Introduction to Statistical Mechanics

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Lecture - 11
Radiation Thermodynamics

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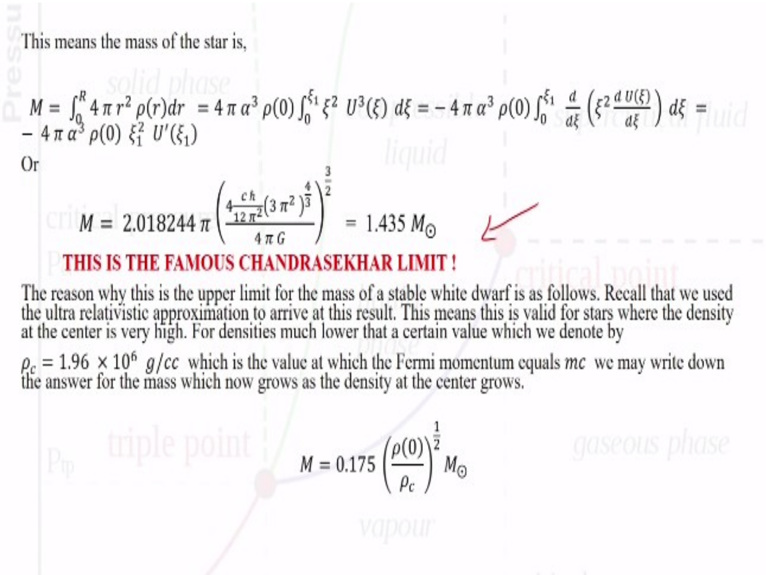

Okay, so this is where we had stopped. We had derived the famous Chandrasekhar limit and we were able to show that there is a mass limit to the white dwarfs, beyond which it cannot be stable okay.

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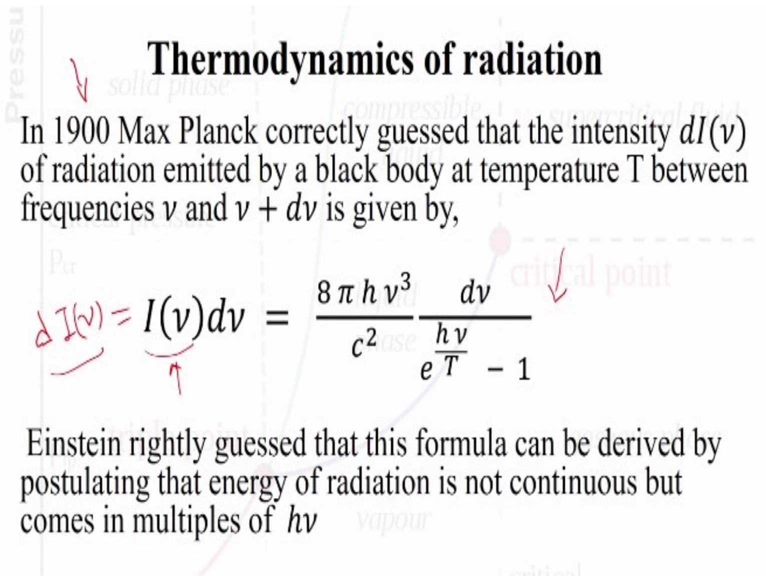

So now, let us get to another topic which is called thermodynamics of radiation. So, in 1900 Max Planck correctly guessed that the intensity of radiation which is emitted by a black body between frequencies say ν and $\nu + d\nu$, so which I denote by dI_ν . So, this is in some sense as intensity per unit frequency. So, I_ν is the intensity per unit frequency and dI_ν is the intensity emitted between frequencies ν and $\nu + d\nu$.

So, Planck by looking at the experimental data guessed that it should have this form and it fitted that experiment perfectly. So, Einstein later on correctly surmised that it is possible to derive this relation, this Planck's guess by postulating that the energy of radiation is not continuous, but comes in multiples of quanta called $h\nu$.

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Consider a cube of length = breadth = height = L with perfectly reflecting inner walls filled with radiation. If there is a small hole in the walls, the radiation emitted from the hole is approximately the radiation of a blackbody (strictly the hole should be both large compared to the wavelength and much smaller than the size of the box. For an ideal black body, all wavelengths are represented which means the hole has to be bigger and bigger which means the size of the box has to go to infinity as well.). Photons are elementary excitations of the EM field and as such the energy contained in them may be written as (harmonic oscillator levels, $r_{k,s}$ is the number of photons with wavevector $\boldsymbol{k}$ and polarization state s)

$$E_{\boldsymbol{k}}(r_{\boldsymbol{k},s}) = \left(r_{\boldsymbol{k},s} + \frac{1}{2}\right) \hbar\, \omega_{\boldsymbol{k}}$$

The wavevector $\boldsymbol{k} = \left(\frac{n_x \pi}{L}, \frac{n_y \pi}{L}, \frac{n_z \pi}{L}\right)$ and $\omega_{\boldsymbol{k}} = c\,|\boldsymbol{k}|$. Furthermore for photons there are two polarization states $s = 1,2$.

Okay, so, let us try to understand how this comes about, how to derive Planck's black body radiation by combining you know, Einstein's quantum hypothesis, namely the energy contained in the radiation is not continuous, but comes in multiples of $h\nu$. So, that is Einstein's quantum hypothesis for radiation. So, that will be combined with our traditional statistical mechanics ideas, which we have been developing till now, and we can try and derive, you know, Planck's black body distribution. So, how do we do that?

So, the idea is that your black bodies, you can think of a black body as you know, black body by definition is something which absorbs all the radiation that is incident on it. Of course, it also emits radiation, so in that sense the blackness does not refer to the fact that it does not emit, but rather it refers to the fact that it absorbs all frequencies equally, in other words, that that whatever falls on it, whatever frequency falls on it, it absorbs.

So, you can sort of simulate this black body in some approximate way in the following sense by imagining a large box in which a small hole is drilled and you pass radiation of all frequencies into that hole. So, all the radiation that goes in will get trapped, so there is no chance of it escaping, so except that, if you want to put a detector near that hole, you will be able to sense what has been trapped inside. So, that is pretty much how you find out what is trapped inside.

So, there are some technical details about how that has to be done with care, so that I will allow you to read this paragraph here, but otherwise that is the basic idea. So you have a huge box and you drill a small hole through it and, you know, feed that radiation through the hole. Because that box is so huge and the inner walls are completely reflecting, so the radiation kind of bounces off the walls continuously and the chances of it exiting through that small hole are very small because of the smallness of the area of that hole okay.

So, that is the basic idea of a black body. Now based upon Einstein's idea, we can think of the energy contained in a certain, so imagine that there is a photon which is moving with wavevector k okay, so that is this k with a bold face. So now, we know from harmonic oscillator type of ideas that if, $r_{k,s}$ is the number, I will tell you what s is later, but this r number, this number r here refers to the number of photons with momentum k, okay.

So you know from your harmonic oscillator types of ideas that if you have n quanta, the energy of those b quanta in harmonic oscillator will be n+1/2 hν or in this case ħω, which is the same thing. So, the energy contained is basically n+1/2 hν. So, that is what I have written here. So now, this is the energy. Now, I will tell you what s is. So, you see electromagnetic radiation is a transfer. So that means, in empty space it transfers radiation, so in other words, the electric field has 2 components perpendicular to the direction of propagation.

So, there are 2 modes. So those modes will contribute separately to the energy. So s can be 1 or 2 depending upon the two perpendicular directions, so to the direction of propagation, which is k. So now, this k itself, so remember that this radiation is actually trapped in that black box, I mean that box which is basically the black body. So there is a huge box in which it is trapped. So as a result, the k vectors are quantized in this fashion as you very well know.

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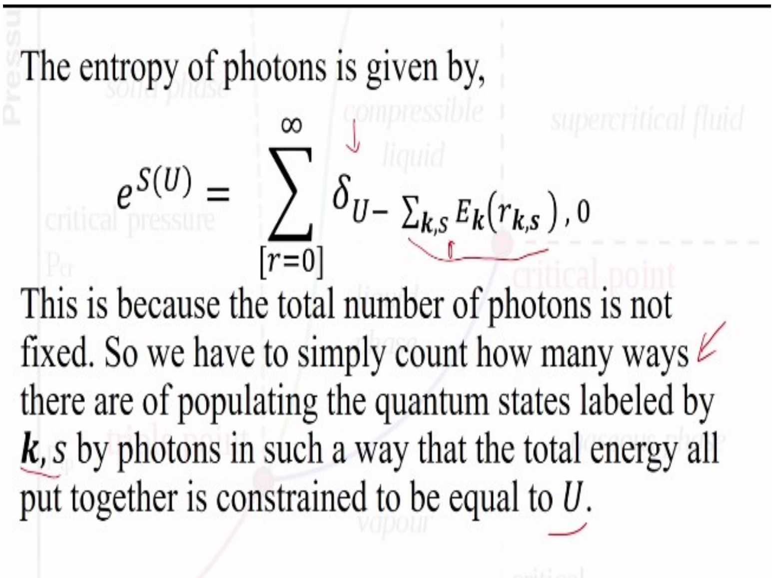

So, now I am going to ask myself how do I calculate the entropy of the system? So the way I calculate the entropy is that I have to, of course, the entropy subject to the constraint that the total energy is U. So the total energy of all the modes put together is U. So this is the total energy of all the modes put together. So, I have to sum over all the wavevectors, which is basically all the frequencies of light that is trapped in the black body and also I have to sum over all the polarizations.

So put together, that is U, so that is signified by this Kronecker delta and then I have to sum over all the number of photons. So remember that in this problem, unlike in the case of marbles on a staircase, where the number of marbles was fixed, here the number of photons are not fixed, it is only the total energy that is fixed. The reason why the number of photons is not fixed is because the photons are being continuously absorbed and reemitted by the walls of the black box.

So, the internal walls of the black box kind of they absorb the photons and reemit, so there is no constraint on how many photons there are, there is only a constraint on how much energy it has. So, we just have to count the number ways in which there are of populating these quantum states labeled by these quantum numbers ks in such a way that total energy is U. So, that is going to be e raised to the entropy of the system.

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The Kronecker delta can be written in the usual integral representation

$$e^{S(U)} = \sum_{[r=0]}^{\infty} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\,\theta\,(U - \sum_{k,s} E_k(r_{k,s}))}$$

This assumes that $Q = U - \sum_{k,s} E_k(r_{k,s})$ is an integer. This not an unreasonable simplification since for systems in thermodynamic limit, Q is macroscopic and therefore arbitrarily close to some integer unless it is actually zero. Hence we write,

$$e^{S(U)} = \sum_{[r=0]}^{\infty} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\,\theta\,U} e^{-i\,\theta \sum_{k,s}\left(r_{k,s}+\frac{1}{2}\right)\hbar\,\omega_k}$$

So as usual, we are going to do it this way that we are going to write it in terms of the Kronecker delta. Of course, that takes the liberty of assuming that this difference is an integer, but then that is a small price to pay, I mean, strictly speaking, you might complain that how do you know this is an integer, but this is an integer for sufficiently large, so if U is sufficiently large, if your system is macroscopic, so any real number is arbitrarily close to an integer.

If you have 1 million point 1, so which is not an integer, you can always approximate it as 1 million without loss of much generality, I mean, you are not paying a huge penalty by approximating 1 million point 1 by 1 million. Of course, if the energy is only 2 or 3, it makes a big difference whether it is 2.1 or 2, but if it is 1 million point 1, it does not make much of a difference if you approximate it by the nearest integer.

So, we are going to pretend that this difference is either 0 or it is an integer. So now, what we are going to do is we are going to substitute the formula for the energy into this expression and we integrate over this θ.

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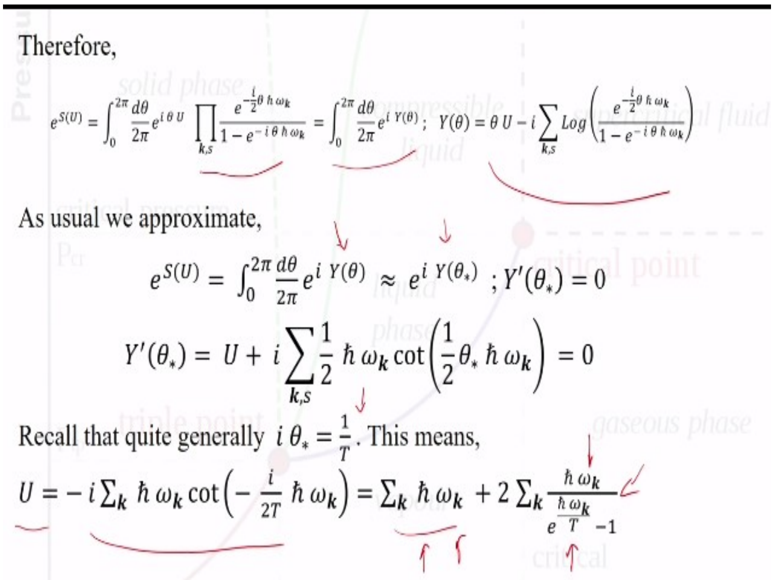

So, but before we do that, we see that we can sum over the r first. So, that is what we have been doing all along, we interchange the order in which we do things. So, this summation becomes a product. So e raised to summation is basically a product of e raised to that thing okay. So, I am going to use that and then do the summation first, summation over r first. So, this is going to be the final answer after doing summation over r. So now, all I am left with is doing an integral over θ.

So that expression that I have to integrate is $e^{iY(\theta)}$, where Y(θ) now has this form. So as usual, I am going to use my saddle point assumption and say that it is legitimate to replace this Y by its most probable value, which is $Y(\theta^*)$ and θ^* is the most probable value of θ. So that is determined by the saddle point, which is $Y'(\theta^*) = 0$. So, recall that this was a very general result, $i\theta^*$was 1 by temperature and which basically allows me to write down a formula for the energy.

So the energy comes out as summation over, this is a zero point energy, so that is because of that n plus half, that half thing is contributing that is a zero point motion of the photons and this is the rest of it. So this how energy of the radiation trapped in a black body is related to temperature and it is related to the frequency of the photons okay.

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The Poynting vector is $\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E} \times \boldsymbol{H} = \frac{c}{4\pi}\boldsymbol{E} \times \left(\hat{k} \times \boldsymbol{E}\right) = \frac{c}{4\pi}\hat{k}\,\boldsymbol{E}^2$

The energy (minus the zero point energy) density is $u = \frac{1}{8\pi}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) = \frac{1}{4\pi}\boldsymbol{E}^2$ (in the time averaged sense).

Hence the Poynting vector for a given **k** is,

$$\boldsymbol{S}_{\boldsymbol{k}} = c\,\hat{k}\,u = \frac{2}{V}\frac{c\,\hat{k}\,\hbar\,\omega_k}{e^{\frac{\hbar\omega_k}{T}} - 1}$$

The intensity due to all wavelengths moving in all directions is,

$$I = \sum_k \hat{k}\,.\boldsymbol{S}_{\boldsymbol{k}} = \frac{2}{V}\sum_k \frac{c\,\hbar\,\omega_k}{e^{\frac{\hbar\omega_k}{T}} - 1} = \frac{2}{V}\sum_k \frac{c\,\hbar\,\omega_k}{e^{\frac{\hbar\omega_k}{T}} - 1} = \frac{2}{(2\pi)^3}\int_0^\infty \frac{c\,\hbar\,\omega_k}{e^{\frac{\hbar\omega_k}{T}} - 1} 4\,\pi\,k^2\,dk$$

But $\omega_k = 2\,\pi\,\nu = c\,k$. Hence,

$$I = \frac{8\pi}{c^2}\int_0^\infty \frac{h\,\nu}{e^{\frac{h\nu}{T}} - 1}\nu^2\,d\nu$$

The intensity in the frequency interval ν and $\nu + d\nu$ is,

$$I(\nu)d\nu = \frac{8\,\pi\,h\,\nu^3}{c^2}\frac{d\nu}{e^{\frac{h\,\nu}{T}} - 1}$$

THIS IS THE FAMOUS PLANCK'S BLACKBODY DISTRIBUTION!

Now, well that is not what Planck calculated, energy is not what Planck calculated, what Planck calculated was, what Planck deduced from the black body data was the intensity of radiation trapped in a black body between frequencies ν and ν + d ν. So how do we derive that? So in order to derive that, we have to look at a quantity in electromagnetic theory, which is related to intensity and you all know that is Poynting vector.

So Poynting vectors in CGS units is

S = c/4π (E X H)

and because it transfers electromagnetic wave, H is nothing but the wavevector direction cross the electric field, and because these E, k, and H form right triad, so they are usually perpendicular vectors, so I am entitled to make this assertion that this is simplifies to this expression. So in other words, the magnitude of the Poynting vector is proportional to the square of the electric field and also it is in the direction of the propagation of the radiation.

So now, notice that I can also find out what is the energy density. So this Poynting vector is E cross H, but then there is also something called the energy density which is 1 by 8 pi, in CGS units, it could be 1 by 8 pi into E squared plus B squared. So in the time averaged sense, this becomes average of B squared is same as average of E squared, so it becomes 1 by 4 pi into E squared. So as a result, you see so by comparing these two, we can conclude that the Poynting vector is basically speed of light times the energy density times the direction of propagation of the radiation.

So it is as simple as that, but now, the energy density if you ignore the zero point motion is basically, we ignore zero point motion because, you know, we want only the energy, we want to count energy beyond the zero point result okay. So, now if you just focus on this and you substitute that expression here for U, and then what you have to do is you in order to find the energy, I mean, rather if you want to find the total Poynting vector that would be the total intensity in all directions. So, this is the intensity in a particular direction for E k.

So, if you want to find the intensity in all directions, you have to take the component of S along k cap and then sum over all the k’s. So if you sum over all the k’s, you get this expression. So then, the summation as you very well know becomes integration and you can write it like this. So now, keep in mind that omega k is nothing but 2 pi nu, which is also ck because omega equal ck is the dispersion relation for a photon. So now this will enable me to instead of k, I am going to substitute in terms of nu now, k is nothing but 2 pi by c into nu, so wherever this k, I am going to substitute nu.

So, then I end up with this is the total intensity over all the frequencies, but then for a given frequency, I can always write like this okay. So, this is nu times nu squared is nu cubed. So, I can always think of this as integral of Iν dν and then I can read off what I ν d ν have been in order for this to be the total intensity over all the frequencies and I get precisely Planck’s black body radiation distribution. So, you see, what we have done, let us recapitulate what we have done, what are the concepts we have had to use to get the thermodynamics of radiation. So we had to invoke Einstein's quanta, that is Einstein's idea that radiation comes in multiples.

So it comes in integer multiple of hν and because of its integer nature, we will get those results that we got, and also we have to invoke our statistical mechanics idea of entropy and while evaluating this, also on the way, we are able to derive energy density of the system. So all these statistical mechanics ideas are also involved along with Einstein's quanta idea, that is, the radiation comes in multiples of hν and it is because of that, that you get this type of Bose-Einstein type of distribution in the denominator here. So, that is the story of the statistical mechanics of radiation okay.

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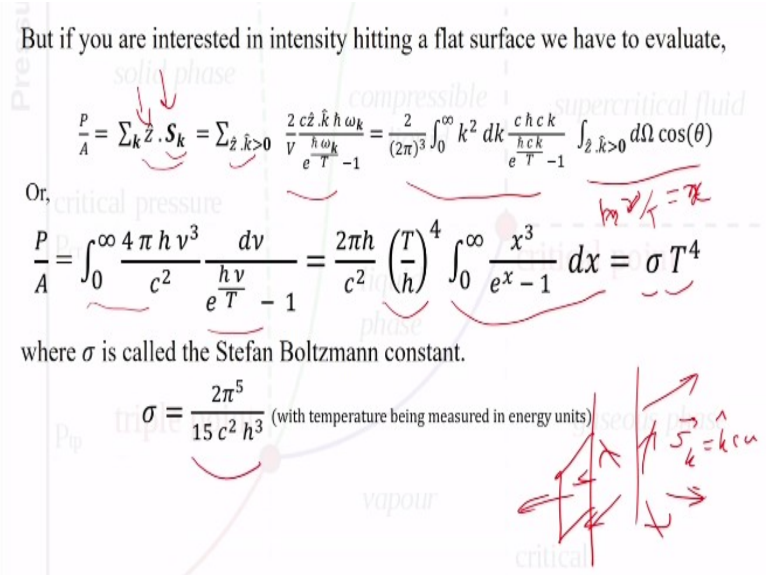

Before I stop here, I can go ahead and derive one more interesting result and that is what is called the total power that is emitted over all the frequencies. So suppose you want to calculate the total power that is emitted, so see the point is that if you have radiation that is being emitted in all directions, so this is sk, so this is k cap cu, so that you can count, well the radiation gets emitted in all directions as you very well know.

So what we have done here is basically calculated the total intensity that is coming out in all directions, but suppose you do not want that, suppose you want only that intensity that is coming out of, say some surface here like this, only in some Z direction, you want to calculate the intensity that is coming out only in that Z direction. So what you have to do is you have to take the Poynting vector and dot product with the Z direction and then you have to make sure that your k vector is.

So you have to ignore the k vectors that are pointing like this, you should only include the k vectors that are pointing in the direction that you are interested in. So in other words, you only look at Z dot k greater than 0 and then you sum over all the case ks where Z dot k greater than 0 and then you look at only the component of s along that Z. So when you do that, you get the total power, so this is basically the power per, intensity is nothing but the power of radiation per unit area you very well know, dimensionally that is what it is.

So, this is what you get when you integrate and so you will have to do this integration and you can do this integration by calling h nu by T as x, yes you remember that, this is what this is that h bar ck by T this is basically h nu by T, which I am going to call x. So, h nu by T is my x and because h nu by T is x, I can delabel, so my temperature goes out. So, I will be able to get the power of the temperature outside and the rest of it is just a pure number. So this pure number, this you can evaluate using your software.

So finally, you can write this, the power per unit area is proportional to the fourth power of the absolute temperature of the black body. So this is called the Stefan-Boltzmann constant and this is the Stefan--Boltzmann law okay. So, this is basically it tells you that the power emitted by a black body is proportional to the fourth power of the temperature of the black body. The total power that is emitted in a given direction is proportional to the fourth power of the temperature of the black body and the proportionality constant is called the Stefan-Boltzmann constant okay.

So now, I am really going to stop here. So in the next class, I am going to discuss the thermodynamics of black holes, which is very interesting and very topical. Because recently, the black hole in galaxy M87 has been imaged using radio astronomy and that was worldwide news as you very well know. So it is very exciting to be discussing the thermodynamics of black holes. So that will be the next topic of my lecture. So, I am going to stop here. Thank you. See you tomorrow.