms such as J1x J2x, J1y J2y, and J1z J2z. Now, of course, the z component will commute with this J1z or J2z, but the x and y components will not commute with these J with this J1z or J2z, and that is why these commutation relations are not equal to 0. And so, let us say, you know, talk about these eigenfunctions corresponding to J1 and J2.

So, these eigenfunctions are are, so this corresponding, I mean, the first one that j1 is equal to j1 m1, which is what we have said just in the last slide. And this for the j2, this will be like a j2 m2. If you are a little more careful because we have this m, the magnetic quantum number that takes values of minus J to plus J or here m1 will take minus j1 to plus j1. But there are these M's corresponding to the L.

That's the orbital part and then the spin part and then this J part. So here, this is what is called as M here. So we are really talking about J, the total angular momentum vector. So we are using this MJ to be equal to M. But remember that if you really want to deal with all three of them, it's best that you the magnetic quantum numbers carry their, you know, respective indices which are either L, S or J and so on.

All right. So, this j1, this takes values which are this corresponding to this. So, the m1 will So, j1 will have some values which are integer values and then m1 will have values which are from j1 minus of j1 to plus of j1 through 0 and m2 will have values which are minus of j2 2 plus of j2.

Let me put the plus sign explicitly and through 0. So, it is like minus j1 minus j1 plus 1 minus j1 plus 2 and all the way to 0 and then so this all the way all this is equal to plus j1. So, this will be equal to the 1 that is before that is equal to j1 minus 1 and so on. So, this will be like 1, 2 and all that okay. So, these are the values and similarly for m2.

So, this these are the values for m1 and similarly you will have values for m2 okay. So, we basically you know we can construct product state for these composite operator. So, we will just tell you what it mean by product states from the single particle states. And in some sense, there is no interaction between these two particles that we are considering.

And that is why we can construct this product states. So, what we mean by product states is the following. So, we have corresponding to particle 1, we can have j1, m1 and written as a ket. And this is going to be multiplied by j2 and m2. That's another kit.

And there is a product. You can write it like this. And this can be written as as a composite state or like a product state has a representation, which is like a j1, m1, j2, m2. You can put a comma in between in both the sides between the indices. In fact, it is probably a good to put a comma everywhere so that we know that these are distinct indices j1m1 and j2m2.

$$[\vec{J}_1, \vec{J}_2] = 0.$$

$m \to \underbrace{m_l, m_s}, m_j \to m$

$$[\vec{J}^2, J_{1z}] \text{ or } [\vec{J}^2, J_{2z}] \neq 0.$$

Eigenfunctions of $\vec{J}_1$ and $\vec{J}_2$ $\quad |j_1 m_1\rangle$ and $|j_2 m_2\rangle$

$m_1 \to -j_1$ to $+j_1$ $\quad m_1 \Rightarrow -j_1, -j_1+1, -j_1+2 \ldots 0, 1, \ldots j_1-1, +j_1$

$m_2 \to -j_2$ to $+j_2$ $\quad$ Similarly values for m_2.

Construct product states from single particle states.

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle = |j_1, m_1, j_2, m_2\rangle$$

So what we have is the following that we have this j1 square. This is that thing acting on j1 m1 will give us a J and it is for the product state as well. I mean this can be written as the complete composite states. j2 m2 and this is equal to j1 into j1 plus 1 h cross square and then it gives you this j1 m1 j2 m2 okay and similarly for the j1 Z on acting on this j1, m1, j2, m2, this is equal to m1, h cross and then j1, m1, j2, m2.

Okay, so that is very clear and similar relations for J2. j 2 and j 2 z or j 2 square and j 2 z you can write that okay. So, now what we do is that so let us work with the composite state. So, it work with that and if you do that then let us write down the composite operator which is equal to j 1 z j 2 z and let us see what that is. So, j 1 m 1

j2 m2. So, this is equal to a J1 Z plus a J2 Z acting on this j1 m1, j2 m2. And this is equal to m1 plus m2 h cross j1 m1 j2 m2. What I am trying to convince you is that this composite state we have formed is a good description of the problem, a good basis, or it can be called an eigenbasis for this problem—that is, to work with this total angular momenta comprising these two particles.

So, this is a sort of basis that makes sense to us. And now you also, you know, sort of you can write it as the J square acting on these. So, J square is if you want to act on these j1, m1 and j2, m2. Now, that is not equal to, say, h cross square if I take common—it is not equal to J1 into J1 plus 1 plus J2 into J2 plus 1 and these states. So, this is incorrect.

So, this is not correct. So, if you write an equality, it will be incorrect. The reason is that we have told that for this J square, this basis is not a correct basis, okay? But it works fine for these, you know, each of these J1 square and J2 square and J1z and J2z. The reason this is not a good basis is that if you expand, you

The j square, this is equal to a j1 plus a j2 whole square and this will give us a j1 square plus a j2 square for which it will work fine but not for the 2 j1 j2. Now, J1 and J2 commute, that is why we can combine the cross terms J1 dot J2 and J2 dot J1, they can be combined and we can write a 2 J1 dot J2. Now, this term is the problematic term because of which these basis is not sort of, you know, an appropriate basis. So now if that is the case, we have these terms, the 2 J1 J2, let me write it explicitly. So this is equal to, let us forget the 2 factor at the moment, but we have a J1 X J2 X plus a J1 Y J2 Y and plus a J1 Z J2 Z.

$$\vec{J}_1^{\,2}\,|j_1, m_1, j_2, m_2\rangle = j_1(j_1+1)\hbar^2\,|j_1, m_1, j_2, m_2\rangle.$$

$$J_{1z}\,|j_1, m_1, j_2, m_2\rangle = m_1\hbar\,|j_1, m_1, j_2, m_2\rangle.$$

Similar relations for $\vec{J}_2^{\,2}$ & J_{2z}.

Composite state

$$J_z\,|j_1, m_1, j_2, m_2\rangle = (J_{1z} + J_{2z})\,|j_1, m_1, j_2, m_2\rangle$$
$$= (m_1 + m_2)\hbar\,|j_1, m_1, j_2, m_2\rangle.$$

$$\vec{J}^{\,2}\,\underbrace{|j_1, m_1, j_2, m_2\rangle} \neq \hbar^2\left[j_1(j_1+1) + j_2(j_2+1)\right]|j_1, m_1, j_2, m_2\rangle$$

$$\vec{J}^{\,2} = (\vec{J}_1 + \vec{J}_2)^2 = \vec{J}_1^{\,2} + \vec{J}_2^{\,2} + \underbrace{2\,\vec{J}_1\cdot\vec{J}_2}$$

$$\vec{J}_1\cdot\vec{J}_2 = \underbrace{J_{1x}J_{2x} + J_{1y}J_{2y}} + J_{1z}J_{2z} \rightarrow \text{do not commute with } J_{1z} \text{ or } J_{2z}.$$

and this does not seem that these two are going to non or do not commute with either J1 Z or J2 Z. Okay, so that is the problem. So, J square does not commute with J2 J1 z and J2 z. So, let me write that down that is important does not commute with J1 z and J2 z and it is an important thing for us. This, you know, emerges two descriptions. What is the other description now?

In case, so we have looked at one description j1, m1, j2, m2. This is one description. But there is another description that is going to work for this thing, which is like a j1, j2, a j m, right? That is going to work. This is also a good description.

or good basis of the problem for this total angular momentum of two particles, the sum of the angular momentum. This is also a good basis. And why is this a good basis? This is a good basis because your, you know, J square would acting on this j1, j2, j m. This is going to give us a j into j plus 1 h cross square and returns me this basis j1, j2, j m.

And also my the total the this j1 z and j2 z the sum of that is which is j1 j2 jm this is equal to you know sort of m h cross and j1 j2 jm. If I forget giving this comma, please do put them. So, this is also a good description. This is exactly what I said. And we have also shown earlier that this is a good description as well.

That is j1, m1, j2, m2. And now we see equally a good description that j1, j2, jm. So, these quantum numbers, the quantum number j1, j2, m1, m2, as well as the quantum numbers J1, J2, Jm, both are the real eigen basis, so to say, for this problem comprising of this J, which is, you know, the sum of the two angular moment of the two particles, okay.

So, for this, once again, this is a bit of repetition that I am doing, but it is probably worth it because you need to understand these well. So, this is equal to J1 square. J2 square, these are the operators, okay? And J1z and J2z, so these are the corresponding operators. And J1, J2, Jm, we have these as operators: J1 square, J2 square, total J square and total Jz. So, these are the operators, and these are their eigenvalues. Both seem to be satisfying or both are good descriptions.

$$\boxed{\vec{J}^2 \text{ does not commute with } J_{1z} \text{ and } J_{2z}.}$$

$$\underbrace{|j_1, m_1, j_2, m_2\rangle}_{\text{good basis}} \rightarrow \underbrace{|j_1, j_2\, j, m\rangle}_{\text{good description.}}$$

$$\vec{J}^2 |j_1, j_2, j, m\rangle = j(j+1)\hbar^2 |j_1, j_2, j, m\rangle$$

$$J_z |j_1, j_2, j, m\rangle = m\hbar\, \underbrace{|j_1, j_2, j, m\rangle}$$

$$\boxed{\begin{array}{lll} |j_1, m_1, j_2, m_2\rangle & \longrightarrow & \vec{J}_1^{\,2},\ \vec{J}_2^{\,2},\ J_{1z},\ J_{2z}. \\ |j_1, j_2, j, m\rangle & \rightarrow & \vec{J}_1^{\,2},\ \vec{J}_2^{\,2},\ \vec{J}^2,\ J_z \end{array}} \Rightarrow \text{Both are good descriptions}$$

Now, if for a given problem, there are two descriptions which are equally valid, and they are, you know, they are the eigenbasis for the problem. Then there has to be a unitary transformation that connects these two bases. And these unitary transformation just let me remind you that if there is a unitary operator which connects say a vector you know to another vector say U y and then you have this U dagger U this is equal to 1 or U U dagger is equal to 1. So, we should have a unitary transformation connecting the two descriptions of the system. If there are two distinct descriptions that are possible, they should be connected by a unitary transformation.

And this unitary transformation takes us to what we know as the Clebsch-Gordon coefficients. All right, so we will tell you what these, you know, these physical meaning of these Clebsch-Gordon coefficients mean, but let me just, you know, sort of take you through this, that what are the allowed values of all these quantum numbers, because these are important We have told you for some of them, like, for example, m1, it starts from this. We have written it explicitly. I'm just repeating it here for completeness.

So it's J1 minus 1, and it goes to minus J1. OK. Or you can write it the other way around as well. and m2 is simply that j2 and j2 minus 1 and then the last term or if you start from the other side, it is the first term. So, those are the two values.

So these are the two magnetic quantum number values, and m, of course, also goes from minus j, or we can write it the way we wrote it here. So it is j minus 1 all the way to minus j, and what about the total j? So J will take values equal to j1 plus j2 j1 plus j2 minus 1 and it will have values which are j1 minus j2 but you have to take the mod because it takes only positive values and so whichever is larger it does not matter I mean the difference between that and you take a difference and then you can take a mod of that okay. So what we are trying to do is that we are trying to build up a connection between these two descriptions. And let me write down the description that we wrote later.

So it's a j1, j2, j m. That description has to connect to for all possible values of m1 and m2 subject to the condition m1 plus m2. This is equal to m. And then we have these conditions. you know, this j1, j2, that is the other description that we have, you know, said that. So, it is m1, m2.

Now, you have to, of course, use this coefficients and that coefficients are known as this Clebsch-Gordon coefficient. So, that is j1 m1 and j2 m2. I hope that is the way we have written. We have written it. Yeah.

So, it is j1 m1 and j2 m2. So, j1 m1 and j2 m2. So, this is where j1, j2, j m. So, what we have done is that we have introduced this vector. And we have really written down something that is nothing but 1, the outer product of this basis sketch.

And then, of course, we have to do a sum m1 and m2, which takes values from minus j1 to minus j2 and so on, subject to the condition that the total m has to be the sum of these individual magnetic quantum numbers, that is m1 and m2. this thing that is this is called as the Clebsch-Gordon coefficients. We can call it as C m1 m2 because those are the things that are summed over and these are the Clebsch-Gordon coefficients. We will write in short Clebsch-Gordon CG coefficients. and this is what we want to find out.

Allowed values of the quantum numbers.

$$m_1 = j_1, j_1-1 \dots -j_1.$$
$$m_2 = j_2, j_2-1 \dots -j_2.$$
$$m = j, j-1 \dots -j$$
$$j = j_1+j_2,\ j_1+j_2-1 \dots |j_1-j_2|$$

CG coefficients.

$$|j_1, j_2, j, m\rangle = \sum_{\substack{m_1, m_2 \\ m_1+m_2=m}} \underbrace{|j_1, j_2, m_1, m_2\rangle \langle j_1, m_1, j_2, m_2|}_{\mathbb{1}} \overbrace{j_1, j_2, j, m\rangle}^{C_{m_1, m_2}}$$

So, you see that the left has one description and the right has this description. So, it is a connection between this description and this description and the coefficient is what you need to find out and these are called as the Clebsch-Gordon coefficients. Okay, so let me draw a table which will probably be simpler for you to understand that what we are, you know, trying to do. Let us take for j equal to j1 plus j2, right? That is the maximum value.

We have said that here. It is j1 plus j2. That is the maximum value and the minimum value is j1 minus j2. So, this is the maximum value. So, we have let us take these values of m and take these values of you know, so this degeneracy is what we have, but then we have these M1 and M2 all possible values of m1 and m2.

So, let us start with this corresponding to this J equal to j1 plus j2. The maximum value of M would be j1 plus j2 because M goes from minus j to plus j. So, the plus J is the largest value which is equal to j1 plus j2. So, what can happen to this m1 and m2 and they have to be simply equal to j1. So, this is just one pair that will satisfy because m1 plus m2 equal to m and so this has a degeneracy or degeneracy of these terms. This is equal to 1.

There is just one kind of terms that you can get. Let us see what is the next one. I mean, it is clear that there is a degeneracy equal to 1 because there is no other way you can choose your m1 and m2 values such that the sum of them gives you the value which is j1 plus j2, which is the maximum value of m corresponding to this maximum value of j, which is equal to j1 plus j2. Let us say the next value of m. So, the next value of m is j1 plus j2 minus 1.

It can come from two ways I mean two more ways than what I just show which I will discuss in just a while. So, it is a j1 minus 1 and j2 is 1. And you can also form it by j1 and j2 minus 1. So, it has a degeneracy 2 at this moment. So, there is no further terms say.

So, M has taken the next lower value, which is j1 plus j2 minus 1. And if you have to choose M1 and M2, because why you have to choose M1 and M2, let me just go back to the expression. You have to calculate this C m1 m2. So, different Choices of m1, m2 you have to you know find out and then take a linear combination of that in order to get this CG coefficients of the Clebsch-Gordon coefficients.

Let me show you the next ones, which is J1 plus J2 minus 2. This can be formed in three ways because J1 minus—say, 2 j2—you can have j1 minus 1, j2 minus 1. And you have j1 and j2 minus 2. So, there are three ways of doing this. And if you really go down to all the combinations and for this value of J, which is written above the table, so the last value that you get or the lowest value that you get is J1 plus minus of J1 plus J2.

For $j = j_1 + j_2$ (Maximum Value)

m	m_1, m_2	degeneracy
$j_1 + j_2$	(j_1, j_2)	1
$j_1 + j_2 - 1$	$(j_1 - 1, j_2)$ and $(j_1, j_2 - 1)$	2.
$j_1 + j_2 - 2$	$(j_1 - 2, j_2)$; $(j_1 - 1, j_2 - 1)$; $(j_1, j_2 - 2)$	3
$\vdots$	$\vdots$	$\vdots$
$-(j_1 + j_2)$	$(-j_1, -j_2)$	1.

And that again has just one combination that is minus j1 and minus j2 and this has again this degeneracy equal to 1. So, this table will for this maximum value of j, so which will be given to you, you know I mean you will have to you will be given a state and the state you will have to find out that what are the linear combinations of this state. And the CG coefficients will make sense because they will give you the probability or rather the probabilities for the state to be found in one given quantum state defined by all these j1, j2, m1, m2 and so on so forth. So, we have taken the maximum value of j and then have done it. You can do the next value of j and can again form a similar table.

Let us call this, you can keep calling it a ket, but we just want to sort of simplify a little notation there instead of writing all these commas, etc. So, let us call this j1, j2, jm to be a function, wave function or this eigenfunction. is j 1, j 2, j m, okay. So, this let us call this, okay. So, we are going to write a j 1, j 2, m 1, m 2, okay.

This is equal to a psi of J1, J2, M1, and M2. So, if you look at it here, where we define the CG coefficients, we have defined these as j1, j2, jm, but we can also write down, you know, a j1, j2, m1, m2 or let us follow the order in which we j1, m1, j2, m2. And then, you know, you can do a summation over m and write this down as j1, j2, jm. This is equal to a j1, j2, jm. m and this is equal to a j1, m1, j2 and m2. So, these will be the Clebsch-Gordon coefficient because this description is connected to this description and now you have m1 and m2.

Allowed values of the quantum numbers.

$$m_1 = j_1, j_1 - 1 \cdots -j_1.$$
$$m_2 = j_2, j_2 - 1 \cdots -j_2.$$
$$m = j, j-1 \cdots -j$$
$$j = j_1 + j_2, j_1 + j_2 - 1 \cdots |j_1 - j_2|$$

CG coefficients.

$$|j_1, j_2, j, m\rangle = \sum_{\substack{m_1, m_2 \\ m_1 + m_2 = m}} |j_1, j_2, m_1, m_2\rangle \overbrace{\langle j_1, m_1, j_2, m_2 | j_1, j_2, j, m\rangle}^{C_{m_1, m_2}}$$

$$|j_1, m_1, j_2, m_2\rangle = \sum_{m} |j_1, j_2, j, m\rangle \langle j_1, j_2, j, m | j_1, m_1, j_2, m_2\rangle$$

So, you have to sum over m which is the summation over m1 and m2. In any case, you can call either of them as phi or psi And because there is a description that is going to, you know, matter and so on. So, your phi of j1, j2, and j1 plus j2, j1 plus j2. So, that's the maximal state.

This is equal to j1, j2 and j1, you know, j2. and this is j1 j2 j1 j2 j1 plus j2 j1 plus j2 and then you have this Psi which is j1 j2 j1 j2. So, we are writing the maximal state j1 j2 and then we have this j1 j2. So, we are writing this maximal states and so on. Okay.

So, both, you know, phi and psi, they are normalized to unity, which says that, you know, all these things, that is j1, j2, j1, j2, these are the values for m1, m2. But for this particular state, we are taking the maximal state and we have a j1, j2, j1 plus j2 and j1 plus j2. This is equal to 1. So, this is the, you know, the sort of normalization condition.

Call $|j_1, j_2, j, m\rangle = \phi^{jm}_{j_1 j_2}$

$|j_1 j_2, m_1, m_2\rangle = \psi_{j_1, j_2, m_1, m_2}$

$$\phi^{j_1+j_2, j_1+j_2}_{j_1, j_2} = \langle j_1, j_2, j_1 j_2 \,|\, j_1, j_2, j_1+j_2, j_1+j_2\rangle \psi_{j_1, j_2 j_1, j_1 j_2}$$

Both ϕ and ψ are normalized.

$$\langle j_1\ j_2\ \underbrace{j_1, j_2}_{m_1\ m_2} \,|\, j_1, j_2, j_1+j_2, j_1+j_2\rangle = 1.$$

And so, now, I mean, you go to this next one. In the table, and then you take this j1 plus j2 minus 1, which is the next state, and that is the next lower value of m, then there are two possible values of m1 and m2 that you can construct. and that will give you this degeneracy of 2. So, you can write again these phi's and psi's and they being normalized and so on, you can write them down and so connect this phi's and psi's and you can define them basically for that state. So, for that next one, at least do this for the next one, next lower value so to say,

of J which is equal to so, m is equal to j1 plus j2 minus 1 and this tells you that this m1 can be equal to j1 or m2 can be equal to j2 minus 1 or it is the other way around. So, m1 equal to J1 minus 1 or m2 equal to J2. That's possible. OK. So we have these fives, you know, j1, j2 and j1, j1 plus j2 minus 1.

That should be a linear combination of, so it's a linear combination of these two things which are given by these combinations, okay? So these should be combinations of psi j1, j2, j1, j2 minus 1. I am forgetting giving these commas, but you can give these commas and psi j1, j2, j1 minus 1 and j2 and so on, okay. So, these are

the linear combinations that these, so there are, you know, going to be two Clebsch-Gordon coefficients which have to be found. We will give an example of that. Now, there is one more thing that we are missing here. What happens when, you know, your m or rather your j is next value of j that is j equal to j1 plus j2 minus 1.

So, that is next. So, this value, so this is the next lower value of j. We were talking about values of, you know, m, different values of m here, but kept the value of j to be equal to the maximum value. But j will also vary from j1 plus j2 to j1. The next value would be j1 plus j2 minus 1.

and it will sort of decrease, ultimately you would reach j1 minus j2 mod and that is the last value that you have. So, this J takes all positive values and we are going to find out for that one. Now, for J equal to j1 plus j2 minus 1, the maximum value of M will start from the second value. Because that is the value of, that will be the maximum value of m for j to be j1 plus j2 minus 1. And for that, you will have these two combinations once again.

So, all these, you know, this latter set would come from, not from this J equal to j1 plus j2, but the latter set will come from the J equal to j1 plus j2 minus 1. And that would actually sit at the top of the table, this table. When we change to, you know, so what I am trying to say is the following. So, if you make another table and so on, now with j equal to j1 plus j2 minus 1 and all these m values and all these, you know, kind of degeneracies that you say. and all these m1 and m2 values.

So, if j is this, the maximum value of m is j1 plus j2 minus 1. And then we can have m1 to be j1 minus 1 and j2, or we can have it as j1 minus 1. In fact, I should write 'and' because there is a linear combination j1 comma j2 minus 1. OK. And which has again a degeneracy, too.

So this sits at the top of the table, just like, you know, we have constructed the other table and so on. So the next one will be j1 plus j2 minus 2. And that will have three combinations and so on, so forth, which is what we have written down. OK. So this, if you need me to complete this, then this is equal to a j1 minus 2 j2 j2.

for the next lower value.

$m = j_1 + j_2 - 1 \Rightarrow m_1 = j_1,\ m_2 = j_2 - 1.$

$m_1 = j_1 - 1,\ m_2 = j_2.$

$\phi^{j_1, j_1+j_2-1}_{j, j_2} \rightarrow$ linear combinations of $\psi_{j_1, j_2, j_1, j_2-1}$ & $\psi_{j_1, j_2, j_1-1, j_2}$.

Next

$\boxed{j = j_1 + j_2 - 1}$ $\quad j = j_1 + j_2 - 1$

m	m_1, m_2.	degeneracies.
j_1+j_2-1	(j_1-1, j_2) and (j_1, j_2-1)	2.
j_1+j_2-2	$(j_1-2, j_2), (j_1-1, j_2-1)$ and (j_1, j_2-2)	3

And j1 minus 1, j2 minus 1, and j1 and j2 minus 2. And this has 3 degeneracy and so on. This is not coming from the maximal value of j, but it is coming from the next lower value of j. And in fact, this state that you get, this j1 plus j2 minus 1, you can actually operate the j this thing. On to this j1, j2, jm, and you can get the state. So, a lowering operator can be operated on these things to lower the value of these maximal j from this maximum value reduces by 1 unit.

I will write down the results for, or rather what happens when you apply J plus or J minus, which is called the lowering operator for the angular momentum. So, I think it is clear that you know these combinations will have to be taken, and there will again be combinations when you have a J1. So, if your j is equal to j1 plus j2 minus 2. Then your m, the maximum value of m, let us call it m max, will be j1 plus j2 minus 2 and will be the third term, which we have written down earlier. The term that you see here will be there at the top of the table, that is in place of this j1, j2.

So, that will be threefold degenerate and so on. So, what are these? So, let me sort of simplify this description, or I mean this notation seems to be the best to me, that this makes these quantum numbers explicit, and that is what is important for you. So, we have two sums m1 and m2, and the only thing that you need to, you know, sort of take into account is m1 plus m2 has to be equal to m, and we have a C m1 m2, that is the Clebsch-Gordon coefficient, and that is equal to j1 m1 j2 m2. So, that is the—so what does this C m1 m2 represent?

What do these Clebsch-Gordon coefficients physically mean? Do they mean anything? Yes, do they mean something? So, these mod square is equal to, so this is the probability that at fixed values of j1 j2, or in fact, it is at fixed values of J and Jz, that is more the way we are writing it, J and Jz, measurement, so if measurement is done, or measurement finds, I mean a particle.

Maybe an electron with J1z and the other, so finds one particle with J1z, I mean this z component of the angular momentum, and the other one with J2z. So, this is the meaning of the Clebsch-Gordon coefficient. So, if at fixed values of j square and jz, if you make a measurement on the particles, the individual particles will be found in their corresponding states. The corresponding, you know, magnetic quantum numbers of them will be m1 and m2. And this is what it is.

$$J_- |j_1, j_2\ \underbrace{j}_{\text{reduced by 1 unit.}}\ m\rangle$$

$$j = j_1 + j_2 - 2.$$

$$m_{max} = j_1 + j_2 - 2.$$

$$|j_1, j_2, j, m\rangle = \sum_{\substack{m_1, m_2 \\ m_1 + m_2 = m}} C_{m_1, m_2} |j_1, m_1, j_2, m_2\rangle$$

$|C_{m_1, m_2}|^2$: Probality that at fixed values of $\vec{J}$ and J_z, measurement finds one particle with J_{1z} and the other one with J_{2z}.

So, let me give you an example. Let us take a state, say, for example, before that, let me just write down this J plus minus, which I may not have written down earlier, but you can find it in any book. So it's h cross and square root of j into j plus 1 minus m into m plus minus 1 and j m plus minus 1. This is for individual, you know, particles. So, if you apply a raising operator, it will have a plus sign here inside the bracket within this m term, the second term, and it will also raise the state or the quantum number of the state, the magnetic quantum number of the state, by 1. And if you apply J minus, it will lower the value of m in the square root by 1, and it will also return you a state where the m value is decreased by 1.

So, definitely these J plus minus will they do not have J and M to be their eigenstates, or J in the other way around. J and M are not eigenstates of J plus minus, whereas they are eigenstates of J square and J Z. But these are important for us. Okay. So, an example, let us take an example and a simple example.

So, let us have j1, j2, j m given as 1, 1, 1 – 1. So, these are the values that are given. and now you have to find the corresponding j1, m1, j2, m2 and which all combinations of m1 and m2 will give rise to this state that will give you the coefficient and the mod square of the coefficients will give you the probability of finding them to be in those states. So, it is very clear that m, which is equal to m1 plus m2. It is m1 plus m2, and this is equal to minus 1.

So, this can happen from m1 equal to 0 to and m2 equal to minus 1, or it can happen from m1 equal to minus 1 and m2 equal to 0. This will give rise to a possibility in which we can write this j1, m1. So, this is j1 is 1 and m1 is 0, and j is of course 1, and m is of course 1. And this will be 1.

So, this will give rise to a state where, in terms of j1, m1, j2, m2. So, this is j2 and m2. So, these are these 1 and minus 1, that is fine. And this one will be 1, minus 1, and 1 and 0. So, these are these combinations here.

So, we will have to take a linear combination of that to express this state for a fixed value of J and M. That is a total, these angular momentum quantum number J and M and these will consist of two such terms. So, we have this 1, 1, 1, minus 1, sort of a little sloppy in writing these commas, but you can put them there. So, we have this as C0 minus 1, this Cm1, m2. So, this is m1 and this is m2.

And we have these 1 0 1 minus 1 and C 0 minus 1 or minus 1 0. This is equal to 1 minus 1 1 0. So, this state can be written the given state. So, this is a given state. for a fixed value of J and M can be written in terms of these states with, you know, m1 and m2 to be given by this and the Cm1, m2.

So, since we have summed over all the possibilities, there is no sum left, but C0 minus 1 mod square will tell you that we get M1 equal to 0 and M2 equal to minus 1 and C minus 1 0 will tell you that mod square of that will tell you that we get a state which is equal to M1 equal to minus 1 and M2 equal to 0. So, what we do, how do we calculate these coefficients? So, we still have to calculate C 0 minus 1 and of course, mod square is one thing that we have to first find these things.

$$J_{\pm}|j,m\rangle = \hbar\sqrt{j(j+1)-m(m\pm1)}\;|j, m\pm1\rangle$$

Example. $|j_1, j_2, j, m\rangle = |1, 1, 1, -1\rangle$

$m = m_1 + m_2 = -1$

$\left.\begin{array}{l} m_1 = 0, \quad m_2 = -1 \rightarrow |1\,0\,1\,-1\rangle \\ m_1 = -1, \quad m_2 = 0 \rightarrow |1\,-1\,1\,0\rangle \end{array}\right\}$.

Given state $|1\,1\,1\,-1\rangle = \underset{m_1\, m_2}{C_{0-1}}|1\,0\,1\,-1\rangle + C_{-10}|1\,-1\,1\,0\rangle$

Have to calculate C_{0-1}, C_{-10}.

So, C m 1 m 2 that you have to find and how do you find that? We can find it by these lowering operator. And the lowering operators are sometimes helpful. Sometimes these raising operators are helpful depending upon whether you adjust one above the lowest value, then you use lowering operator to make it zero. If you adjust below the maximum value, then you use a J plus and make it equal to zero because you cannot, you know,

increase it beyond that. So, if you are in a, sorry, you are in a maximally aligned state, that is a maximum value of, you know, j, then you use a j plus to make it 0. And if you are at the minimum value, then you make it operated by a j minus so that it becomes 0. Now, you see that for this particular problem, when you have j equal to 1 and m equal to minus 1, we are already at the minimal state. So, you put a j minus, that will become equal to 0, because you cannot lower for a given value of j, which is 1, you cannot have anything, the m value, the total m value cannot be anything lower than minus m.

So, that is what the idea is and then you operate it on this, minus 1 and that is equal to 0. So, this is given by j1 minus plus a j2 minus and this is equal to, you know, C 0 or the way we have written it is C 0 minus 1 and then we have these 1 0 1 minus 1 that is a J 1 M 1 J 2 M 2 plus a C minus 1 0 and we have 1 minus 1 1 1 0 and so on. and this is equal to 0. So, you have to operate that and if you do that carefully, so you will have a J1 minus and J2 minus and use this relation for these individual.

$$J_-|1 1 1 -1\rangle = 0 = (J_{1-} + J_{2-})\left[C_{0-1}\,|\underset{j_1\,m_1\,j_2\,m_2}{1\,0\,1\,-1}\rangle + C_{-10}\,|1\,-1\,1,0\rangle\right] = 0.$$

So, remember that this is J1, M1, this is J2, this is M2 and so on the same thing J1, M1 and J2, M2 and so on. So, we have this J minus operating on 1 1 1 minus 1 that gives us you know a root 2 and gives us this. So, this root 2 operating this J minus on this things or J 1 minus and J 2 minus which are C 0 1 minus plus C minus 1 0. and will give us this state, which is 1, minus 1, 1, minus 1, and so on. And this, if we put equal to 0, then you have a C0 minus 1 is equal to C minus 1, 0, and C0 minus 1, and so this root 2 comes from that square root that you have here, this square root.

okay. So, this C 0 minus 1 is equal to minus of C minus 1 0 and so on. So, this tells you that these Clebsch-Corton coefficients C 0 minus 1 equal to 1 by root 2 equal to minus C minus 1 0 and so on. So, that means that there are equal probabilities Because these two mod square is, you know, equal to half, there are equal probabilities to find, you know, m1, m2 in the 0, minus 1, minus 1, 0 and so on, okay?

$$J_-|1 1 1 -1\rangle = \sqrt{2}\,(C_{0-1} + C_{-10})\,|1 -1 1 -1\rangle = 0.$$

$$C_{0-1} = C_{-10}.$$

$$C_{0-1} = -C_{-10} \Rightarrow C_{0-1} = \frac{1}{\sqrt{2}} = -C_{-10}.$$

There are equal probabilities to find $(m_1\, m_2) \rightarrow (0,-1);\ (-1,0)$.

That is what So, this is the way to find the Clebsch-Gordon coefficients and let us take another example and so this I find the Clebsch-Gordon coefficients second example. And so, let us say the state is given as 1, 1, 2, 0. So, which means that j1 equal to 1, j2 equal to 1, j equal to 2 and m equal to 0. So, m1 now can go from minus 1 to plus 1

That is basically minus 1, 0, and plus 1, and m2 can also go from minus 1 to plus 1, which means minus 1, 0, and plus 1, and then subject to the condition that m1 equals m1 plus m2. So, all these combinations of m1 and m2 have to finally give you an m equal to

0. So, what we do is that we write a slight bit of, you know, okay, so we can write the full notation, this j1, j2, jm, which is equal to my, the Clebsch-Gordon coefficient C, m1, m2, and then we have this j1, m1, j2, m2, and so on, and then sum over m1 and m2 subject to the condition that m1 plus m2 will have to be equal to m. So, let me only use the relevant things. If you, you know, allow me to just shorten the notation, this will be like C m1 m2.

This will make it a little simpler notation: m j1 m1, and the other one will have just j2 m2. So, then we have. This is a shorthand notation. So, I go from this to this notation. So, a little change in notation and a little bit of shorthand notation.

So, we have 2 2, which is equal to C 1 1, 1 1, 1 1. So, the a 1, m 1 are 1 1, and j 2, m 2 are 1 1 as well. So, that is a 2 2. Now, we have these 2 1. This is equal to C 1 0. 1 1 1 0 and C 0 1 to be equal to 1 0 and 1 1. That is 2 1.

<u>Example</u> $|1\,1\,2\,0\rangle$

$j_1 = 1, \quad j_2 = 1, \quad j = 2, \quad m = 0.$

$m_1 \to -1 \text{ to } +1, \quad -1, 0, +1$

$m_2 \to -1 \text{ to } +1, \quad -1, 0, +1.$

$m = m_1 + m_2$

$$|j_1 j_2\, j m\rangle = \sum_{m_1, m_2} C_{m_1, m_2}\, |j_1, m_1\, j_2\, m_2\rangle$$

notation

$$\boxed{|jm\rangle = \sum C_{m_1 m_2}\, |j_1 m_1\rangle |j_2 m_2\rangle.}$$

$$|22\rangle = C_{11}\, |11\rangle|11\rangle$$

$$|21\rangle = C_{10}\, |11\rangle|10\rangle + C_{01}\, |10\rangle|11\rangle$$

Let me write the 2 0. We can write because 2 0 has many terms. So, we have these 2 0 that is J m equal to 2 0.

that will be equal to C 1 minus 1, 1 1, 1 minus 1 plus C minus 1 1, 1 minus 1, 1 1 plus C 0 0, 1 0, 1 0, 1 0. plus C again as 0 0 will be equal to 1 0 1 0. So, there are four terms there and so this is you know equal to a C 1 minus 1 and 1 1 1 minus 1 and a C minus 1 1 So, 1 minus 1, 1, 1 and plus this is really 2, 0, 0, 1, 0, 1, 0. And so if you normalize it, it becomes because there's 1 here, there's 1 here, there's 2 there.

So 1 square plus 1 square plus 2 square, which is 4, is 6. So this comes with a 1 by root 6. And then we have this 1, 1, 1 minus 1. And so I hope this 0 0 is clear because there are these two combinations which are given. That's why this is written twice.

OK. And so this is again 1 by root 6. So it's 1 minus 1, 1 plus, you know, 2 by root 6. So it's 2, and then 1, 0, 1, 0, and so on. OK.

$$
\begin{aligned}
|20\rangle &= C_{1-1}|11\rangle|1-1\rangle + C_{11}|1-1\rangle|11\rangle + C_{00}|10\rangle|10\rangle + C_{00}|10\rangle|10\rangle. \\
&= C_{1-1}|11\rangle|1-1\rangle + C_{11}|1-1\rangle|11\rangle + 2C_{00}|10\rangle|10\rangle. \\
&= \frac{1}{\sqrt{6}}\left[|11\rangle|1-1\rangle + |1-1\rangle|11\rangle + 2\,|10\rangle|10\rangle\right].
\end{aligned}
$$

So, this is the way to find it. Now, this is a distinct problem from the last one. In the last one, we had to operate by a j minus or, in some cases, as I said, if it is at the maximal value, you will have to apply a j plus. We did not have to do anything here because it was at the middle value. So, you had a 0 here.

So, you had to do it twice. You can still do it. Your m was equal to 0. If M is equal to 0 and the maximum value of m can go to 2, you have to do this twice. Instead of that, you wrote down all the linear combinations by inspection of what can be the m1 and m2.

It looks a little complicated, but it is not actually. If you just sit down and sort of clear up in your mind that what are these indices for a given value of m, what are the different combinations of m1 and m2 that we can find, you can simply write it down. We have shown two examples. There will be more examples in the assignment. And this is how one actually finds out the Clebsch-Gordon coefficients, which, as I said, denote the probability for a given J and M, that is, which are the eigenvalues for j1, j2,

not J1m, J and m, that is, the eigenvalues for J square and Jz. What are the different combinations of M1 and M2 that could be there? So, if you make a measurement, what are the probabilities that you would find them in distinct m1, m2 states? m2 states, I mean, m1, m2 states that are defined by distinct indices. That is what Clebsch-Gordon coefficients physically mean.

Okay. So, we will stop here for today and continue with some new topics next time.