ELEMENTS OF MODERN PHYSICS

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Lec 12: Operator Methods for Harmonic Oscillator

You remember that we have done this harmonic oscillator, the simple harmonic oscillator, using the solution of the Schrödinger equation for these half k omega, k x square, which was written as half m omega square x square kind of potential. And then it involved a lot of, you know, variable transforms and finally getting the solution in terms of these Hermite polynomials. And what we got is this energy has a particularly simple form, which is H n plus h-bar omega, where n can take values 0, 1, 2, and so on. So even at n equal to 0, you still have an energy, which is called the zero-point energy. And that was sort of an algebraic solution of this problem where the Schrödinger equation was solved for this potential.

There is a simpler way of doing it, and let us review that. And so that is why it is written as harmonic oscillator revisited. And we will revisit it in terms of some notations called raising and lowering operators. Which are written as, okay, the raising operator is written as A dagger and the lowering is written as A. And these operators will be helpful in writing down the Hamiltonian and also will be helpful in writing down the wave functions corresponding to this Hamiltonian. Which you already know, but there is a much simpler way of arriving at things.

And then we'll define a number operator in terms of A and A dagger. And we'll review some of the commutation relations, which are helpful in this regard. And then finally, we'll be building up the eigenstates of the Hamiltonian. All right. So, what we have is, just to remind you, that this is the simple harmonic oscillator Hamiltonian.

So, we have a p square over 2m plus a half m omega square x square. The notations are standard. So, p is the momentum, m is the mass, and x is the displacement from the equilibrium position. And this is the Hamiltonian that we had to solve, and we did that.

And just to remind you of these essential commutation relations, this is equal to I h cross. And so this has to be satisfied by these P and X operators. So, they cannot be simultaneously determined. And the reason that commutation relation is non-zero is because these are infinite-dimensional vectors. So, just to give you a short introduction of these A and A dagger operator, okay—before that, let me write down these things in terms of the A and A dagger operator.

So, what we do is that we make some dimensionless variables where x tilde is nothing but root over m omega by h cross, omega is the angular frequency and x, and we have p tilde, that is a new variable and is a dimensionless variable, is equal to p divided by root over m omega h cross, okay. So, these are the two relations that we have, or rather dimensionless variables that we want to work with. It is exactly the same as the p and x, except that we have rendered them, you know, as dimensionless variables. They are still variables, but they are dimensionless variables.

And instead of this, I have cross x tilde p tilde; they obey the commutation relation which is given by I. OK. And nicely, we can write down the Hamiltonian. That's again a dimensionless Hamiltonian, which is nothing but half, you know, p tilde squared plus x tilde squared. And this is a very nice form, or rather a very convenient form, because both these variables that we see there are quadratic, and there's no other factor there apart from this half.

And the way we can do it is that we can write down this a which is equal to h by h cross omega, and that's equal to, you know, p square over 2 m h cross omega plus half m omega square x square divided by h cross omega. And so, your E n tilde, which is the eigenvalues of this Hamiltonian, would come out as simply n plus half without any dimension of energy, which is h cross omega associated with it. So, we have x tilde, p tilde, and h tilde, all of which are dimensionless operators. This makes the algebra easy.

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

$$[x, p] = i\hbar$$

$$\tilde{x} = \sqrt{\frac{m\omega}{\hbar}}\, x \quad , \quad \tilde{p} = \frac{p}{\sqrt{m\omega\hbar}}$$

$$[\tilde{x}, \tilde{p}] = i$$

$$\tilde{H} = \frac{H}{\hbar\omega} = \frac{p^2}{2m\hbar\omega} + \frac{1}{2}\frac{m\omega^2 x^2}{\hbar\omega}.$$

$$\tilde{H} = \frac{1}{2}\left(\tilde{p}^2 + \tilde{x}^2\right)$$

$$\tilde{\epsilon}_n = \left(n + \frac{1}{2}\right)$$

$\tilde{x}, \tilde{p}, \tilde{H}$ all are dimensionless operators.

$$\tilde{H}|\psi_n\rangle = \tilde{\epsilon}_n |\psi_n\rangle$$

There is no other reason for introducing that. So, your H tilde acting on all these wave functions will give us E n tilde, returning you back the wave functions where E n tilde is what we have just written there. So, it just makes life a little easier, and there is one short word of caution: your H tilde is equal to, as you saw, that it is equal to x square plus p square or p square plus x square—it's the same thing—but this is certainly not equal to half of x tilde plus i p and x tilde minus i p tilde. And this is true because your X tilde and P tilde, the commutation is not equal to 0, and it is equal to I, which is what we have said.

Now, we introduce these operators, these raising and lowering operators. So, let us call them as lowering and raising operators. And in some books, they call it a Dirac notation. They mean the same thing. So, your A operator is 1 by root 2 x tilde plus i p tilde and say had you not made this x tilde and p tilde you will have to carry around this m h cross omega etc.

So, this looks simpler at least and A dagger this is equal to 1 by root 2 x tilde minus i p tilde. And I stopped short of explaining what is a and a dagger. So a is, you know, a vector in general, a vector with some entries like this. So it's like a 0, a 1, a 2 and all the way up to a n. And a dagger is basically a row vector with all this a 0 star, a 1 star, a 2 star and so on till a n star, okay.

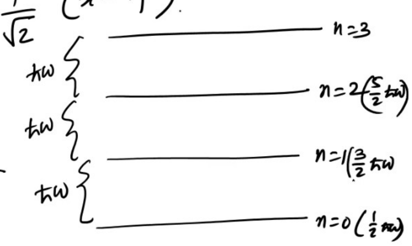
So what you do in dagger is that just like any matrix, when you take a dagger, you take a transpose and the conjugate and this is exactly what you do there. So this a and a dagger have these forms and you can do a sort of reverse transform or rather write this x tilde and p tilde in terms of a and a dagger, which are going to be useful for us. So, it is x tilde equal to 1 by root 2 a plus a dagger and p tilde is equal to 1 by root 2 a minus or rather this is let us write this dagger first. So, this will be a dagger minus a and this a dagger plus a for x tilde and p is equal to 1 by root 2 a dagger minus a. So we are going to use these operators to solve this problem of quantum harmonic oscillator.

And though it looks like that, you know, these ladder operators—or these are sometimes called ladder operators—these raising and lowering operators that you see are particularly important because, you know, I'm just a priori telling you this, that we have n equal to 0, then n equal to 1, and then n equal to 2. And then n equal to 3. These are the energy levels of the harmonic oscillator. This is h cross omega, this is h cross omega, this is h cross omega, this is of course well known. But if you think in terms of these indices, this looks like that this small n equal to 1, our small n equal to 0 here, small n equal to 1 here, n equal to 2 here, and 3 here, they correspond to oscillators because

$$\tilde{H} = \frac{1}{2}\left(\tilde{x}^2 + \tilde{p}^2\right) \neq \frac{1}{2}\left(\tilde{x} + i\tilde{p}\right)\left(\tilde{x} - i\tilde{p}\right). \qquad [\tilde{x}, \tilde{p}] = i$$

Lowering and raising operators.

$$a = \frac{1}{\sqrt{2}}\left(\tilde{x} + i\tilde{p}\right) \quad ; \quad a^\dagger = \frac{1}{\sqrt{2}}\left(\tilde{x} - i\tilde{p}\right)$$



$$a = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \qquad a^\dagger = \begin{pmatrix} a_0^* & a_1^* & a_2^* & \cdots & a_n^* \end{pmatrix}$$

$$\tilde{x} = \frac{1}{\sqrt{2}}\left(a^\dagger + a\right)$$

$$\tilde{p} = \frac{1}{\sqrt{2}}\left(a^\dagger - a\right)$$

So suppose this has no oscillator, so its energy is half h cross omega. And this has one oscillator, it's, you know, three half h cross omega. And then it's five half h cross omega and so on. So these are the energies of these oscillators. things and it looks like that these n actually denotes the number of oscillators that you have each oscillator will have an energy equal to you know half h cross omega and then you can just simply build up all these states by assuming that these are occupied by three oscillators and four oscillators and five oscillators and so on, and their energies will be given by this n plus half h cross omega, where n is this number that we are talking about. So this n really has to be, you know, equated—or rather, there has to be a correspondence between n and the number operator, and that most simply comes out from this analysis that we are talking about. So let us look at, with these as the backbone of this formalism, which is truly a sort of parallel formalism to the algebraic method that one can do—which we have done sketchily. We have not done it very elaborately, but it can be done. It is in the interest of keeping the mathematics alive.

a little brief and also you know to emphasize on the main results we have cut down on the solving the differential equation and all those variable transforms but nevertheless we got the answers which are the for the eigenstates and the eigenenergies. We are going to get the same thing here as well okay. So what kind of operators are these a and a dagger? Are they commuting with each other or they are non-commuting? What kind of eigenstates do they have?

Are they the eigenstates of these n operators that we are going to talk about and so on? So these are some of the questions that we are going to ask and it would correspond to the commutation relations or rather they would be given by the commutation relations. So, let us see this commutation relation a and a dagger. And this is equal to half. And then we have a x tilde plus ip p tilde comma.

So this is that the first term, which is A. And so this is X tilde minus I P tilde and this commutation. So X will, of course, commute with X and P will commute with P. But what will not commute is that X will not commute with P or P will not commute with X. So we will have two terms which are non-zero which are P tilde X tilde and minus I by 2 X tilde P tilde. So these are the cross terms that do not commute. and this is equal to minus ih cross and this is equal to plus ih cross.

So, that gives us i by 2 into minus ih cross and minus i by 2 into ih cross. So, this minus i square becomes equal to 1 and this is equal to, sorry, I will use this as simply equal to, because x and p commutation is not i h cross but it is simply equal to i which is what we have said. So, this h cross will not be there in the in this operators. So, this is equal to half and this is equal to minus of minus half.

So, this is equal to half as well. So, this is equal to 1. So, a and a dagger do not commute. and their commutation relation is 1 just like P and X do not commute but now the commutation relation is not like X and P but non-zero but it is equal to 1. So, let us see an important quantity called as a dagger a and see if it is something meaningful and a priori telling you that this quantity is nothing going to be nothing but the number operator which when acts on a given state will count the number of oscillators that are there in the state.

So, this is going to be an important quantity in our discussion. So, it is x minus ip into x plus ip, all these with a tilde and so on. So, this is equal to half x tilde square plus p tilde square plus i x tilde p tilde. and one has this as, so now this is known to be equal to, you know, so this is equal to i and so this is minus, so this a, this is nothing but equal to minus 1. And so this is equal to halfx tilde square plus P tilde square minus 1.

And if you remember that half of x tilde squared plus p tilde squared is nothing but the Hamiltonian, which is what we have written down earlier. So this H tilde is equal to a dagger a plus half h. Or it is also equal to A dagger minus half, okay? And that gives you because you can show that, since you have a a dagger equal to 1, so a a dagger minus a dagger a equals 1. So, if you want to interchange, then you have to have a minus sign coming in. So, what I did was replace a dagger a with a a dagger.

So, a dagger a is equal to a a dagger minus 1, and that makes it a a dagger minus half. So, that is the Hamiltonian in terms of these raising and lowering operators. And we are talking not about the bare Hamiltonian that we started with, but about a dimensionless Hamiltonian whose eigenvalues are the dimensionless energies. So, if we return to the Hamiltonian that we had started with, then it is a dagger a plus half. h cross omega that immediately tells us that this quantity, which is a dagger a, is nothing but the number operator.

$$[a, a^\dagger] = \frac{1}{2}\left[(\tilde{x}+i\tilde{p}), (\tilde{x}-i\tilde{p})\right]$$

$$= \frac{i}{2}\underbrace{[\tilde{p},\tilde{x}]}_{-i} - \frac{i}{2}\underbrace{[\tilde{x},\tilde{p}]}_{i} = \frac{i}{2}\times(-i) - \frac{i}{2}\times i$$

$$= \frac{1}{2} + \frac{1}{2} = 1.$$

$$a^\dagger a = \frac{1}{2}(\tilde{x}-i\tilde{p})(\tilde{x}+i\tilde{p}) = \frac{1}{2}\left(\tilde{x}^2 + \tilde{p}^2 + i\underbrace{[\tilde{x},\tilde{p}]}_{i}\right)$$

$$= \frac{1}{2}\left(\tilde{x}^2 + \tilde{p}^2 - 1\right).$$

$$\boxed{\tilde{H} = a^\dagger a + \frac{1}{2} = a a^\dagger - \frac{1}{2}}$$

$$[a, a^\dagger] = 1.$$
$$a a^\dagger - a^\dagger a = 1.$$
$$a^\dagger a = a a^\dagger - 1$$

$$H = (\underbrace{a^\dagger a}_{} + \tfrac{1}{2})\hbar\omega.$$
$$= (N + \tfrac{1}{2})\hbar\omega$$

N: number operator
$= a^\dagger a.$

So, this is the number operator corresponding to so, n is the number operator and is written in terms of these a and a dagger as equal to a dagger a. And so, let us see what this property of n is. Now, H is of course Hermitian, so N has to be Hermitian. So, let us see that n is a Hermitian operator. And how we can show this is that we have n dagger, which is, with n equal to a dagger a, and N dagger is a dagger a dagger. Now, when you do that, what you have to do is note that (A B) dagger is equal to B dagger A dagger.

And that can be easily proved. You can take any two Hermitian matrices and then you can just multiply them or use them as a basis to show that when you take a dagger or a transpose, you have to take it from the opposite direction. So this tells you that this dagger is equal to a dagger a, which is equal to n. Now, that means that n dagger is equal to n. So, this is the Hermitian property, right? So, it is a Hermitian operator.

And so, the eigenvalues are real and physical observables. And so, your H dagger becomes nothing but almost equal to n, except for a factor of half. So that dimensionless Hamiltonian is just n plus half, and we are now very sure that the eigenvectors of H tilde and n are the same, okay. And you know that because H tilde n is equal to 0.

Okay. And this is easily seen from here because n will commute with n, and half is just a number. So n and H tilde will commute with n. And if that happens, then the eigenvectors of H and n are the same. And of course, vice versa. Just a quick check because we need this for later use.

So some of the commutation relations that we have been talking about. So what's the relation commutation relation with n a and that's equal to a dagger a and a which is equal to a dagger a a and plus a dagger a a okay so I've just used this a b c it's equal to a b c plus a c b So that is what I have used and this is of course equal to 0 because a commutes with a and a dagger a is equal to minus 1 and so this is equal to minus a. So the commutation relation between n and a are not same or rather not 0 and because they are not 0, the eigenfunctions of N and eigenfunctions of a are different. So, the eigenstates of n are not the eigenstates of a or a dagger.

Let me also show this the same thing for a dagger. So, a dagger is equal to a dagger a a dagger and we have this as a dagger a a dagger plus a dagger a dagger a, okay. This is certainly equal to 0 and this is equal to 1. So, that gives you a dagger.

So, the commutation relations of n, the number operator that we have identified or we have called that as a dagger a. So, a dagger a and a A, they do not commute. In fact, the commutation relation gives something a finite, which is minus a. And n also does not commute with a dagger and gives you just a dagger. So, these two are important. Let me write that.

N is a Hermitian operator. $\quad N = a^\dagger a \qquad (AB)^\dagger = B^\dagger A^\dagger$

$$N^\dagger = (a^\dagger a)^\dagger = a^\dagger a = N.$$

$N^\dagger = N$ — Hermitian operator.

$$[\tilde{H}, N] = 0. \qquad [AB, C] = A[B,C] + [A,C]B.$$

$$\tilde{H} = N + \frac{1}{2}.$$

Eigenvectors of $\tilde{H}$ and N are same.

Commutation relations

$$[N, a] = [a^\dagger a, a] = a^\dagger \underbrace{\cancel{[a, a]}}_{0} + [a^\dagger, a]\, a = -a.$$

$$[N, a^\dagger] = [a^\dagger a, a^\dagger] = a^\dagger \underbrace{[a, a^\dagger]}_{=1} + \underbrace{[a^\dagger, a^\dagger]}_{=0} a = a^\dagger.$$

$$\boxed{\begin{aligned} [N, a] &= -a \\ [N, a^\dagger] &= a^\dagger \end{aligned}}$$

So, n a is equal to minus a, and n a dagger is equal to simply a dagger. These are important relations for us. Okay, so let us see some more properties of this. So, the first property of these various operators and their eigenstates. So, the eigenvalues of n

N, let us call them as n, small n. They are positive or 0. They can be 0, but they are positive. And how do we do that? Let us see A acting on some state, okay, psi n, and the square of that. So, what we do is that if we take this state,

and take the norm of this vector. So, this is a vector because there is an operator acting on the state. So, it is a vector, and what we do is that we take this A psi n, that is the norm of the vector, and call this norm of the vector, of course, it is a length. And this length has to be greater than or equal to 0 at most if there is a null vector, okay. So, this thing can be written as this mod square is psi n a dagger a psi n, this is greater than or equal to 0, okay.

So, that tells us that psi n, n psi n is greater than or equal to 0. So, I think it's clear that a psi n mod square is like a psi n, okay, and the conjugate of that, so that will be a bra psi n and a dagger, and that's why we have written that, okay. So, this means that the expectation value of n between these states, psi n states, is greater than 0, which tells you that if we have a relation where... n acting on psi n gives you an eigenvalue equation, that is, psi n's are the eigenstates of this capital N operator, yielding you the eigenvalue, which is small n, and returns you the eigenfunction, this psi n. Then, from this, it is clear that this n has to be greater than or equal to 0.

So, that tells you that the eigenvalues of n are positive definite, or they could be 0 in the worst case, or rather in some special cases when n equals 0. Okay, so now, the n equal to 0 state is So, how do we sort of characterize the n equal to 0 state, or how do we understand that? So, let me do this: for an arbitrary state phi, let us say this is equal to 0. This phi is arbitrary.

So, if we take a dagger a and left multiply by this equation, this is equal to 0 as well. So, which means that n acting on phi gives you 0. So, the eigenvalue of these operators of n on phi is equal to 0, okay. So, which tells you that this phi, now let us say we call it psi 0, because we are writing psi n, and this is the ground state of the system, and the ground state has no oscillators, and the energy of this state is equal to half h cross omega.

Properties.

1. The eigenvalues of N (Let's call them as n) are positive or zero.

$a|\psi_n\rangle \rightarrow ||a|\psi_n\rangle|^2 \geq 0.$ $\quad \langle\psi_n|a^\dagger a|\psi_n\rangle$

$\langle\psi_n|a^\dagger a|\psi_n\rangle \geq 0.$ $\quad N|\psi_n\rangle = n|\psi_n\rangle$

$\langle\psi_n|N|\psi_n\rangle \geq 0$

$\boxed{n \geq 0.}$

2. $n=0$ state.

$a|\phi\rangle = 0.$ $\quad |\phi\rangle$ is arbitrary.

$a^\dagger a|\phi\rangle = 0.$ $\quad$ E-value of N on $|\phi\rangle = 0.$

$N|\phi\rangle = 0 \Rightarrow$

$|\phi\rangle = |\psi_0\rangle$: Ground state. $\rightarrow \frac{1}{2}\hbar\omega.$

So, what we have proved is that there is a n equal to 0 state which is also special. It contains no oscillators but it has certain energy which is equal to half h cross omega and the way we prove that is that we take an arbitrary ket. and apply the a operator which is a lowering operator on this gate and then suppose this relationship exists and then we left multiply it or left operate it by this a dagger so a dagger a on 5 will be 0. a dagger a is nothing but n and that tells you that the eigenvalue of these state that is operation of n on this particular state is 0 and let us call that state as the ground state of the system and which has no oscillators okay. So, then let us also do this number 3 property number 3 if n is greater than 0.

then the vector a psi n is a non-zero vector of, non-zero eigenvector of n with eigenvalue n minus 1. This really justifies the name that we have given it to is a lowering operator. So, which means that suppose we take an n for which it is n is greater than 0 because for n equal to 0 what happens we just saw but for any other n which is larger than 0. If A acts on that state or that eigenstate, I mean that eigenstate of n, what it will do is that it will reduce the eigenvalue by n minus 1 and that will not be an eigenvalue equation. So, A acting on psi n will not give you n minus 1 psi n minus 1.

or rather psi n. So, this is untrue. What I am saying is that A psi n is not equal to n minus 1 psi n. And the reason is that this is wrong because psi n's are not eigenstates of a. And the reason is for that is that capital N does not commute with either a or a dagger. So, this psi n which are eigenstates of n are not eigenstates of either of a or a dagger. So, let us see how we can show this or so n a that is a commutator acting on this psi n we have found out that the commutator is minus a psi n.

So, we can, you know, open this bracket and write it as n a psi n, where n a means the number operator multiplied by the lowering operator, equal to a n, which is the other term. So, I have taken it on the other side: a n psi n minus a psi n. And what we can do is, because n acting on psi n will give me a n psi n, and this will return a psi n. I mean, 'return' means this will not be—so let me erase this, because, you know. And so this is true: this is a psi n, so this is equal to n minus 1 a psi n. So, this is exactly what we are saying—that n acting on a psi n gives you—and so n acting on this state will give us n minus 1, lowering the number of oscillators, and returns me this a psi n. So, that tells you that a psi n is an eigenvector of n. with eigenvalue equal to n minus 1, okay?

That is an important thing. Similarly, for the other one—that is, again, for n greater than 0 a dagger psi n is an eigenvector of n with eigenvalue n plus 1. Okay? And how you can show that is—it's exactly in the same spirit.

You take this n a dagger, and once you do that, it becomes equal to a dagger psi n. And then open up the bracket again. And what you have is n a dagger psi n. So, n a dagger psi n, and this is equal to n plus 1 a dagger psi n. So, it's the same thing. You know, the relation that you have for a. So, a dagger psi n is an eigenvector of n, returns the eigenvalue n plus 1, and returns me the same ket, okay. So, we have, you know, acquired a lot of properties of this.

3. If $n > 0$, vector $a|\psi_n\rangle$ is a non-zero e-vector of N with e-value $(n-1)$.

$$[N, a]\,|\psi_n\rangle = -a|\psi_n\rangle.$$

$$Na|\psi_n\rangle = aN|\psi_n\rangle - a|\psi_n\rangle = an|\psi_n\rangle - a|\psi_n\rangle$$

$$N\left[a|\psi_n\rangle\right] = (n-1)\left[a|\psi_n\rangle\right]$$

$a|\psi_n\rangle$ is an eigenvector of N with e-value $= (n-1)$.

$a^\dagger|\psi\rangle$ is an e-vector of N with e-value $(n+1)$.

4. If $n > 0$

$$[N, a^\dagger]\,|\psi_n\rangle = a^\dagger|\psi_n\rangle$$

$$N\left[a^\dagger|\psi_n\rangle\right] = (n+1)\left[a^\dagger|\psi_n\rangle\right]$$

So, let us see the properties of these eigenstates of the Hamiltonian. So, we know that A acting on the ground state will give me 0. So, apart from raising and lowering, they are also called creation and annihilation operators, which is probably more contextual in this particular case. A dagger creates an oscillator in a given state and takes the state to the next higher state, while A destroys or annihilates an oscillator and brings the system from a higher state to a lower state. There is no destruction or annihilation possible in the ground state because that corresponds to n equal to 0, which means the number of oscillators is equal to 0.

So, you are left with a psi 0 equal to 0, and a psi 1 is simply equal to a c 1 a dagger and a psi 0. So, what I did here is write down a coefficient. So, C is an unknown coefficient at this point; we have to find it using normalization. But what is important here is that the first excited state or the state— So, N equal to 0 is the ground state, and we are trying to see how we can normalize—

get the first excited state. So, what you do is create one oscillator in this state by using a dagger, and then you go to the psi 1 state, the first excited state, and this is exactly what we have done. Now, this has to be normalized, so this is the normalization constant, and it has to be found out from normalization. Let us see how we can do that. So, we can do this normalization as psi 1 psi 1, which is equal to a C mod squared.

And then we have these psi 0 and we have a a dagger psi 0. Now, a a dagger is nothing but c1 square psi 0. So, this is a dagger a plus 1. This is from the relation that we have written down earlier that a a dagger is equal to, so this is the one that we can change this order. So, when you go from a a dagger to a dagger a, you simply pick up a plus sign.

We have written that somewhere here. So, this is equal to psi 0 and so this is nothing but equal to C1 mod square and a psi 0 a dagger a psi 0 and psi 0 psi 0 assuming everything to be normalized this is equal to 1. And this is nothing but equal to 0 because a dagger is a number operator that when acts on the ground state gives you 0. So, we have this is equal to and the left hand side equal to 1 as well. So, 1 equal to C1 mod square into 1 that tells us that C1 is equal to 1.

Eigenstates of the Hamiltonian

$$\left.\begin{array}{l}\text{Creation}\\ \text{Annihilation}\end{array}\right\} \text{operators}$$

$$a|\psi_0\rangle = 0$$

$$|\psi_1\rangle = c_1\, a^\dagger|\psi_0\rangle. \qquad c_1 : \text{an unknown coefficient}$$

$$\langle\psi_1|\psi_1\rangle = |c_1|^2 \langle\psi_0|a a^\dagger|\psi_0\rangle$$

$$= |c_1|^2 \langle\psi_0|(a^\dagger a + 1)|\psi_0\rangle.$$

$$= |c_1|^2 \Big[\underbrace{\langle\psi_0|a^\dagger a|\psi_0\rangle}_{=0} + \underbrace{\langle\psi_0|\psi_0\rangle}_{1}\Big] \Rightarrow 1 = |c_1|^2\cdot 1$$

$$\Rightarrow c_1 = 1.$$

$$\boxed{|\psi_1\rangle = a^\dagger|\psi_0\rangle}$$

So, what we get is the first excited state of this system to be equal to a dagger psi 0. So we are given the ground state and then we are trying to, you know, successively build the eigenstates of the Hamiltonian. So the first excited state you build up from the ground state by once operating the creation operator or the raising operator. How do we do the next one, the second excited state, which is, so let us have this picture in mind all the time. So, this is n equal to 0, n equal to 1, n equal to 2, they are all equidistant from each other in terms of energy.

So, we want a psi 2. Which is equal to a C2, and we have to have an a dagger and then a psi 1, right? And we again do this psi 2, psi 2. That normalization gives you a C2 squared, and this is equal to a psi 1. a a dagger psi 1, and again I change that C2 mod squared is equal to a psi 1 a dagger a plus 1, and then you have a psi 1 there. So now we have this C2 mod squared, and we have two terms.

One is psi 1 a dagger a. So this is a psi 1 plus psi 1 psi 1. This is normalized, and the left-hand side is equal to 1, and this is nothing but 1 because n acting on psi 1 gives me these 1 and psi 1. So, when you take an overlap with psi 1, it gives you 1. So, this is equal to C2 squared, and we have 1 plus 1. So, that gives you C2 is equal to 1 by root 2.

So, my psi 2 comes out as 1 by root 2 a dagger psi 1, and that will be 1 by root 2. And if I do it twice, I can generate this psi 2 state if I do it twice on the ground state. So, that tells you that psi 2 dagger can be generated from the psi 0. By with this, you know these kinds of normalization and a dagger squared, and so we successively—so when we are the ground state and we want to go to the second excited state, we have to create two oscillators, which means that we successively operate the ground state by these a dagger twice, okay. So similarly, an nth eigenfunction of this oscillator can be written as psi n. So this is, you know, similarly, or you can say that by induction.

$$|\psi_2\rangle = c_2\, a^\dagger |\psi_1\rangle$$

$$\langle\psi_2|\psi_2\rangle = |c_2|^2 \langle\psi_1| a a^\dagger |\psi_1\rangle$$

$$= |c_2|^2 \langle\psi_1| a^\dagger a + 1 |\psi_1\rangle$$

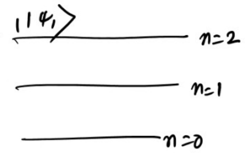
$$1 = |c_2|^2 \left[\langle\psi_1| a^\dagger a |\psi_1\rangle + \langle\psi_1|\psi_1\rangle\right]$$

$$= |c_2|^2 [1+1] \Rightarrow c_2 = \frac{1}{\sqrt{2}}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} a^\dagger |\psi_1\rangle = \frac{1}{\sqrt{2}} (a^\dagger)^2 |\psi_0\rangle$$

$$\boxed{|\psi_2^\dagger\rangle = \frac{1}{\sqrt{2}} (a^\dagger)^2 |\psi_0\rangle}$$

$$N|\psi_1\rangle = 1|\psi_1\rangle$$



So this is C n and a dagger, and it is on psi n minus 1, and so this is equal to psi n plus psi n and this is c n mod square psi n minus 1 a dagger a plus 1 psi n minus 1 and so on and then again I calculate this a dagger a acting on psi n minus 1 will give me n minus 1, and this will be like cn mod square, and then we have this psi n minus 1, n psi n minus 1, and plus psi n minus 1, psi n minus 1, okay. So this will give me n minus 1. So this is like a C n square.

And then there's a n minus 1 plus 1. So this is equal to this whole thing is equal to 1. So C n is equal to. So this will cancel. And this is like 1 by root n. So a psi n.

that is an nth arbitrary n, for an arbitrary n, the nth eigenstate can be generated from the ground state by successively operating the creation operator that many times. So, this is like 1 by root n and a dagger to the power n and psi of n. I mean, here, of course, we have these, we will write this as a dagger and acting on psi n minus 1. And so this can be written as 1 by, you know, root n. And we'll have n minus 1 and then we have a a dagger square.

Well, I should not write it here. It should still be n. a dagger square and then we have also a root over n minus 1 and then we have a psi n minus 2. So, if you want it to be from the psi 0, so psi n can be generated from the this the ground state as psi 0, sorry psi 0. and you operate it by a dagger to the power n n times.

So, a dagger operated on n times on this psi 0 will produce the nth eigenstate of the problem and you see a 1 by n factorial because if you collect all these root over n, n minus 1, n minus 2, etc. that will give you the 1 by root over n factorial which comes as the this normalization constant. And of course, these are normalized, these psi n's, each of the psi n's are normalized and so on. So, we have the psi n and psi nm and this is equal to delta nm and each one of them is normalized, we know, and this psi n, the outer product is equal to 1.

Similarly, $|\psi_n\rangle = c_n a^\dagger |\psi_{n-1}\rangle$

$$\langle\psi_n|\psi_n\rangle = |c_n|^2 \langle\psi_{n-1}|(a^\dagger a + 1)|\psi_{n-1}\rangle$$

$$= |c_n|^2 \left[\langle\psi_{n-1}|N|\psi_{n-1}\rangle + \langle\psi_{n-1}|\psi_{n-1}\rangle\right]$$

$$= |c_n|^2 \left[n - \cancel{1} + \cancel{1}\right] = 1$$

$$c_n = \frac{1}{\sqrt{n}}$$

$$|\psi_n\rangle = \frac{1}{\sqrt{n}}(a^\dagger)|\psi_{n-1}\rangle = \frac{1}{\sqrt{n}}(a^\dagger)^2\frac{1}{\sqrt{n-1}}|\psi_{n-2}\rangle$$

$$\frac{1}{\sqrt{n(n-1)(n-2)}}$$

$$\boxed{|\psi_n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|\psi_0\rangle}$$

Okay, so this is in a nutshell what we wanted to discuss. Let me see a little more properties of this A and A dagger which will help you to appreciate this ongoing discussion more. So, these application of A and A dagger. And what we do is that we will simply show that that let me write down this result first and then we will show this. So, n plus 1 psi n plus 1.

So, even if psi n is not an eigenstate of a dagger, it still gives you a relation which is known to us. a psi n is equal to root over n psi n minus 1. So, a or a dagger acting on the eigenstates of n, that is psi n, changes the state n gives you an eigenvalue which is not eigenvalue rather that gives you a coefficient which depends on n. Because I cannot call it an eigenvalue equation because it is not returning me back the state, it is giving me some other states. So, the coefficient that comes has a dependency on n. So, let me prove these relations.

So, a psi n is equal to 1 by root n a a dagger psi n minus 1. So, I am proving the second one first. Because psi n plus 1, so a psi n is a, now the psi n I am generating from n minus 1 by operating it by a square, I mean with a dagger, okay, a dagger. A dagger acting on psi n minus 1 and then of course will give me a psi n and with this normalization that you have 1 by root n.

So we have this a psi n which is equal to that. So this is equal to 1 by root over of n a dagger a plus 1. This is a psi n minus 1. Okay, so this is equal to 1 by root over of n, 1 by root over of n and you have this as you know a dagger a which is an eigenstate of psi n or psi n minus 1 plus psi n minus 1 and so this gives you 1 by root n And then there is a n minus 1 psi n minus 1 and plus.

So there's a psi n minus 1. So we have plus 1, and then there is a psi n minus 1. Okay, there. So, this is equal to a root over n psi n minus 1, which is what we have claimed. So, a acting on psi n gives you a different state and returns you a coefficient, which is root over n, but changes the state, and it is definitely not an eigenstate of psi n or, rather, psi n is not an eigenstate of a.

Application of a and $a^\dagger$

$$a^\dagger|\psi_n\rangle = \sqrt{n+1}\,|\psi_{n+1}\rangle$$

$$a|\psi_n\rangle = \sqrt{n}\,|\psi_{n-1}\rangle.$$

$$a|\psi_n\rangle = \frac{1}{\sqrt{n}}\, a\,\underbrace{a^\dagger|\psi_{n-1}\rangle}_{|\psi_n\rangle}$$

$$= \frac{1}{\sqrt{n}}(a^\dagger a + 1)|\psi_{n-1}\rangle = \frac{1}{\sqrt{n}}\left[a^\dagger a|\psi_{n-1}\rangle + |\psi_{n-1}\rangle\right]$$

$$= \frac{1}{\sqrt{n}}\left[n-1+1\right]|\psi_{n-1}\rangle.$$

$$\boxed{a|\psi_n\rangle = \sqrt{n}\,|\psi_{n-1}\rangle.}$$

Similarly, we can show the other relation, that is, a dagger, okay. I just, you know, leave it to you to, you know, do it, but it is the same thing. You take a a dagger of psi n, and then these a dagger, you can, I mean, the psi n, you can write it in terms of n and so on, or if you want it, I mean, we can do it in just… one or two lines. So, we have these adjoint equations of these ones that we have done: psi n a is equal to root over n plus 1 psi n plus 1, and psi n of a dagger is equal to root over n psi n minus 1.

And so basically, a decreases. You can again show that what we have written here is that A dagger acting on psi n. is equal to root over n plus 1 and psi n plus 1, okay. That is coming from the top expression that you have. And finally, you know, so we have these X. Are these X and P? Do they have the psi n as the eigenfunctions? And it does not look like because since a and a dagger, their eigenfunctions psi n's are not eigenfunctions of them.

So x and p will also not have psi n's of the eigen functions. Well, let us see that. So, x psi n is, now we have gone back to the original x operator, which is h cross over m omega, and then there is a 1 by root 2. So, you can combine the root 2 here itself. So, it is h cross by 2 m omega, and it is a dagger plus a psi n. Now, it is very clear that for x and p also, we are not,

talking about these eigenfunctions. I mean, the psi n's are not eigenfunctions of x and p either. So, but H is formed of x and p, but luckily H is formed of n plus half. So, the psi n's are eigenfunctions of n and hence they would be eigenfunctions of x. So, p and x are combined in such a manner that even though x and p do not have eigenfunctions of psi n, but H has eigenfunctions, psi n as a eigenfunctions.

So, this is equal to H cross by 2m omega and then we have this a dagger acting on this, we give n plus 1. By the way, in some books, you will have psi n written as n, just again n ket, but it means the same thing. So, psi n plus 1 and plus root over of n psi n minus 1 and so these are that and so basically, so P psi of n is equal to I m omega h cross by 2 and this you can show that it is equal to n plus half psi n plus 1. minus root over n psi n minus 1. So, in the same spirit that psi n is not an eigenfunction of x, it is not an eigenfunction of p as well.

$$\langle\psi_n|a = \sqrt{n+1}\,\langle\psi_{n+1}| \qquad |\psi_n\rangle = |n\rangle$$

$$\langle\psi_n|a^\dagger = \sqrt{n}\,\langle\psi_{n-1}|.$$

$$a^\dagger|\psi_n\rangle = \sqrt{n+1}\,|\psi_{n+1}\rangle$$

$$x|\psi_n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,(a^\dagger + a)\,|\psi_n\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\,|\psi_{n+1}\rangle + \sqrt{n}\,|\psi_{n-1}\rangle\right]$$

$$p|\psi_n\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\left[\sqrt{n+1}\,|\psi_{n+1}\rangle - \sqrt{n}\,|\psi_{n-1}\rangle\right].$$

And one last thing that I want to show you is that what are the matrix representations for a and a daggers. Okay, so we have these relations that is psi n prime, this which is some state which is different than n maybe. So in this particular case, it's definitely this is different than n. So it's root over n delta n prime n minus 1. And similarly, for the psi n prime, a dagger psi n, this is equal to root over n plus 1 delta n, so this n prime, n prime n plus 1, okay? So, what is the matrix representation for a?

So, a matrix representation and now this is say written like this. So, this way you have n running this way and n prime running this way, okay. So it is the matrix elements are root over, you know, n into n root over n. But it's connecting states which are n prime will be n minus one. So you have all the diagonal elements to be equal to zero.

And the off diagonal elements are like root over one here. root over 2 here, and root over 3 here, and so on, okay? And there's been nothing. So, there's 0 here, there's a 0 here, and the only non-zero will be like this. So, it's in this n, n prime basis because your relationship is, so you have a n prime, now I'm writing it psi n prime as n prime,

n prime a n this is equal to root over n delta n prime n minus 1, which means if n prime is equal to n minus 1. So if n prime is say 1, then n has to be 2. That's why this way n runs and this way n prime runs. So the 1, 2 element survives, which has a value root over 1, and similarly this 2, 3 element survives, which has a value root over 2, and so on. And because there is nothing in the diagonal element, it is very clear that these psi n's are not the eigenstates of a and a dagger. So similarly, we can write it for a dagger. dagger as well, and then we again have 0, 0, 0, 0, and so on. Now we have a root over 1 here and a root over 2 here, and so on, because this relation is again n prime is equal to n plus 1.

So, you have a 1, 2 kind of, I mean, so this will have n prime a dagger n is equal to root over n plus 1, and we have a delta n prime n plus 1. So, if n prime is equal to 2, then n is equal to 1. So, you have this 2, 1 element and 3, 2 element, etc. They survive.

Matrix representations for a & $a^\dagger$

$$\langle \psi_{n'} | a | \psi_n \rangle = \sqrt{n}\, \delta_{n', n-1}$$

$$\langle \psi_{n'} | a^\dagger | \psi_n \rangle = \sqrt{n+1}\, \delta_{n', n+1}.$$

$$\langle n' | a | n \rangle = \sqrt{n}\, \delta_{n', n-1}$$

$$\langle n' | a^\dagger | n \rangle = \sqrt{n+1}\, \delta_{n', n+1}$$

$$a = \begin{array}{cc} & \xrightarrow{n} \\ \downarrow & \begin{bmatrix} 0 & \sqrt{1} & & & \\ & 0 & \sqrt{2} & & 0 \\ & & 0 & \sqrt{3} & \\ 0 & & & 0 & \\ & & & & \ddots \end{bmatrix} \end{array}; \quad a^\dagger = \begin{bmatrix} 0 & & & \\ \sqrt{1} & 0 & & \\ & \sqrt{2} & 0 & \\ & & & 0 \\ & & & & \ddots \end{bmatrix}$$

So these are the matrix representations, and at least if we do the ground state wave function and the first excited state wave function, at least have a form for them that will make the story complete because you have been able to picture it as these various energy levels of the harmonic oscillator. They are occupied by the number of oscillators corresponding to which is exactly the same as the quantum number that is associated with it, namely the N. So, let us at least do one or two wave functions. So, let us start with the ground state wave function and try to get an idea of what is And so, we start with A psi 0 equal to 0, which is known because you cannot do any more destruction of particles or annihilation of particles; there is no particle at all.

And we take this in the original operator. Now, we need these original operators x and p because we need all these m omega, etc. They are present so that you can have a one-to-one correspondence with what you have learned through this algebraic method. So, this is equal to x plus I divided by root over m omega h cross p, and that acting on psi 0 is a 0. What I have done is that I have written a in terms of x and p, which is what you see here.

We have written somewhere x and p in terms of, so we have written that at the beginning where we have written, you know, a and a dagger in terms of x and p, and also the inverse relation, that is, x and p in terms of a and a dagger. So here we are using that we are writing a in terms of x and p. So we will write p as minus ih cross d dx or del del x. So if you do that, then what we have is that m omega by h cross x plus d dx, that is equal to, and we have this psi 0x. So it's in the coordinate space.

And this is equal to 0. So, if you solve this, you have a derivative of x. So, what you do is that you bring this x down. So, it is like a psi 0 prime divided by psi 0 prime x, which is d dx of psi 0 x divided by, and so this d psi dx, and then there is a psi. So, we have a psi 0 prime and a psi 0 x, and this, if you solve, what you get is that the psi 0 x is equal to some constant exponential minus m omega by 2 h cross x squared.

This is very comforting because indeed the lowest eigenstate, or the ground state wave function of an oscillator, is simply a Gaussian, which we have seen earlier. So it is basically that ground state, which is a Gaussian; we have to draw it a little. Okay, so that's the ground state we got by this method. Okay, and you can get the first excited state as well by operating it on this a dagger and then multiplying by. So I sort of left these one or two steps here, but they can be filled in by you easily.

So now these normalization constants have to be found out, and this normalization constant can be found out by taking this equal to 1 and. 1 and this thing comes out as m omega by pi h cross whole to the power 1 fourth. So what you do is that you take a c square exponential minus m omega by 2 h cross. So it becomes h cross x square dx. And write down this integral as exponential minus alpha x square from minus infinity to plus infinity is nothing but root over pi by alpha.

Ground State Wavefunction.

$a|\psi_0\rangle = 0$

$p = -i\hbar \frac{d}{dx}$

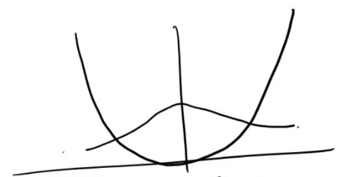
$\frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\omega}{\hbar}}\,x + \frac{i}{\sqrt{m\omega\hbar}}\,p\right]|\psi_0\rangle = 0$

Coordinate Space

$\left(\frac{m\omega}{\hbar}x + \frac{d}{dx}\right)|\psi_0(x)\rangle = 0$

$\frac{\psi_0'(x)}{\psi_0(x)}$

$|\psi_0(x)\rangle = C e^{-\frac{m\omega}{2\hbar}x^2}$



$\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{\alpha}}$

$\langle\psi_0(x)|\psi_0(x)\rangle = 1$

$c^2\int e^{-\frac{m\omega}{\hbar}x^2}\,dx$

$C = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$

This is called a Gaussian integral. So using this Gaussian integral, you can do that. And now, once you have done, or rather found, this ground state wave function, all the other eigenstates of the oscillator can be generated. And how you do that is the following: you can write down this psi n of x, which is a 1 by root over n factorial, and you can write it as. psi 0 x or rather this is in the coordinate space.

So, this is like a a dagger whole to the power n and the psi 0 x. So, do it for n equal to 1 and you will get this state. That is the psi 1 and so on so forth. And once you do, I mean, everything and then do a normalization of that as well, what you get is that 2 to the power n, n factorial, it is h cross by m omega whole to the power n and this whole to the power half. m omega by pi h cross n whole to the power one-fourth and we have this exponential function. Okay, so this is that a dagger which can be now written as m omega by h cross x minus d dx whole to the power n

And we have this round state, which is minus m omega by 2 h cross x squared. And you can really get all these things. So, just a few lowest conditions. So, psi 1 is, if you do it once, then it comes out as 4 over pi and m omega by h cross whole cube, this whole to the power 1 fourth and x exponential minus m omega by 2 h cross x square and so on so forth. So, psi 2x is equal to, this is equal to m omega by 4 pi h cross whole to the power 1 fourth.

And we have this 2m omega by h cross x square minus 1 e to the power minus m omega by 2h x square. So, this is the Hermite polynomial H1. This is the Hermite polynomial

H2 and so on. Okay, so you can actually generate all of them. And in terms of these, like whatever we have seen it there and.

$$|\psi_n(x)\rangle = \frac{1}{\sqrt{n!}} (a^+)^n |\psi_0(x)\rangle$$

$$= \left[\frac{1}{2^n n!}\left(\frac{\hbar}{m\omega}\right)^n\right]^{1/2} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left[\sqrt{\frac{m\omega}{\hbar}}\, x - \frac{d}{dx}\right]^n e^{-\frac{m\omega}{2\hbar}x^2}.$$

$$\psi_1(x) = \left[\frac{4}{\pi}\left(\frac{m\omega}{\hbar}\right)^3\right]^{1/4} \underbrace{x}_{H_1(x)} e^{-\frac{m\omega}{2\hbar}x^2}.$$

$$\psi_2(x) = \left(\frac{m\omega}{4\pi\hbar}\right)^{1/4} \Big(\underbrace{\frac{2m\omega}{\hbar}x^2 - 1}_{H_2(x)}\Big) e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\vdots$$

So these will give you the entire spectrum as well as the eigenfunctions of that. Now, one last quick thing that we want to do here is talk about the degeneracy of these. oscillators. Now, the oscillators, of course, do not have degeneracy in 1D. So, let us see what kind of degeneracies we get in 3D.

So, what we have is that the energy is given by n plus 3by 2 h cross omega, and this is equal to nx plus 1 by 2 plus ny plus 1 by 2 plus nz plus 1 by 2. So, in each direction, we have this. So, it is nx plus ny plus nz plus 3 by 2. So, which means that nx plus ny plus nz is equal to n, not 3 by 2. So, the degree of degeneracy will have to obey this.

I mean, I am so sorry, this is equal to n, not 3 by 2. So, this is equal to n. So, that is why the 3 by 2 comes from these three halves. Okay. So, let's first choose nx to have any of these values. 0, 1, 2, and this n, right?

I mean, it can go all the way up to n. In that case, your ny and nz, for the last value that you see here, your ny and nz will be equal to 0. So, we have ny plus nz, that is equal to n minus nx, okay? So, what are the different possibilities for ny, nz? So, this ny, nz pair, have possibilities as n minus nX plus 1.

How this is coming? It will be 0 n minus nX that is for when nX is equal to 0. So, this is 1. then it is 1 n minus nx minus 1, then it is 2 n minus nx minus 2 and so on and this will be the final one will be n minus nx and 0. So, these are the possibilities for this ny and nz.

And these, if you count, they are n minus nx plus 1 because it started from 0. So, the degeneracy can be now calculated. So, degeneracy, let us call it as g n. So, g n is equal to n x equal to 0 to n, n minus n x plus 1. That is the pair n y n z can take all these values and where n x will go from 0 to n. So if you do the simplification, so this becomes n plus 1 and sum over nx equal to 0 to n and sum over 1.

Degeneracy.

In 3D. $E_n = \left(n + \frac{3}{2}\right)\hbar\omega = \left(n_x + \frac{1}{2} + n_y + \frac{1}{2} + n_z + \frac{1}{2}\right)\hbar\omega$.

$n_x + n_y + n_z = n$.

Choose n_x to have any of $0, 1, 2 \ldots n$

$n_y + n_z = n - n_x$.

$\{n_y, n_z\} \Rightarrow (n - n_x + 1)$.

$\{0, n - n_x\}, \{1, n - n_x - 1\}, \{2, n - n_x - 2\} \ldots \ldots \{n - n_x, 0\}$.

That's the last term and last term is n plus 1 term. So the first term and the last term. So this is n plus 1 and then you have a sum over 1. And now you have n x equal to again 0 to n sum of n natural numbers. But now there are n plus 1 of them.

So this is n plus 1 into n plus 1. You sum 1 for n plus 1 times. So it's n plus 1 into n plus 1 minus this is n which is n plus 1 times. So it's n into n plus 1 by 2. And so this is equal to n plus 1 whole square and this is n into n plus 1 by 2.

And if you do a simplification of that, it becomes equal to, so if you take a n plus 1 common, so you have a n plus 1 minus n by 2. So this will become equal to n plus 1. and 2n plus 2 minus n divided by 2. So, this is equal to what n plus 2. So, it is n plus 1 into n plus 2 divided by 2.

So, the ground state is, of course, degenerate and so on. So, for n, which is nx plus ny plus nz. So, suppose n equals 3, how many-fold degenerate? So, the degeneracy of that is equal to 3 plus 1, multiplied by 3 plus 2, divided by 2. So, that is like 4 multiplied by 5.

Divided by 2, so that is like 10-fold degenerate, okay. So, the harmonic oscillator in higher dimensions—now we have shown it for three dimensions—has a degeneracy which goes as n multiplied by n plus 1, n plus 1 multiplied by n plus 2, divided by 2. And so, each of these energy levels, which was non-degenerate in 1D, in 3D, they become degenerate, with the degeneracy for a given n we have found to be equal to 10-fold. So, that completes the discussion of this harmonic oscillator. We have done both, you know, this algebraic method of solution, and then we also have, you know, figured out a solution using these a and a dagger operators.

Degeneracy

$$g_n = \sum_{n_2=0}^{n} (n - n_2 + 1) = (n+1)\sum_{n_2=0}^{n} 1 - \sum_{n_2=0}^{n} n_2$$

$$= (n+1)(n+1) - \frac{n(n+1)}{2}$$

$$= (n+1)^2 - \frac{n(n+1)}{2}$$

$$= (n+1)\left[n+1 - \frac{n}{2}\right]$$

$$= \frac{(n+1)}{2}\left[\underbrace{2n+2-n}_{n+2}\right] = \frac{(n+1)(n+2)}{2}$$

$$n = 3 \rightarrow \frac{(3+1)(3+2)}{2} = \frac{4\times 5}{2} = 10 \text{ fold.}$$

They are all very neat; one does not have to go through very rigorous mathematics, but just by looking at the properties of these a and a dagger, which are called raising and lowering, but more conveniently, or rather more, you know, more usually or more frequently, they are called the creation and the destruction operators or annihilation operators. Creation and annihilation operators. Sometimes they are called creation and destruction operators. And this gives you a complete description of the problem without resorting to the algebraic method that we have seen earlier. So, I will stop here for today.

Thank you.