Group Theory Methods in Physics
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Lecture - 38
SO(n) and Lorentz group

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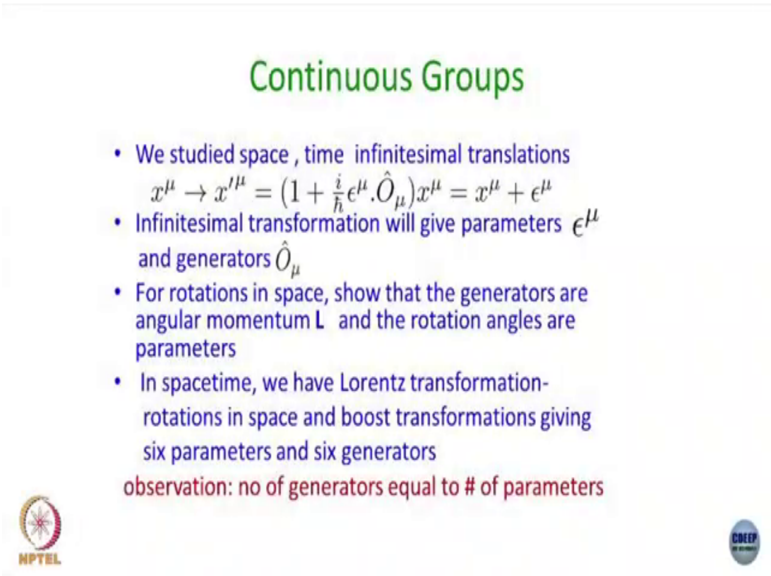

So, we went through all this infinitesimal translation then I entered through parameter space, then the generators associated with such a transformations. And whatever is the number of parameters, you will also have the number of generators, I am just warming you up before we get on to, ok.

And then I said that rotations also in the last lecture we saw how the generator is angular momentum, orbital angular momentum and then the rotation angles are the parameters and you have three independent rotation angles. And then we then I try to explain what happens in space time. Space time you can have two inertial frames, where one is moving with respect to the other with constant velocity or you can have rotations of the frame itself.

So, Lorentz transformation will involve rotations in space and also the boost transformation which involves one space and one time. Am I right? Everybody is synced in rhythm with whatever I have been talking so far, ok. So, there will be six parameters and six generators. Basically, whenever you have a transformation you do see that the number of generators is equal to the number of parameters, ok.

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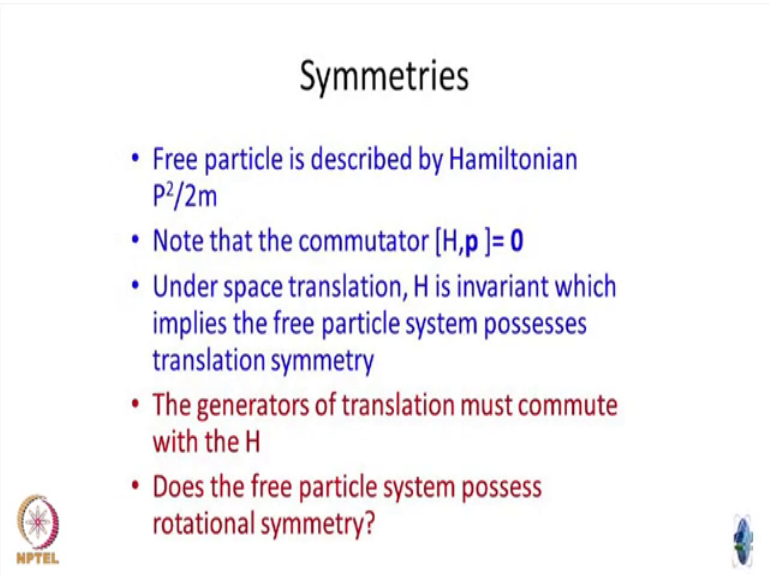

And then I said that what is the symmetry of a system, you can look at what is the symmetry of the system. Harmonic oscillator is not translation invariant, but your free particle has both translation symmetry and rotational symmetry. Then the corresponding generators of those transformation should commute with the Hamiltonian, is a condition for looking at what is the group symmetry possessed by the system, ok.

So that I left it and I we did I did tell you to check whether Hamiltonian P 2 over 2 m commutes with l x l y l z and check it out whether it is 0 or not. I do not know how many of you have done it, but please check it out.

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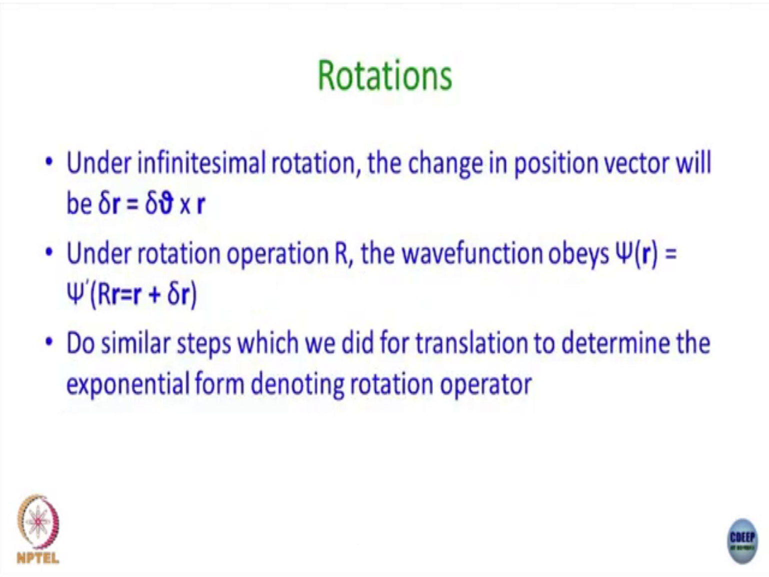

And then we went through this explicit infinitesimal rotation how a position vector, the boldface refers to position vector notation. Even though I am not putting a vector here this is the position vector and it is a cross product of two vectors, ok. And then using the usual principle of how the wave function is going to remain. The functional form of psi will change to psi prime r will also change to r prime and using this you can find what is the transformation which relates your wave function under rotation, ok.

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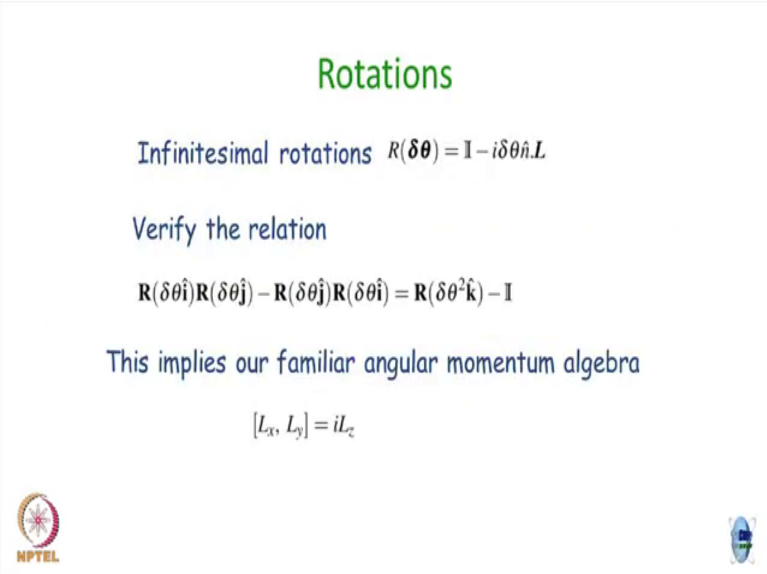

So, this also we went through and for an infinitesimal rotation besides deviation from identity is the orbital angular momentum. I have suppressed the H cross, but remember that could also be a H cross, ok.

And then I said that you can try to find that the order of infinitesimal rotation. Most of the book says infinitesimal rotation is commuting which is true as long as you keep up to order delta theta, but if you keep up to order delta theta squared then you see a deviation and if you try to write explicitly using the first expression, identity minus parameter delta theta which is having a direction n hat unit vector dotted with the orbital angular momentum vector.

So, if you put in this explicit expression into this then you will see that the difference at order delta theta squared is. How many of you have checked this? Did anybody you have checked, its correct, and that automatically gives you the conventional algebra for angular momentum, orbital angular momentum, the commutator between L x L y turns out to be i L z, ok.

This is familiar from various directions, but for the purpose of this course; whatever I have doing here I could have done for any transformation and then I should end up trying to get a algebra for the generators. By algebra I mean the commutator brackets amongst the generators must always give you one of a linear combinations of the generators ok.

So, that is what we mean by an algebra. Commutator bracket of the generators, any two generators because here you have only three generators commutator bracket of any two generator will give you the third generator here, but in general it could also be 0. If you had momentum linear transformation translation which was momentum P x p y commutator was 0, ok. So, it could be either 0 or it could be your linear combinations of the generators which are in that set which constitutes a transformation. Is this clear, ok?

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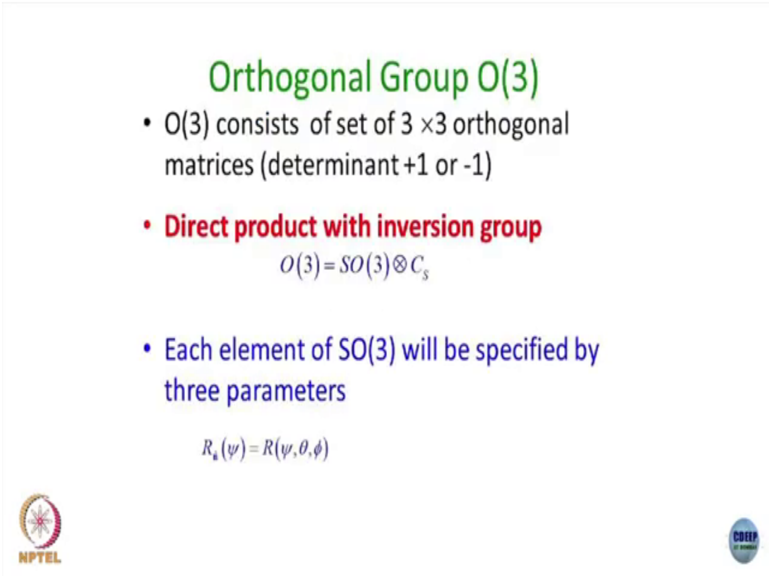

And then we went on to our familiar orthogonal group of rotations in 3 dimensional space and then I said that there can be proper rotations or improper rotations. Improper rotations will have a determinant to be minus 1 and O 3 includes proper and improper rotations. So, it can have determinant plus 1 and determinant minus 1. And someone came and asked me what is this direct product this C s is just a set which involves identity and an inversion matrix, ok.

So, this is a discrete group, the C s is a discrete group and it is not a continuous group. Why it is not a continuous group? To go from identity to minus identity, you cannot continuously go if you have to continuously go you should be able to write that minus identity as identity plus infinitesimal transformation times generator. You cannot do that for an inversion.

Inversion is a discrete transformation, you have a vector like this, it is not continuously going to this place, it is you know it makes an abrupt transformation. So, you cannot achieve such an abrupt inversion by infinitesimal steps. The way we did for translation, the way we did for rotation is a finite rotation can be seen as n steps of infinitesimal transformation that cannot be done for inversion or parity, time reversal those are all discrete groups the way we saw in the first half of the semester ok.

But what you can show is that the O 3 is a direct product of S O 3 times C s; C s has two elements. So, you will have for every element it will give you two elements; one with determinant plus 1, the other one with determinant minus 1. Is that clear?

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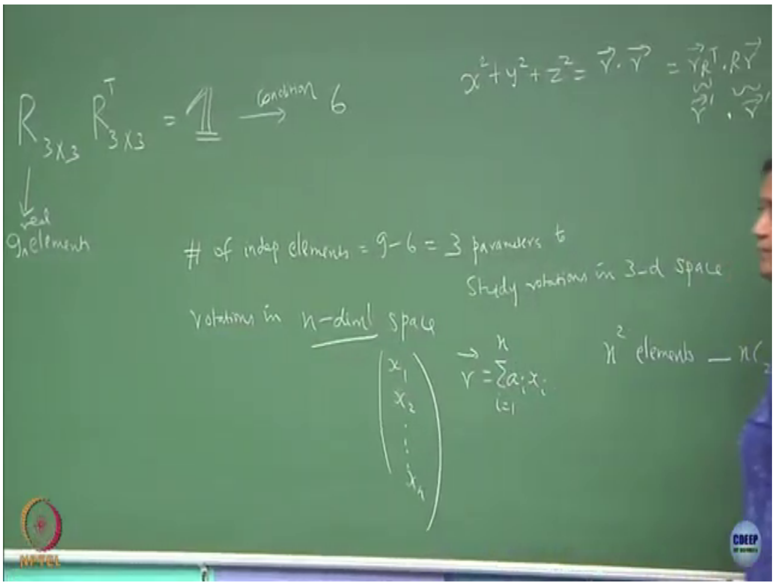

And I also specified that S O 3. So, you can also look at why there are 3 parameters by writing r which is a 3 cross 3 matrix, right. And you have this condition that R R transpose 3 cross 3 is identity ok. So, R is a 3 cross 3 matrix with all the entries being real, right.

So, you have this to be this has 9 elements to start with, right. And this condition so, this is a condition or constraint you can call. How many conditions are there? So, you will have diagonal element should be 1 off diagonal elements of this is 0, ok. So, you will see that how many conditions you will get out of this there will be 3 diagonal equations and 3 off diagonal equations.

So, the number of conditions is 6, ok. So, in general any 3 cross 3 matrix will require 9 real if it is real entries will require 9 real parameters to define a 3 cross 3 matrix, but because they represent rotations, what is rotation doing.

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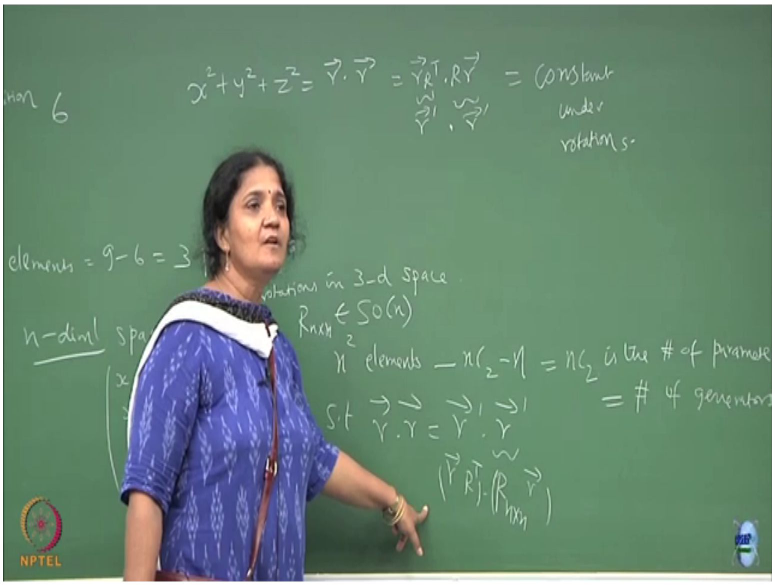

Rotation takes care that x 2 plus y 2 plus z 2 which is r vector dot r vector remains same under rotations, ok. So, this means it is r dot R transpose dotted with R r. So, this one is your r prime and this one is your r prime and there is the dot product. So, this is what we mean by saying it is constant or invariant under rotations, ok. So, this is going to be invariant under rotations, this objects are going to be invariant under rotations.

These are 3 dimensional real vector space R r can be treated like operators 3 cross 3 matrix representations of linear operators acting on the three dimensional vector space and these operators or linear operators have 9 real elements, ok.

It requires 9 real elements to specify your rotation matrix; however, they are not independent they have to satisfy this constraint which is 6 constraints, ok. So, the number of independent elements turns out to be 9 minus 6 which is three and that is why you require 3 parameters to study rotations in 3 dimensional space.

If you do rotations in 2 dimensions can you say the same argument what will be the argument, there would be 4 elements required for r matrix 2 cross 2 matrix and then how many constraints.

Student: 3 constraints.

3 constraints. So, you will have only one which is rotation about an axis perpendicular to the plane, clear. What happens if you go to 4 dimensions 4 will be 4 cross 4 constraints in general can be written as n C 2, right. So, suppose I look at rotations in n dimensional space. What do I mean by n dimensional space. You do not have only x y z coordinate, you have a vector space which is not just x y z, it is x 1 x 2 up to x n, right.

Any arbitrary position vector will be a i times x i summation over i running from 1 to n that is the meaning of n dimensional real vector space each of these coefficients are real coefficients and you can have linear operators acting on this vector space. So, you will have n squared elements minus constraint what is the number of constraint n C 2 what does that give us n C 2 or n into n plus 1 by 2.

Student: (Refer Time: 13:33).

N into because the diagonal elements should also be included this is only off diagonal elements. We should also include the diagonal because this condition diagonal equation plus the off diagonal equations, off diagonal is n C 2, but you will also have a diagonal. So, finally, what do we get from here? I do not know check it out whether you get n into n minus 1 by 2. Anybody can check it. So, what is that going to be for n is 2, 2 C 2 is 1 n is 3, 3 c 2 is 3 for n equal to 4 how much it is 4 C 2 is 6 and so on, ok. So, these are the n C 2 is the number of parameters which is also equal to number of generators.

So, the R which is n cross n is an element of is an element of S O n such that r vector which is this dotted with r vector should be r prime vector dotted with r prime vector where r prime is this one is R n cross n on the r vector and this one is R transpose. You understand what I am saying, right. If you remember this 3 dimensions extrapolating it to n dimensional vector space real vector space is straightforward.

So, you can ask why am I doing this, ok. So, many times it happens that these symmetries are seen in systems, ok. So, we want to always exploit such symmetries and get more physics out of the system like hydrogen atom.

There is a symmetry which is called Runge Lenz vector classical mechanics, if you have done you would have seen that there is a quantity called Runge Lenz vector which is conserved just like Hamiltonian sorry, just like Hamiltonian commutes with Ls components it also commutes with the Runge Lenz vector. So, you want to combine them and make it look like some kind of an abstract symmetry.

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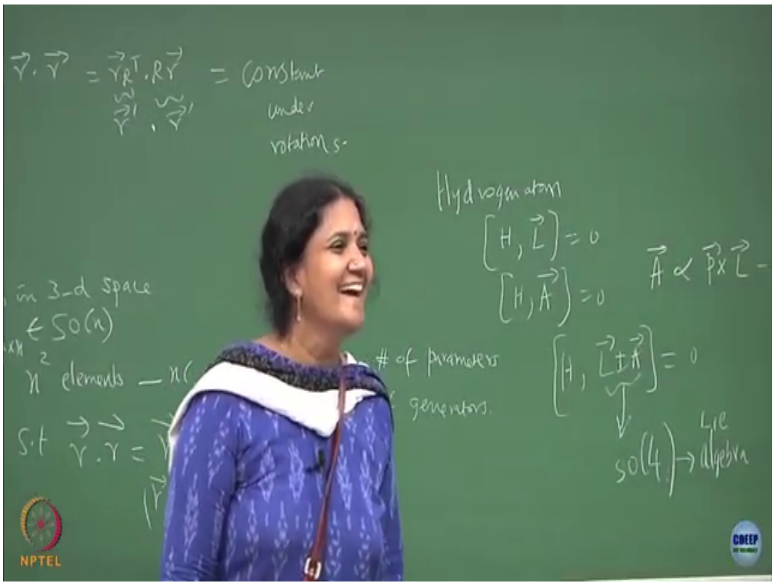

So, you will see at some point that I can show a hydrogen atom Hamiltonian will commute with Ls it will also commute with Runge Lenz vectors ok, but what you can show is that the Hamiltonian with L plus or minus A, ok.

This will turn out to be satisfying an algebra which we will call it as S o 4, small s o 4 is to denote the algebra and you will also see algebra is for the generators and if you exponentiate it the generators with parameters you will get the corresponding group. When I write the group, I will put the S and O to be caps bigger letters, ok. So, that is the notation which is followed that this is small s small o this is algebra lie algebra, ok.

So, you can write the lie algebra for them which have not done here, but at some point I will give it as assignment problems. Where you will do the algebra L x L y L z you know the algebra, but you do not know how a x a y a z algebra is you should also know the algebra between L and a, ok.

So, then only it will become a closed algebra a closed algebra is any commutator bracket between the generators. there are six generators SO4 has fixed generators. There are 6 generators I want to write the algebra for this 6 generators, clear.

Student: Maam A.

A is the Runge Lenz vector, you know what is an Runge Lenz vector that which keeps a par bit closed and it is proportional to p cross L minus some constant times r or something, ok. So, r hat so, there are some interesting aspects of looking at is this a polar vector or an axial vector question what so much doubt.

Student: (Refer Time: 19:21).

Some of you are saying axial, some of you are saying polar it cannot be both it is not Jaqen hight.

Student: (Refer Time: 19:30).

L is an axial vector, p is a polar vector polar vector with an axial vector is going to be under parity what happens? It changes sign this any way changes sign this is a unit vector, position vector. If this changes sign, you know it is a polar vector you cannot add a polar vector and an axial vector L is an axial vector, but p cross L is a polar vector. So, this is going to be a polar vector ok, ok. So, this is for going from S O 3 to an abstract So n group, ok. So, these will be the S o n group now also want to slightly tell you about space time, ok.

So, far I looked at it as a 3 dimensional space and an abstract n dimensional real vector space, but I want to do a slightly variant that I want to look at space time. So, let me do the same thing for space time.

Student: Maam why it is S 4 S o 6 component (Refer Time: 20:42).

Where is 6 components?

Student: L x L y and L z (Refer Time: 20:47) a x a 1.

So, I argued with you that n C 2 decides for you they are S o n group number of parameters. This is also equal to number of generators, there are 6 generators n C 2 will gives 6 only for n equal to 4. S o n equal to 4 is what will give you 6 generators, is that clear? Number of generators is not the n of s o n number of generators decides for you. How many independent elements?.

You can have in the matrix which are linear operators acting on that vectors and that will turn out to be with this constraint. It will turn out to be n 2 minus n, these are the constraints diagonal elements and off diagonal elements. And what you get is the number of independent parameters and that decides for you the number of generator which we have seen by examples, we did Lorentz transformation also, we did rotations in 3 dimension. I am only extrapolating for general S o n group how to do it, is that clear. Now, I am going to do it for space time.

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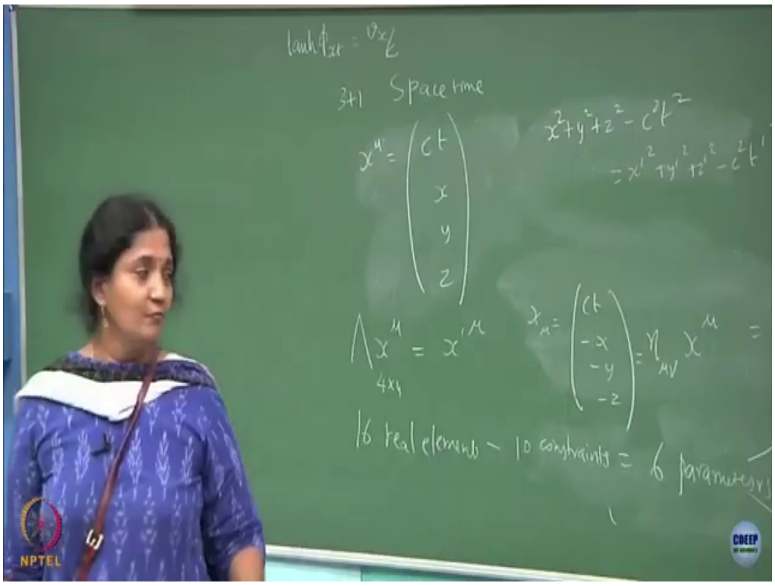

So, space time means what? So, let me look at 3 plus 1 dimensional space time, ok. So, what we do is we write x mu by x mu I mean c t x 2, sorry x y and z. This is the meaning of the 4 dimensional still a real vector space, right. And I will do a transformation which I call it as lambda acts on x mu to give me an x prime mu and lambda what is the dimension degree of these matrices 4 cross 4 4 cross 4 matrices acting on x mu will give you a new space time vector.

What is the property required, you require x 2 plus y 2 plus z 2 minus c 2 t 2 to be same as x prime 2 plus y prime 2 plus z prime 2 minus c 2 t prime 2 c is the velocity of light and you require this under any transformation between two inertial frame if this is preserved then you will get the you know the system has preserves this symmetric that is why you have all the equations of motion having the same form f equal to m a will remain as f prime equal to m a prime using a subset of this which is rotations, but you can write all your Maxwell's equations in this way, ok.

So, this is the requirement. So, in this notation which I have written here x mu as a column vector this way I can write this also as a dot product just like I wrote here I can do that there. So, what is the dot product going to be you have to remember if x mu is this way. We define small x mu to be c t minus x minus y minus z, or equivalently you can also define it as eta mu nu x mu where eta mu nu is a diagonal matrix. What is the diagonal matrix?

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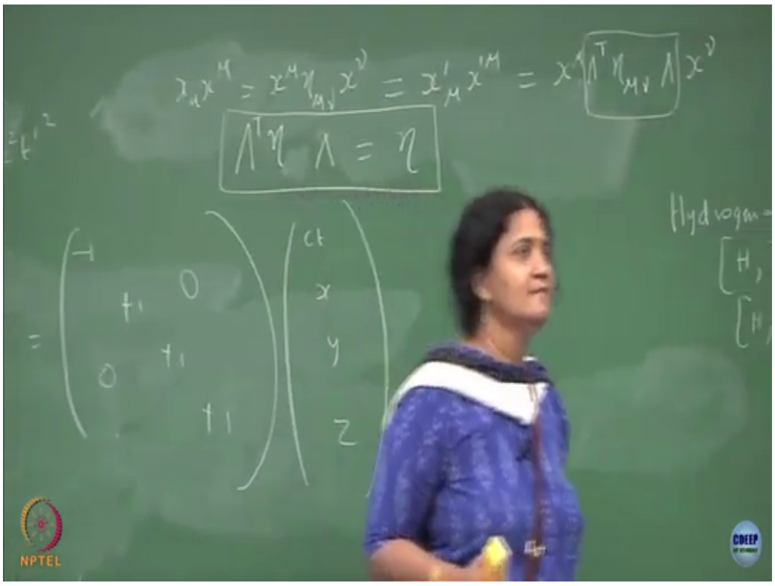

Let me write it above diagonal matrix is 1 minus 1 minus some books follow the reverse, but there is an overall negative sign, ok. So, just follow one notation, ok. So, that will give you back your x mu. So, there are ways of writing this equation. How will you write it just like the way I wrote here, I can write it as x mu x mu or equivalently I can write this as x mu eta mu nu x nu all are one and the same the repeated index has to be summed up, ok.

So, this explicit form is x 2 plus y 2 plus z 2 minus c 2 t 2, ok. So, I wrote the other way around here I think probably you can put this to be negative see this is what I make mistakes, ok. So, then you will get this to be the same thing. As I said overall negative sign can be taken out.

And what do I am what am I saying if you do this transformation lambda which is a 4 cross 4 matrix it takes x mu to x prime mu, but you can say that this thing should be same as the corresponding dot product is defined using eta mu eta mu does not change, ok.

So, what is that give me it gives me that lambda; lambda eta mu nu. So, lambda transpose eta mu nu lambda let us write that. So, this gives you a condition that your lambda eta lambda should be eta, clear. So, now, you have some earlier when I did rotations, I did not introduce an eta, but if I had I introduced an eta what will that eta be, it will be just an identity matrix.

Now you see that the time coordinate is distinguished from space coordinate and that is very essential for cause and effect, effect to follow cause should happen in space time, ok. So, that is why that this sign difference is important you cannot say in space time it is plus c 2 t 2 you could have asked me why do not I put plus c 2 t 2, but if you do that in one inertial frame if you had event 1 and event 2 where one was a cause and the other one was an effect.

When you go to another inertial frame it should be the both the events are seen by both the observers. So, the cause and event, event should follow the cause in any inertial frame. That forces this negative sign in your space time. If I had put a plus then there is no distinction between S o 4 or the Lorentz transformation. Lorentz transformation has to respect the cost effect sequence and that is inbuilt in this negative sign and that is inbuilt in your eta matrix there are some of them which is plus some of them which is minus, ok.

So, that gives you a constraint that you have 4 cross 4 matrix how many independent entries are there, how many are there? 16. These are the constraints any change in the number of constraints just because I inserted an eta just that earlier all the diagonal elements for one now some diagonal elements are one another diagonal elements are minus 1.

So, the number of constraint is going to be mean the same clear number of constraints is not going to change just because I introduced an eta that eta keeps track of one with a sign and negative sign and 3 with positive signs, this is the way I am taking it, clear. So, the number of constraints again here is going to be how many? 10 4 diagonal and six off diagonal that is 10. So, 16 minus 10 is going to be so, there are 16 real elements minus 10 constraints which adds up to give you 6 parameters.

And we have already seen what are the parameters, I called it as phi x t phi y t phi z t which are the boost where one inertial frame moves with respect to the other inertial frame with velocity v x. How is velocity v x related to phi tan hyperbolic phi of x t is v x by c right, clear. So, those are 3 parameters other 3 parameters are your conventional rotations about axis which we called it as phi x y in the x y plane phi y z and phi x z, ok.

So, it is clearly fitting in that the six generators which we saw elaborately for Lorentz group last time ok. I did go over all the algebra of writing the generators finding the algebra which I left it to you. If you do it, there are 6 parameters and 6 generators and which is a consequence of this, ok. Is this clear?