

**Basic Quantum Mechanics**  
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**Module No. # 08**  
**Angular Momentum – II**  
**Lecture No. # 06**  
**Clebsch-Gordon Coefficients**

In our last lecture, we were discussing the addition of two angular momenta and we had taken this specific case for  $j_1$  equal to half and  $j_2$  equal to half. So, we have something like a neutron-proton problem. Both are both have an intrinsic spin angular momentum of half  $\hbar$  cross.

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$j_1 = \frac{1}{2} \quad \& \quad j_2 = \frac{1}{2}$

$m = m_1 + m_2 = 1, 0, -1$

$| \phi(j, m) \rangle = | \psi(j_1, j_2, m_1, m_2) \rangle$

$| \phi(1, 1) \rangle = | \psi(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \rangle$

$| \phi(1, 0) \rangle = \frac{1}{\sqrt{2}} [ | \psi(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}) \rangle + | \psi(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) \rangle ]$

$| \phi(1, -1) \rangle = | \psi(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) \rangle$

$| \phi(0, 0) \rangle = \frac{1}{\sqrt{2}} [ - | \psi(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}) \rangle + | \psi(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) \rangle ]$

So, how will they add up? We found that the total angular momentum  $j$  will take 1 and 0 and therefore, we have a total of four states and we wrote down that  $j_1$  equal to half. So, therefore,  $m_1$  is equal to half and minus half and  $j_2$  is equal to half. So,  $m_2$  is equal to plus half and minus half. So, therefore, the  $m$  value which is equal to the sum of the 2  $m_1$  plus  $m_2$  will take the values 1 0 0 and minus 1 half plus half half minus half minus half half minus half minus half.

Then, we wrote down, first we wrote down that  $\phi_{1,0}$ . This is common to all  $j=1$  is. So, this is  $j$ , this is the first digit is the value of  $j$  and this is the value of  $m$ . So, let me write it down again. This is  $\phi_{1,1}$ . So,  $j$  value is 1 and  $m$  value is 1. So, when  $m$  is 1,  $m=1$  has to be half and  $m=2$  has to be half. So, therefore, this will be equal to  $\psi_{j-1, j-2}$ . Let me write it down here half, half, half, half. These are the values. The last two digits are the values  $m=1$  and  $m=2$ .

Then, what I did was that I operated this by  $j$  minus and I will get root 2 of  $\phi_{1,3}$  and when I took the root two outs here, so I get  $1/\sqrt{2} \psi_{\frac{1}{2}, \frac{1}{2}}$  and this will become  $j=1$  minus operating on this will be half minus 1. So, this will be minus half and this will be half plus  $1/\sqrt{2} \psi_{\frac{1}{2}, \frac{1}{2}}$  minus half. So, these coefficients  $1/\sqrt{2}$  which relate the  $\phi$  functions to the  $\psi$  functions are known as the Clebsch-Gordon coefficient.

Now, if I operate this again by  $j$  minus, so as you may recall that  $j$  minus ket  $j$   $m$  will be  $j$  plus  $n$ , that is  $1$  plus  $0$  and then, you will have  $j$  minus  $m$  plus  $1$ , that is under root of 2. So, again under root of 2 will come, but a little algebra will show that  $\phi_{1,-1}$  will come out to be you have to do a little algebra. I leave that as an exercise. You will find that this will be half, half, minus half, minus half and that you should have expected because when  $m$  is minus 1,  $m=1$  has to be minus half,  $m=2$  has to be minus half.

Now, how do I get  $\phi_{0,3}$ ? You can do by operating this by this function  $j$  minus or  $j$  plus because the  $j$  value cannot change. So, you use the fact that this has to be orthogonal to that. So, you write down  $\phi_{0,0}$  will be a linear combination of these two, such that  $a$ , it is normalized and  $b$ , it is orthogonal to this. So, that is given by  $1/\sqrt{2} \psi_{\frac{1}{2}, \frac{1}{2}}$  minus half plus  $\psi_{\frac{1}{2}, \frac{1}{2}}$  minus half. So, we finally, actually the values of  $j=1$  and  $j=2$  are repeated everywhere and therefore, one omits them and one writes as  $\psi_{\frac{1}{2}, \frac{1}{2}}$ . So, let me rewrite this.

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$$\begin{aligned}
 | \phi(1, 1) \rangle &= | \psi(\frac{1}{2}, \frac{1}{2}) \rangle = \uparrow\uparrow \\
 | \phi(1, 0) \rangle &= \frac{1}{\sqrt{2}} [ | \psi(-\frac{1}{2}, \frac{1}{2}) \rangle + | \psi(\frac{1}{2}, -\frac{1}{2}) \rangle ] \\
 &= \frac{1}{\sqrt{2}} [ \uparrow\downarrow + \downarrow\uparrow ] \\
 | \phi(1, -1) \rangle &= | \psi(-\frac{1}{2}, -\frac{1}{2}) \rangle = \downarrow\downarrow \\
 | \phi(0, 0) \rangle &= \frac{1}{\sqrt{2}} [ - | \psi(-\frac{1}{2}, \frac{1}{2}) \rangle + | \psi(\frac{1}{2}, -\frac{1}{2}) \rangle ] \\
 &= \frac{1}{\sqrt{2}} [ -\downarrow\uparrow + \uparrow\downarrow ] \\
 &\quad \uparrow \\
 &\quad \text{Singlet state}
 \end{aligned}$$

So, I get phi 1, 1 is equal to psi half, half. These are the values of j and m and these are the values of m 1 and m 2 and you have phi 1, 0 is equal to under root of psi minus half half plus psi half minus half. Let me leave some space and then, this is phi 1, minus 1. This will be psi minus half minus half and finally, the singlet state 0, 0 is equal to 1 over root 2 minus here psi minus half half plus psi half, minus half.

I could have easily well chosen plus here and minus here, but it is a matter of convention, it is chosen minus here and I will tell you why. This state is often represented as spin up state. So, this is up down. This is up up. Both are pointing upwards. This is 1 over root 2 down up plus up down. This is down down and this is 1 over root 2 of minus down up plus up down. These three states are the triplet states and this state is known as the singlet state.



Let me give you another example. I hope this is clear. So, we consider the case, the hydrogen atom problem, the spin orbit interaction. So, we assume that  $j_1$  is equal to  $l$  is equal to 1. So, as we know that the wave functions, the hydrogen atom wave functions are  $r$ , say  $n$  is equal to two states. So, this is  $r_{n1}$ , that is  $r_{21}$  or  $r_{Y1m}$ . This is the  $\theta$   $\phi$ , where  $m$  can take values 1 0 and minus 1 and then, you have this spin up  $s$  is equal to half. So, the  $m_2$  value, this is  $j_2 m_2$  value can take half and minus half. So, therefore, we start with the ket  $\phi_{j_1 j_2}$ . I forget. So, that is 1, half and as I told you that the maximum value of  $m$  is you see  $m_1$ , sorry this is  $m_1$ . So,  $m_1$  is 1 0 minus 1. So, the maximum value is 3 by 2. So, the  $j$  is 3 by 2 and  $m$  is 3 by 2. So, this is the  $j$  value and this is the  $m$  value.

Now, this is equal to the  $\psi$  function corresponding to 1. This is  $j_1$ , this is  $j_2$  and  $m_1$  and  $m_2$  must be equal to one and half because  $m_1$  plus  $m_2$  must give me 3 by 2. Now, in order to get the Clebsch-Gordon coefficients, I must operate this by the operator  $j$  minus. So,  $j$  minus will be  $j$  plus  $m$ . So, 3 by 2 plus 3 by 2 square root of 3 and then,  $j$  minus  $m$  plus 1, that is just 1.

So, if I operate this, if I operate  $j$  minus on this, so I will get  $\phi_{1, \frac{1}{2}, \frac{3}{2}}$  and then, half and this will be this is  $j_1$  and this is  $j_2$ . So,  $j$  minus is equal to on the right hand side I operate this by  $j_1$  minus plus  $j_2$  minus. So, if I operate this by  $j_1$  minus, so this will be  $j_1 j_1$  plus  $m_1$ , that is square root of 2 and then,  $m_1$  will decrease by 1. So, this will be  $\psi_{1, \frac{1}{2}, \frac{3}{2}}$  plus. If I operate this by  $j_2$  minus, so this will be half plus half, that is 1 and  $j_2$  minus  $m_2$  is 3 plus 1. So, that is just 1. So, this will be  $\psi_{1, \frac{1}{2}}$ , this will be 1 and this will be minus half.

So, please see  $m_1$  is 3,  $m_2$  is half. So,  $m$  is half, automatically  $m_1$  is 1. So, here now the Clebsch-Gordon coefficients are square root of 2 by 3. If I divide this by, so if I take this on this side, so square root of 2 by 3 and 1 over root 3 and if I operate this again, I will get three halves and I leave this as an exercise minus half. So, I will get an expression for  $\phi$ . Let me, no I leave that as an exercise. That is very simple. So, you will operate this by  $j$  minus. If I operate  $j$  minus on this, I will get here, so half to minus half. So, then, it will become 3 and minus half and minus 1 and plus half and again, we will be able to get Clebsch-Gordon coefficients.

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The image shows handwritten mathematical equations on a whiteboard. At the top, there is a label  $2 \text{ } (j=3/2, m=3/2) = (0)$ . Below it, the first equation is:

$$|\phi(1, \frac{1}{2}, \frac{3}{2}, \frac{1}{2})\rangle = \sqrt{\frac{2}{3}} R_{2,1}(r) Y_{1,0}(\theta, \phi) + \sqrt{\frac{1}{3}} R_{2,1}(r) Y_{1,1}(\theta, \phi)$$

The second equation is:

$$|\phi(1, \frac{1}{2}, \frac{3}{2}, -\frac{1}{2})\rangle = R_{2,1}(r) \left( \sqrt{\frac{1}{3}} Y_{1,-1}(\theta, \phi) - \sqrt{\frac{2}{3}} Y_{1,0}(\theta, \phi) \right)$$

At the bottom left of the whiteboard, there is a logo for NPTEL.

Let me write down the exact wave functions. Please see this is  $l$  is equal to 1 and  $m$  is equal to 1. So, this will be and this is the spin up state or  $z$  up state. So,  $R_{2,1}$  of  $r$   $Y_{1,1}$  multiplied by  $\frac{1}{\sqrt{3}}$ . This wave function is known as the Pauli wave function corresponding to this state. So, therefore, the Pauli wave function for say,  $n$  is equal to 2 doublet  $l$  is equal to 1, that is  $p$  and I have taken  $3$  by  $2$  and for  $m$  is equal to  $3$  by  $2$ . When  $j$  is equal to  $3$  by  $2$ , then  $m$  can have  $3$  by  $2$  half minus half minus  $3$  by  $2$ . So, the Pauli wave function is this is known as the Pauli wave function  $R_{2,1}$  of  $r$   $Y_{1,1}$ . Let me do the second case.

So, this is the Pauli wave function corresponding to  $m$  is equal to half. So, the  $\phi$  function corresponding to one half three halves half will be the Pauli wave function will be square root of  $\frac{2}{3}$ . Please see this  $\frac{1}{\sqrt{3}}$ . So, therefore,  $n$  is 2. So,  $2, 1$  of  $r$   $Y_{1,0}$  and the spin up state, that is  $\frac{1}{\sqrt{3}}$  plus plus square root of  $\frac{1}{3}$   $R_{2,1}$  of  $r$   $Y_{1,1}$  and then, the spin down state is  $\frac{1}{\sqrt{3}}$ . So, therefore, the Pauli wave function will become under root of  $\frac{2}{3}$   $R_{2,1}$  of  $r$ . Actually,  $r$  can take  $R_{2,1}$  of  $r$   $Y_{1,0}$  and under root of  $\frac{1}{3}$ ,  $R_{2,1}$  of  $r$   $Y_{1,1}$ .

Now, I leave this as an exercise for you to show that  $\phi$  if I operate this by  $j$  minus operator, then I will get  $\phi(1, \frac{1}{2}, \frac{3}{2}, \frac{1}{2})$  and if I operate this by  $j$  minus half will reduce to by 1. So, this will be minus half. This will come out to be  $R_{2,1}$  of  $r$ . If I take outside and I leave this as an exercise for you to show this will be square root of  $\frac{1}{3}$   $Y_{1,0}$ .

1, minus 1 of theta phi. These are the spherical harmonics and then, square root of 2 by 3 y 1, 3 theta phi.

Now, you should understand that this is the value of m. So, that must be equal to m 1 plus m 2. So, m 1 is minus 1 and this is the spin up state. So, m 2 is plus half. So, minus 1 plus half is minus half this multiplied by the spin down state. Why? Because this is in the second row, so 3 minus half.

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The image shows a whiteboard with handwritten equations in purple ink. The equations are:

$$J_- |\phi(j_1, j_2, j_1 + j_2, j_1 + j_2)\rangle = |\psi(j_1, j_2, j_1, j_2)\rangle$$

$1, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}$ 
 $1, \frac{1}{2}$

$$|\phi(j_1, j_2, j_1 + j_2, -j_1 - j_2)\rangle = |\psi(j_1, j_2, -j_1, -j_2)\rangle$$

$1, \frac{1}{2}, \frac{3}{2}, -\frac{1}{2}$ 
 $1, \frac{1}{2}$

NPTEL logo is visible in the bottom left corner of the whiteboard image.

Now, you can in this way you can take l is equal to 2 and l is equal to 3 and you can take any function and l 1 l 2, you have to first start phi j 1 j 2 j and m. So, the maximum value of m m maximum is j 1 plus j 2 because the maximum value of m 1 is j 1. So, the maximum value of j is also j 1 plus j 2. So, therefore, phi j 1, j 2. You understand this. This is m 1 plus m 2, m 1 plus m 2. This will be equal to psi of j 1 j 2, then m 1 m 2. I am sorry the maximum value of m 1 is j 1. So, therefore, phi j 1 j 2 j 1 plus j 2, this is the maximum value of j and the maximum value of m is j 1 plus j 2. This must be equal to psi j 1 j 2 and the maximum value of m 1 is j 1 and the maximum value of m 2 is j 2. So, one must always start off with this equation. Sorry let me cover this.

So, if j 1 is 1 and this is half, then this is 3 by 2, 3 by 2 and this will be one and half. So, let us suppose j 1 is 1. So, 1, half this is 3 by 2, 3 by 2 and this will become 1, 3 by half. So, 1 plus half is this. So, if there will be only one function like that now you keep on operating this by j minus j minus j minus and you will get all the Clebsch-Gordon

coefficients. Finally, if you keep on operating this, then the last function will be  $\phi_{j_1 j_2 j_1 + j_2}$ . This is  $j$  value and the minimum value of  $m$  will be  $j_1 - j_2$ . This is the minimum value of  $m$ ; this will come out to be  $\psi_{j_1 j_2 j_1 - j_2}$ . So,  $m_1$  will be  $j_1 - j_2$  and  $m_2$  will be  $j_1 + j_2$  and in between you have to operate a large number of times by  $j$  minus operator.

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$j_1 =$	$m_2 = \frac{1}{2}$	$m_2 = -\frac{1}{2}$
$j_1 + \frac{1}{2}$	$\sqrt{\frac{j_1 + m + \frac{1}{2}}{2j_1 + 1}}$	$\sqrt{\frac{j_1 - m + \frac{1}{2}}{2j_1 + 1}}$
$j_1 - \frac{1}{2}$	$-\sqrt{\frac{j_1 - m + \frac{1}{2}}{2j_1 + 1}}$	$\sqrt{\frac{j_1 + m + \frac{1}{2}}{2j_1 + 1}}$

$j_1 = 1, j_2 = \frac{1}{2}$   
 $\phi(1, \frac{1}{2}, \frac{3}{2}, m) = \sqrt{\frac{\frac{3}{2} + m}{3}} \psi(1, \frac{1}{2}, m - \frac{1}{2}, \frac{1}{2})$   
 $+ \sqrt{\frac{\frac{3}{2} - m}{3}} \psi(1, \frac{1}{2}, m + \frac{1}{2}, -\frac{1}{2})$



In almost all books, the table is given. So, for example, for an arbitrary value of  $j_1 j_2$ , let us suppose  $j_2$  is equal to half. Then, this table gives us the values of the Clebsch-Gordon coefficients. Let me take an example. Let us suppose  $j_1$  is equal to say 1. So, the  $j$  value can be either 1 plus half or 1 minus half, that is 3 by 2 or half.

So, this will be this tells us that the phi function. So, let me take 3 by 2. So,  $j_1$  is equal to half and  $j_2$  is equal to  $j_1$  is equal to 1 and  $j_2$  is equal to half. So, you have  $\phi_{j_1 j_2 j_1 + j_2}$  is half  $j$  is let us suppose 3 by 2 and let us suppose  $m$ . Then, this tells us that this is  $j_1$  is equal to, so 1 plus half 3 by 2 plus  $m_2$  into 1 plus 1, that is 3. This will be multiplied by  $\psi_{m_2}$  is half. So,  $m_1$  will be  $m$  minus half. So,  $\psi_{j_1 j_2 j_1 - j_2}$  that is 1, half and  $m_1$  must be  $m$  minus half half plus this number. This is how to use the Clebsch-Gordon. The table corresponding to Clebsch-Gordon coefficients. So, this will be 1 plus half that is 3 by 2 minus  $m$  divided by 3  $\psi$  of 1, half  $m$  plus half minus half.

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$$j_1 = 1, j_2 = \frac{1}{2}$$

$$\phi(1, \frac{1}{2}, \frac{3}{2}, m) = \sqrt{\frac{\frac{3}{2}+m}{3}} \psi(1, \frac{1}{2}, m-\frac{1}{2}, \frac{1}{2}) + \sqrt{\frac{\frac{3}{2}-m}{3}} \psi(1, \frac{1}{2}, m+\frac{1}{2}, -\frac{1}{2})$$

$n=2$   
 $l=1$

$$m = \frac{1}{2}$$

$$\phi(1, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}) = \sqrt{\frac{2}{3}} \psi(1, \frac{1}{2}, 0, \frac{1}{2}) + \sqrt{\frac{1}{3}} \psi(1, \frac{1}{2}, 1, -\frac{1}{2})$$

$$= \sqrt{\frac{2}{3}} R_{21} Y_{10} \begin{pmatrix} \uparrow \\ 0 \end{pmatrix} + \sqrt{\frac{1}{3}} R_{21} Y_{11} \begin{pmatrix} \downarrow \\ 0 \end{pmatrix}$$

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So,  $j_1$  is equal to 1. These are the wave functions. For example,  $m$  is equal to let us suppose, let me consider the case when  $j$  is equal to 3 by 2 and  $m$  is equal to half. If  $m$  is equal to half, then you will have the Pauli wave function  $\phi(1, \frac{1}{2}, 3 \text{ by } 2 \text{ half})$ . So, you will have 3 by 2 plus half is 2. So, this is 2 by 3. So,  $\psi(1, \frac{1}{2}, m - \frac{1}{2}, \frac{1}{2})$ . So, this is 3 half plus 3 by 2 minus  $m$   $m$  is half. So, 3 by 2 minus half is 1 under root of 1 by these are automatically normalized  $\psi(1, \frac{1}{2}, m \text{ is half})$ . So, half plus half is 1 minus half. So, always remember that 3 plus half is half 1 minus half is half.

So, this will tell you, this will have that. Let us suppose we consider the state  $n$  is equal to 2  $j_1$  is equal to 1. So,  $l$  is equal to 1. So, you will have square root of 2 by 3  $R_{21}$  of  $r$   $Y$ . If we take hydrogen atom wave functions  $y_{13}$  spin up state, this is the spin up state plus under root of 1 by 3  $R_{21} Y_{11}$  spin down state. So, this is my 1 3 state and this is the 3 1 state.

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$$j_1 = l = 2$$
$$\phi\left(2, \frac{1}{2}, \frac{5}{2}, \frac{1}{2}\right) = \sqrt{\frac{3}{5}} \psi\left(2, \frac{1}{2}, 0, \frac{1}{2}\right) + \sqrt{\frac{2}{5}} \psi\left(2, \frac{1}{2}, 1, -\frac{1}{2}\right)$$

Similarly, you can take another example. Let us suppose  $j_1$  is equal to 2. If  $j_1$  is equal to 2, then  $j$  can take 2 plus half and 2 minus half 5 by 2 and 3 by 2 and you can immediately write. Let me just write this up and then, so if  $j_1$  is equal to say 1 is equal to 2, so you will have 2 plus half 2 plus half is 5 by 2. Let me consider the phi function corresponding to 2. So,  $j_1$  is 2,  $j_2$  is half. This is 5 by 2 and let me consider the function corresponding to  $m$  is equal to half. Let me consider half. So,  $m$  value let us suppose it is half. So, this will be equal to this will be under root of  $j_1$  is 2, that is 2 plus half plus half, that is 2 plus 1 is 3 and this is 2 into 2. I hope I am doing the algebra correctly. 3 by 5 and this will be psi.

Please see this 2, half 2, half and  $m_1$   $m_2$  is half. So, this should be 3 because this plus, this must be equal to this plus, this number is  $j_1$  is equal to 2 minus half plus half that is 2 by 5. So, it is normalized square of this plus square of this is 1. So, this is psi 2, half,  $m_2$  is minus half. So, if  $m_2$  is minus half  $m_1$  has to be plus 1 because this plus this must be equal to this. So, this is my Pauli wave function.

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$$\begin{aligned}
 &= \sqrt{\frac{3}{5}} R_{32} Y_{20} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 &+ \sqrt{\frac{2}{5}} R_{32} Y_{21} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
 &= \begin{pmatrix} \sqrt{\frac{3}{5}} R_{32} Y_{20} \\ \sqrt{\frac{2}{5}} R_{32} Y_{21} \end{pmatrix}
 \end{aligned}$$

$n=5$   
 $j_1=3, j_2=\frac{1}{2}$   
 $j=\frac{5}{2}, m=\frac{3}{2}$   
 $j=\frac{7}{2}, m=\frac{5}{2}$

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So, the wave function will come to be very simple. So, the corresponding Pauli wave function will be square root of 3 by 5. Let us suppose n is 3. So,  $R_{32}$  is 2 and this will be  $Y_{23}$  spin up state, that is 1 3 plus under root of 2 by 5  $R_{32} Y_{21}$  spin down state.

So, you have therefore, this will be the corresponding Pauli wave function is square root of 3 by 5  $R_{32} Y_{23}$  under root of 2 by 5  $R_{32} Y_{21}$ . One can go on and on and using this table that I had shown. I leave it as an exercise for you to show that for j 1, say let me consider the case when n is equal to say 5.  $J_1$  is equal to let us suppose 3 and  $j_2$  is equal to half. So, the value of j will be either, 3 plus half or 3 minus half. So, let us suppose I take 3 minus half. So, this will be that j b equal to 5 by 2 and m is equal to 3 by 2. Calculate the Pauli wave function. Similarly, you can have j is equal to 7 by 2 and m is equal to say 5 by 2. Calculate the Pauli wave. You can frame any number of problems using that.

So, I have given you the complete theory of calculation of the Clebsch-Gordon coefficients. You start with the maximum value of j and the maximum value of m, relate to the corresponding psi function which are simultaneous Eigen functions of  $j_1^2, j_2^2, j_1 z$  and  $j_2 z$  and the constant coefficients which relate this as a unitary transformation. You go from one set unit vectors to another set of unit vector. The coefficients which correlate from one set of function to the other set are known as the Clebsch-Gordon coefficients. I have given you the recipe for calculating that and also in

most cases, one has to use this particular table to calculate the Clebsch-Gordan coefficient.

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$j_2 = \frac{1}{2}$		
$j =$	$m_2 = \frac{1}{2}$	$m_2 = -\frac{1}{2}$
$j_1 + \frac{1}{2}$	$j_1 + m + \frac{1}{2}$	$j_1 - m + \frac{1}{2}$

$j_1 \geq j_2$

$j \quad \underline{j_1 + j_2}, \quad j_1 - j_2$

This corresponds to  $j_2$  is equal to half. In many books  $j_2$  is equal to 1 including in our book and of course, that is the table has been taken from shortly for  $j_2$  is equal to 1. The tables are given. The point that you should remember is that the value of  $j$  goes from the value. If  $j_1$  and  $j_2$ , the value of  $j$  will go from  $j_1 + j_2$ , I am assuming  $j_1$  is greater than or equal to  $j_2$ . So,  $j_1 + j_2$  in units of 1 to  $j_1 - j_2$ .

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JWKB Approximation

1-d Schrodinger Eq.

$$\frac{d^2\psi}{dx^2} + \frac{2\mu}{\hbar^2} [E - V(x)] \psi(x) = 0$$

$k^2(x)$

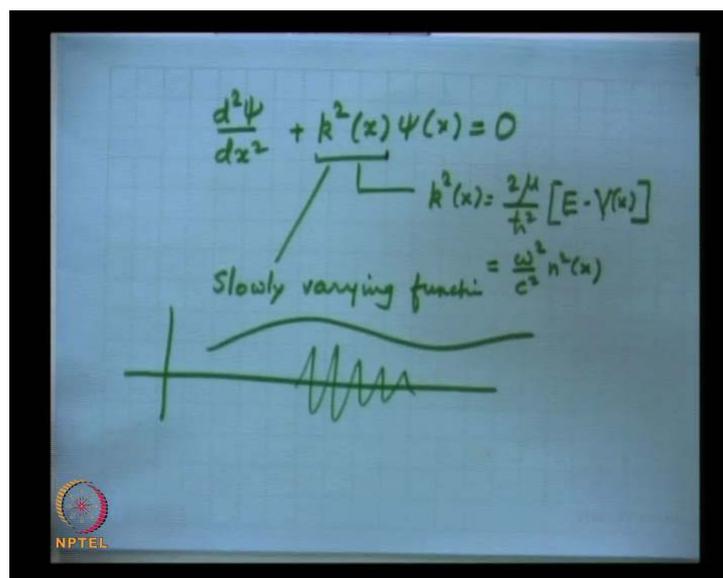
$$\frac{d^2\psi}{dx^2} + \frac{\omega^2}{c^2} n^2(x) \psi(x) = 0$$

Our next thing that we would like to discuss is one of the most fascinating topics in quantum mechanics and that is the JWKB approximation. I will just outline you the method that we will develop and we will discuss the details in our next lecture.

So, this method, this approximate method, was first put forward by Jeffrey's around 1887 or 89 and then, when quantum mechanics was developed in 1926, this method was simultaneously published by Wetzell Krimmers and Brillouin in the year 1926 itself. Many books write this as WKB approximation, but since it was Jeffrey's who first put forward the method as a mathematical solution to a differential equation, I think it is appropriate and many people do that to write this as the JWKB approximation.

Now, our Schrödinger equation, the one-dimensional Schrödinger equation. One-dimensional Schrödinger equation is in the form of  $\frac{d^2 \psi}{dx^2} + k^2(x) \psi(x) = 0$ . So, this function is a known function of  $x$ . So, we denote this by  $k^2$  of  $x$ . This is also a wave like equation. Even in plasma physics or electromagnetic theory, we have when a electromagnetic wave propagates through a inhomogeneous medium. We have a wave equation, we have an equation like this plus  $\omega^2$  by  $c^2$   $n^2$  of  $x$   $\psi$  of  $x$  is equal to 0, where  $n^2$  of  $x$   $\omega$  is the angular frequency of the wave,  $c$  is the speed of light in free space and  $n^2$  is the square of the refractive index.

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So, in many problems, even in waveguide theory, you have an equation which has the form  $\frac{d^2 \psi}{dx^2} + k^2(x) \psi = 0$ . Now, the WKB, the JWKB approximation gives us an approximate solution of this equation when  $k^2(x)$  does not vary too much. So, we will develop this. So, in the Schrödinger equation case  $k^2(x)$  is equal to  $2\mu(E - V(x))$ . In a plasma physics problem, it is  $\omega^2 - c^2 n^2(x)$ . In waveguide theory problem, it has a different expression, but in each case, it is a function of  $x$ . Of course, here as you can see that if  $E$  is less than  $V(x)$ , then  $k^2$  will become negative.

We will discuss the solution when  $k^2$  becomes negative, but right now let us assume that  $k^2$  is positive and it is a slowly varying function. What is meant by slowly varying? I will explain in a moment. It does not vary too much, that is if I plot  $k^2(x)$ , it varies slowly, something like that and it does not go through any 3s or something like that. So, we are not considering functions  $k^2(x)$  which are very rapidly varying functions, that is we will show, we will derive the condition for the validity of the WKB JWKB solutions.

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The image shows a whiteboard with handwritten mathematical derivations. At the top, the wave equation is written as  $\frac{d^2 \psi}{dx^2} + k^2 \psi(x) = 0 \Rightarrow \psi(x) \sim e^{ikx}$ . Below this, it says "Let us try  $\psi = A e^{iu(x)}$ ". The next line is  $\psi(x) = A e^{iu(x)}$ . The derivative is given as  $\psi'(x) = i u' \psi(x)$ . The second derivative is  $\psi'' = i u'' \psi(x) + (i u')^2 \psi(x)$ . Finally, the equation is substituted into the wave equation to get  $i \frac{d^2 u}{dx^2} - \left(\frac{du}{dx}\right)^2 + k^2(x) = 0$ . An NPTEL logo is visible in the bottom left corner of the whiteboard.

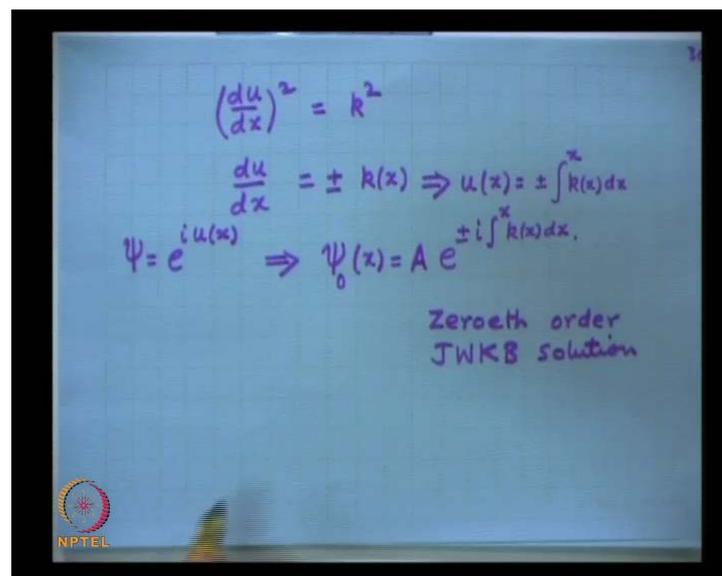
So, the first approximation, let us assume that  $k^2$  was totally independent of  $x$ . So, if  $k^2$  is totally independent of  $x$ , that is independent of  $x$ , so  $k^2 \psi(x)$  is equal to 3. If this is not a function of  $x$ , then we know the solutions. I can write down in terms of sin cosine or  $i k x e$  to the power of  $i k x$ , where  $k$  is the under root of this and  $x$

is of, this suggests that we let us try out a solution of this form  $e^{iu(x)}$ . So,  $\psi$  is given by  $e^{iu(x)}$ .

So, let me brute force differentiate. So,  $\psi = e^{iu(x)}$ . So,  $\frac{d\psi}{dx}$  which is  $\psi'$  will be equal to  $i u'(x) e^{iu(x)}$  and then, if I differentiate this, so I get  $\psi'' = i u'' e^{iu(x)} + i u'(x)^2 e^{iu(x)}$ . If I differentiate this again, so you get  $\psi'' = i u'' e^{iu(x)} + i u'(x)^2 e^{iu(x)}$ . So, this will be plus  $i u'(x)^2 \psi$ . I hope you understand.

So, I substitute this. I still do not know what  $u$  is. So, I substitute this with this equation and I will get  $\frac{d^2 u}{dx^2} + k^2 = 0$ .  $\psi$  will cancel out with every  $i^2$  is minus 1 minus  $\frac{d^2 u}{dx^2} + k^2 = 0$ . So, I substitute this in this equation and I obtain this. I must mention that if I am able to solve this equation, then this will be a rigorously correct solution of this equation. So, I am assuming  $k^2$  to be a function of  $x$ . Once again that if I assume of this equation, a solution of this type, then I obtain that  $u(x)$  satisfies an equation of this equation. If I am able to solve this equation, then this will be a rigorously correct solution of this equation because I have not yet made any approximation. Now, when  $k^2$  of  $x$  is independent of  $x$ , then this tells us that  $u(x)$  is equal to  $kx$ . So,  $\psi'' = -k^2 \psi$ .

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So,  $u'$  will be  $k$  and  $u''$  will be  $3$ . This suggests that if  $k^2$  of  $x$  is slowly varying, then let me in the first place neglect this term. If I neglect this term, then

I will obtain  $du$  by  $dx$  whole square. If I take  $k$  square outside and minus minus sign, so this is  $k$  square or  $du$  by  $dx$  is equal to plus minus  $k$  of  $x$ . So, therefore, I can integrate this and I obtain  $u$  of  $x$  is equal to plus minus integral  $k$  of  $x$   $dx$ . Since, I had assumed the solution of the form of  $e$  to the power of  $i$   $u$   $x$ , then we obtain zeroeth order solution as  $\psi_3$  of  $x$  is some constant. Let us suppose a  $e$  to the power of plus minus  $i$  integral  $x$   $k$  of  $x$   $dx$ . This is known as zeroeth order JWKB solution.

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$$\psi_0(x) = A e^{+i \int k(x) dx} \quad (1) \text{ Phase Integral Method}$$

$$\frac{d\psi_0}{dx} = +i k(x) \psi_0$$

$$\frac{d^2 \psi_0}{dx^2} = +i \frac{dk}{dx} \psi_0 + (ik)^2 \psi_0$$

$$\frac{d^2 \psi_0}{dx^2} + \left[ k^2 - i \frac{dk}{dx} \right] \psi_0(x) = 0 \quad (2)$$

$$\frac{d^2 \psi}{dx^2} + k^2 \psi = 0$$

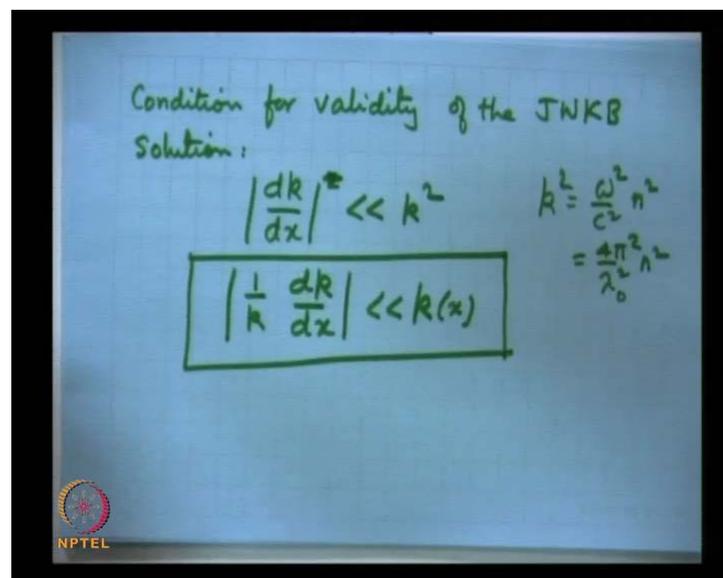
The only that thing that I have done is neglected the second derivative here, neglected the second derivative of  $u$ . Other than that, that is the only approximation that we have made. Now, let me find out what is the equation that  $\psi_3$  satisfies. So, let me differentiate this. Let me just take the plus sign. So, you get, so if I take the plus sign, let me write it down again that  $\psi_3$  of  $x$  is equal to I can do it with the minus sign also, plus I integral  $k$  of  $x$ . Therefore, this is known as here in this approximation only the phase changes and therefore, this is also some people call is as the phase integral method as being propagates only the phase changes and therefore, many people call this as the phase integral method.

So, let me find out what is the equation that  $\psi_3$  of  $x$  rigorously satisfies. I have here  $d$  is a  $\psi_3$  by  $dx$ . If I differentiate this, so this will be the differentiate coefficient of this will be  $k$  of  $x$ . So, this will be plus  $i$   $k$  of  $x$  times, the entire quantity  $\psi_3$ . If I differentiate this again, then I get  $d^2 \psi_3$  by  $dx$  square. So, this is equal to plus  $i$   $dk$  by

$\frac{d}{dx} \psi^3 + \psi^3 \text{ prime} + i k \psi^3 \text{ prime}$  and  $\frac{d}{dx} \psi^3$  is  $i k \psi^3$ . So, this becomes minus, this thing into  $\psi^3$ . So, this becomes minus  $k^2$  and if I take this also on this side, so you have  $\frac{d^2}{dx^2} \psi^3 + k^2 \psi^3 - i k \psi^3 \text{ prime} = 3 \psi^3$ .

Please see that this solution is a rigorously correct solution of this equation. Equation 1 is a rigorously correct solution of equation 2, but what we wanted is the solution of this equation  $\frac{d^2}{dx^2} \psi + k^2 \psi = 3 \psi$ . So, therefore, if this term, the magnitude of this term is very small compared to this term to  $k^2$ , then our solution should be reasonably good.

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Once again this function  $\psi^3(x)$  is a rigorously correct solution of equation 2, but what we had wanted to solve was this equation and therefore, this will be an accurate solution of equation 3 provided we are able to neglect this term. That means, the  $\frac{dk}{dx}$  must be less than  $k^2$  and therefore, we write down that the condition for validity of the JWKB solution, the zeroth order JWKB solution is that  $\frac{dk}{dx} \text{ mod }^2$  is less than  $k^2$  or  $\frac{1}{k} \frac{dk}{dx}$  should be less than  $k$ . This is the condition for the validity, that is  $k(x)$  should not vary so rapidly that  $\frac{1}{k} \frac{dk}{dx}$  is less than  $k(x)$ .

In optics or in electromagnetic theory, when I apply that  $k^2(x)$  is equal to  $\frac{\omega^2}{c^2} n^2(x)$  and therefore, this is equal to  $\frac{4\pi^2}{\lambda_0^2} n^2(x)$ .

$\lambda^3 n^2$ . This condition means that the refractive index should not change too much in a distance of the order of wavelength. So, that is the condition for the validity of zeroth order WKB approximation.

So, in our next lecture, what we will do is go try to improve on this solution and develop what is usually referred to as the first order WKB solution. In fact, people use just the first order WKB solution and we will then use this solution to solve the one-dimensional Schrödinger equation. These solutions are extensively used not only in quantum mechanics, but many diverse areas of physics. Thank you.