Physics of Biological Systems
Prof. Mithun Mitra
Department of Physics
Indian Institute of Technology, Bombay

Lecture - 17
Introduction to the navier stokes equation

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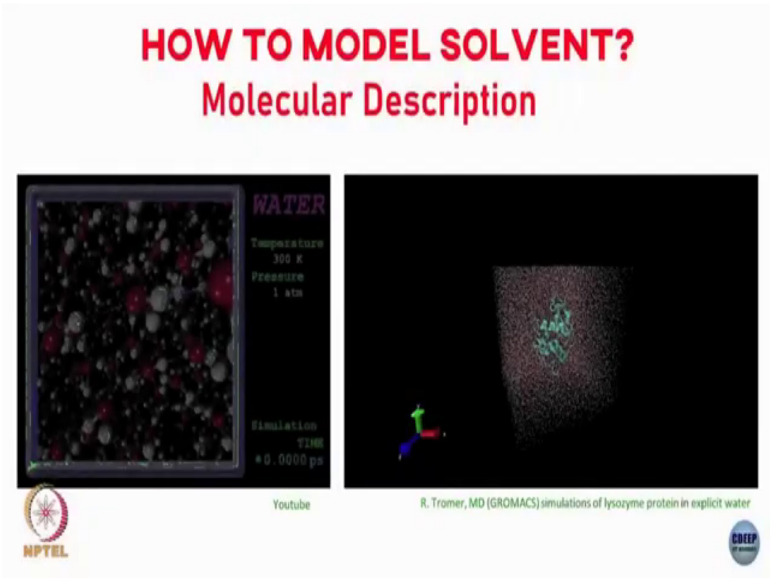

So, the question is therefore, how to model this fluids? Now, there are two answers to that one is very sort of optimistic or molecular description. So, you can say that well if I talk of water for example, it consists of; so, these fluids if I look deep inside consists of water molecules H 2 O. And, I could sort of simulate each of these individual water molecules in a sort of molecular dynamic simulation and then try to see how this water molecules affects the dynamics of whatever else that is present in this medium.

So, for example, this is the protein that is present in this water, you can do an explicit simulation. So, this is a lysozyme protein in explicit water, you can do this molecular dynamic simulations to see, exactly how the solvent is going to affect this the states of the confirmations of this protein, that is a very detailed answer.

It has certain advantages in that you are taking into account explicitly all the forces that these individual water molecules are going to exert on each other and on whatever protein or organism of interest. On the other hand these simulations are very resource intensive, if you do molecular dynamics simulations you cannot go; you cannot continue with simulations for a very long time. So, we are restricted to sort of observing events which take place over a very short timescale, ok.

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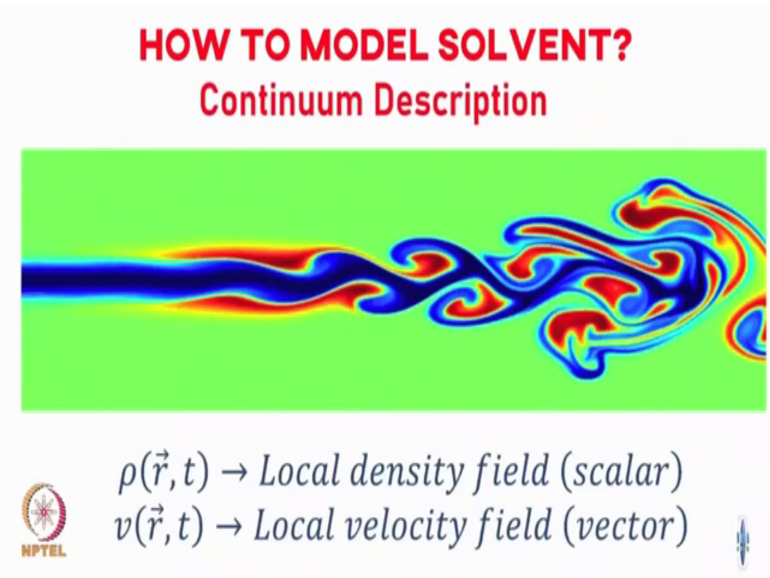

So, the molecular description has its advantages and its disadvantages what we will do is, we will not focus on this molecular description, but what we will do is we will focus on a continuum description. And, like I said in this continuum description what we will do is that will parameterize the fluid by these two quantities.

A local density field which says how dense how many molecules of that fluid are there and a local velocity field which says how fast the fluid is moving, ok. The local density field is a scalar, the local velocity field is a vector. So, you have a scalar field and a vector field and we will use these two to sort of develop our equations of motion, ok.

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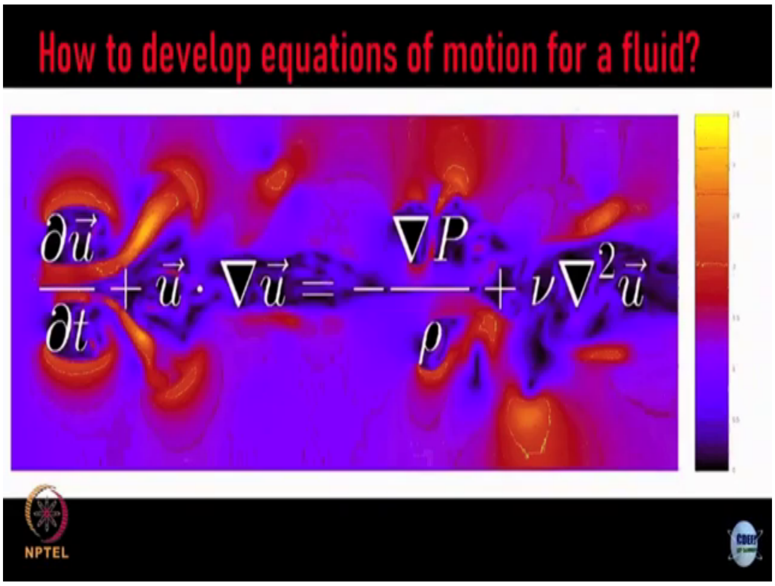

And, eventually what we will come up with is this Navier-Stokes equation. So, before I go into the derivation this is the form of the Navier-Stokes equation, over here are the sort of material derivative on the left hand side. These are the on the right hand side of the forces up on this fluid. So, there is a force due to pressure gradients, there is a force due to viscosity and this eta sort of is what is called the kinematic viscosity; it has the viscosity and the density together ok. So, that is what we will try to do ok.

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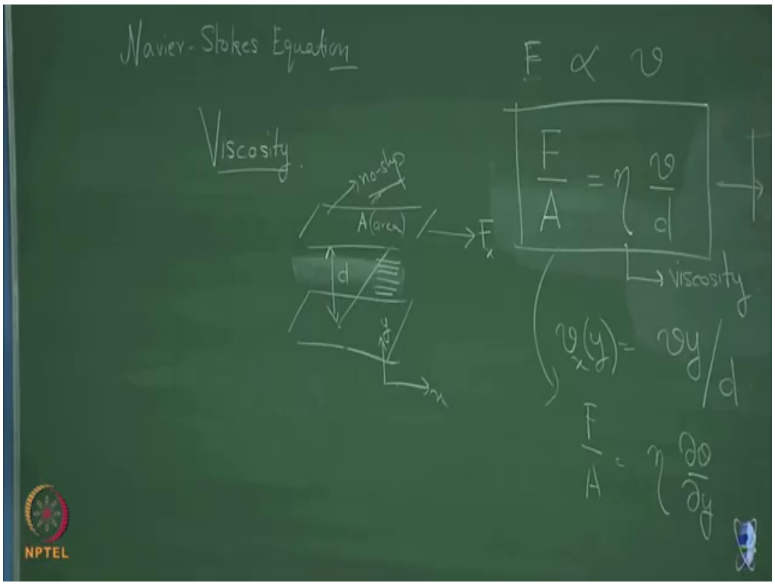

So, before we start the derivation let us just clearly define what we mean when I say viscosity, ok. So, the canonical way of defining is that you assume you have let us say two parallel plates like this and in between you have your fluid, ok. Let us say you pull one of these plates with some amount of force, let us call it F and as you pull it sort of exerts a shear force on this fluid. And, in response it starts moving with a velocity and if you do experiments what you will find and let us say that these plates have some area A, this is the area.

And, if you do experiments what you will find is that for many liquids not all provided your forces are low and so on, you will see that this force is let us say this force is going to be proportional to the velocity with which these platelets. And, if you write out all the constants what it turns out is that the force per unit area which is the stress is going to be equal to the velocity divided by the distance between the plates. So, this is let us say d and the constant of proportionality is what I call as my viscosity, ok; so, this is my viscosity.

We assume many things, we assume for example, there is a no slip boundary condition which means that at the top layer of contact of the fluid with the plate there is 0 relative velocity, which means the top layer of the fluid moves with the plate ok. It is mostly true or rather its the most common boundary condition to take, there are cases where it is not valid for example, in inviscid flow such as superfluid helium and so on.

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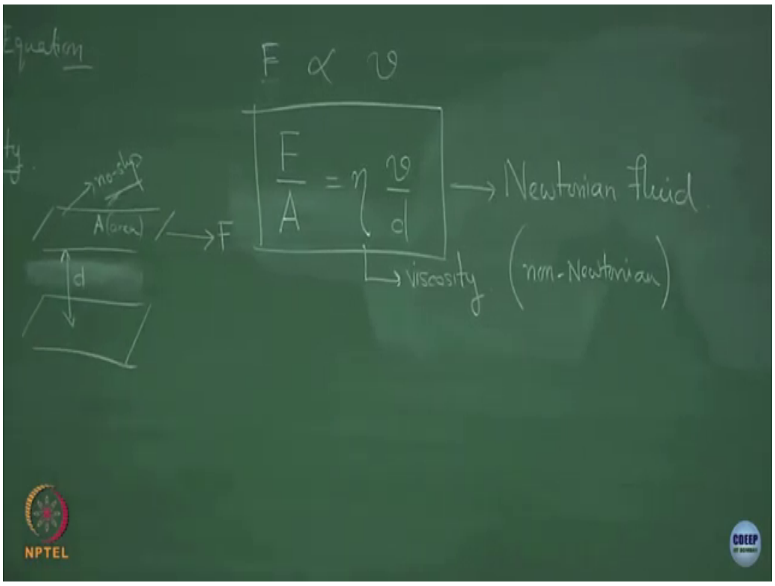

Any fluid which obeys this equation that the force is gone proportional to the velocity and this constant of proportionality appears in this way of this called the viscosity. This is what is called a Newtonian fluid; this is called a Newtonian fluid. There is a whole class of fluids which are non-Newtonian fluids which are non-Newtonian fluids, but provided your forces are small or your velocities are small most fluids to a first approximation you can start off treating as a Newtonian fluid and see what that gives you.

There are some obvious non-Newtonian fluids such as paint and so on, biological media is also non-Newtonian, but depending on the question that you are asking you can do a certain approximation treat it as Newtonian fluids. So, generally whenever you have let us say a pure fluid such as water and you mix in stuff in it, such as proteins or whatever any additives in that sense. You sort of disrupt the pure Newtonian character; it will become non-Newtonian to a certain extent in that this relation will not be generally valid. However, to a first approximation will deal with it deal with biological fluids as Newtonian fluids and then if it turns out that does not match what you are seeing we will try to relax our approximations and see what non-Newtonian fluid would do, yes.

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You want to see the non-Newtonianness of a biological fluid, you can do stress strain relationships in a microscope. You extract the sort of cell fluid and you can do stress strain experiments on that and plot how the stress goes as a function of strain and see whether it is linear. But, perhaps you are asking in a more organic context offhand I do not, I will see if I can think of something which in vivo sort of a system shows the non-Newtonianness fluids, alright. So, what we will do is that we will stick to Newtonian fluids for the time being which is that we will assume; so, this definition by it is this equation by itself defines for me what a Newtonian fluid is.

Any fluid which obeys this sort of a relation and the velocity is proportional to the force or the force is proportional to the velocity with this constant of proportionality is what I will call a Newtonian fluid, ok. And in fact, in a setup like this you have where you have one plate fixed and you have the upper plate moving, the bottom plate fixed; you can actually solve for this full velocity profile. Let us say this is my x direction, this is my y direction you can see you can show that the velocity profile will increase linearly. So, at the bottom plate is this is stationary. So, the velocity is 0, at the top plate the velocity is v and in between you can write this v as a function of y.

So, this is my y axis and its flowing let us say in the x direction because that is my shear direction, you can show that this is going to be a simple linear sort of relationship. So, when y is equal to 0 at this bottom plate it is not going to move, when y is equal to v at sorry when y is equal to d at the top plate is going to move with the velocity, ok. So, if you think about this if you think about this fluid is composed of sort of layers, well each layer is also sort of sliding past one another; you can also write this as F by A is equal to sum eta del v by del y, ok.

So, each as each layer of fluid sort of slides past the other, there is an associated friction caused by the sliding. And, the rate of change of velocity as you move from one layer to the next will be proportional to the force and again with this constant of proportionality given by the viscosity. We can write down what is the dimensions of this viscosity and write down what is the dimensions of this viscosity.

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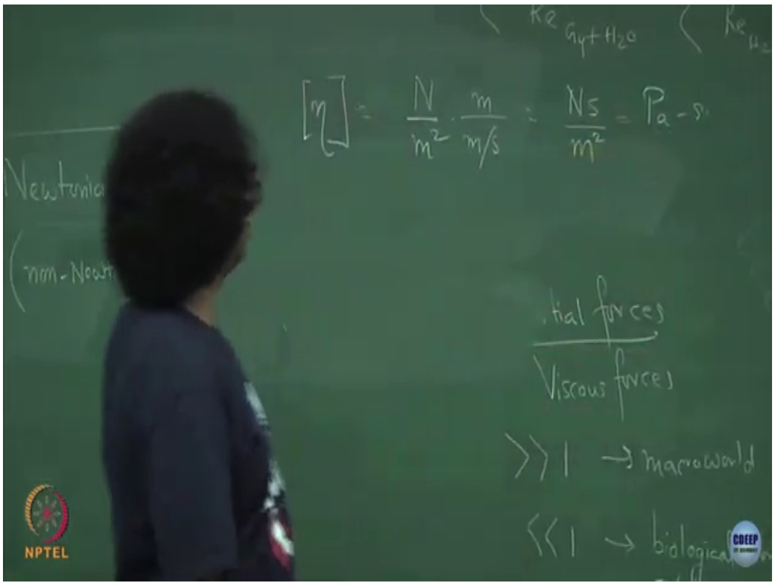

So, eta is force per unit area. So, this is let us say Newton per meter square and into the distance divided by the velocity meter per second. So, this is Newton’s second per meter square which is basically Pascal seconds, ok; Newton per meter square is Pascal’s and then seconds. So, that is what I call the standard unit of viscosity and the viscosity of water is like I said, the viscosity of water is around 10 to the power of minus 3 Pascal seconds.

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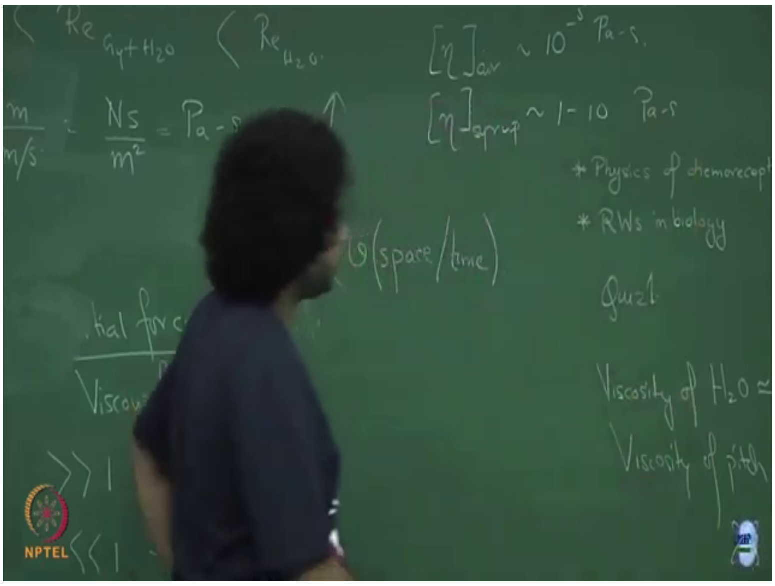

You can write down other things for example, if you want to look at the viscosity of air that is roughly of the order of 10 to the power of minus 5 Pascal seconds. Whereas, this syrup this corn syrup or glycerine would be of the order of say syrup would be something around 1 to 10 in that range Pascal seconds and pitch as I wrote down is even much more viscous.

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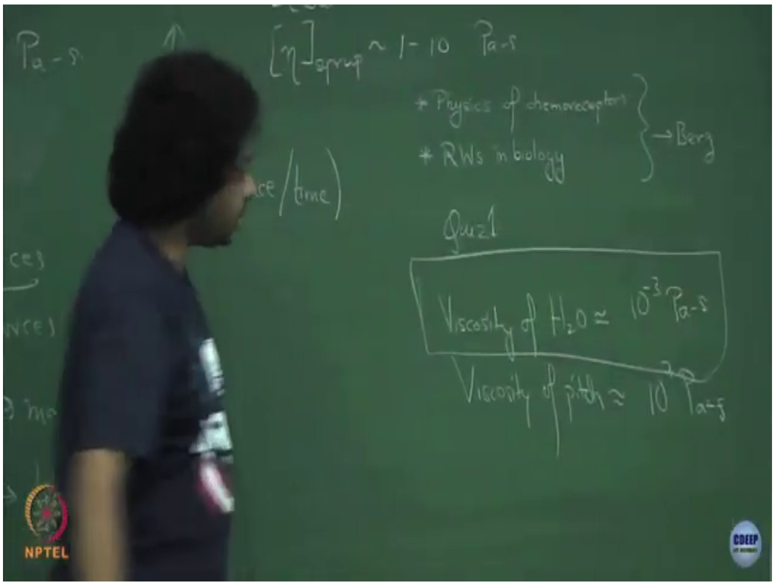

So, so for the time being the, what we will assume what will interest ourselves in is; so, the viscosity is with which lie in this range of water; so, around 10 to the power of minus 3 Pascal seconds. But, like I said it is not just the viscosity which will determine your flow properties, it is together with the velocity as well as the length scales that you are looking at. So, for example, for us as human beings water is not such a viscous object, it is more viscous than air yes, but it is still not something that we would think of as very difficult to sort of swim through.

Whereas, if you are talking about a microorganism which is on the scale of microns; the same environment water is going to look at its going to seem to this microorganism as if its swimming through pitch and that is where this Reynolds number is going to come in. So, with this definition of the viscosity let us try to then write down this equation, this Navier-Stokes equation.

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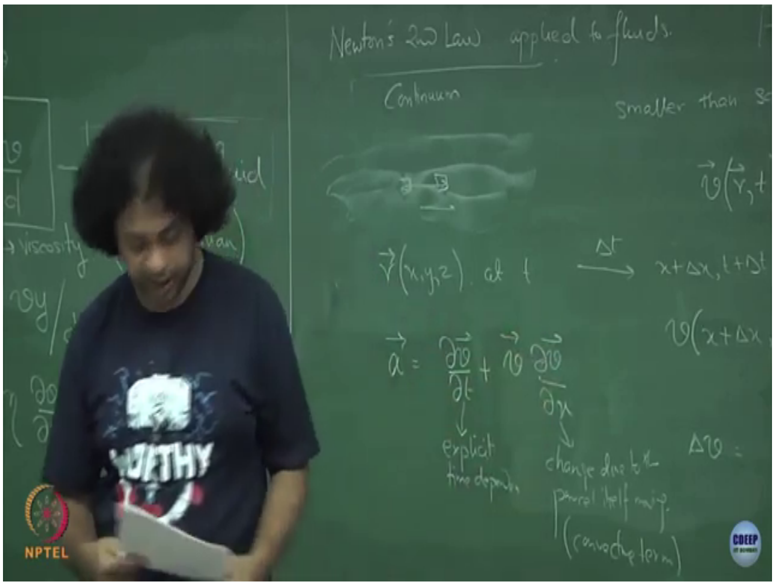

So, the Navier-Stokes that it is nothing, but Newton's second law, it is Newton's second law. So, that is good we all know Newton's second law is just Newton's second law applied specifically for the case of fluids. So, you can derive it in lot of ways. So, what I will do is very loose almost hand waving derivation ok, just to give you a sense of where these terms come from.

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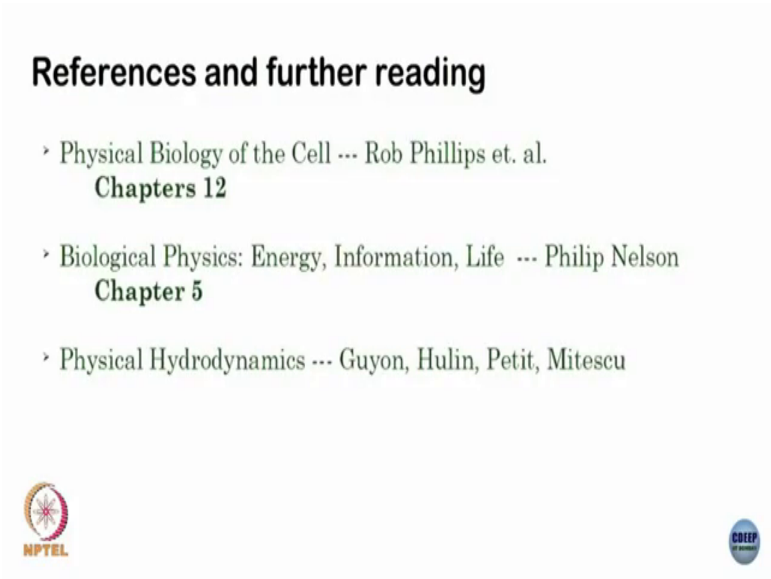

If you are interested I will show a couple of books at the end I can show it now, for example, this is a nice book. This third book is a nice book for if you are interested more deeper in the mathematics or a formal description of this Physical Hydrodynamics, ok. The second book Nelson has a very nice sort of intuitive explanations of different phenomenon for in viscous highly viscous fluids or in less viscous fluids and so on. So, that is a more intuitive description, this third book is a more mathematical description, alright.

So, this is what we will try to do we apply Newton's second law to fluids and see what that gives us.

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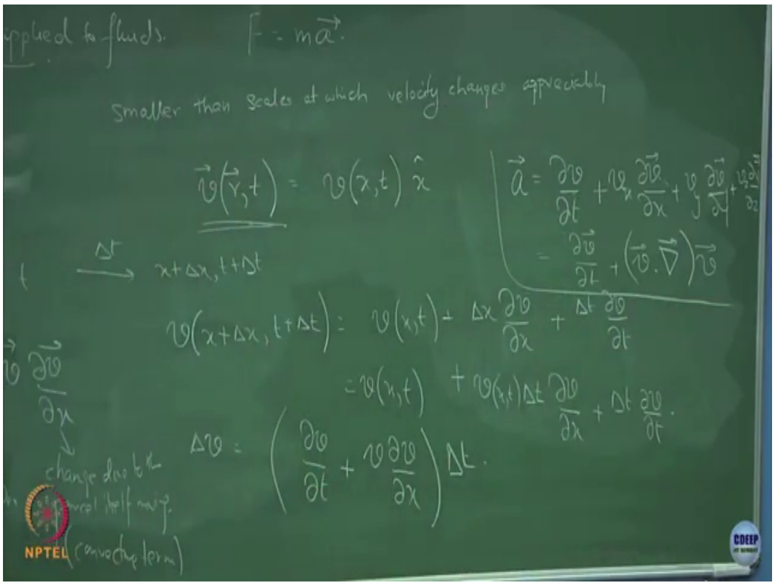

So, what we will write as force is equal to mass times acceleration, but for a parcel of; so, let us say I have some fluid, ok. In this fluid I select a small parcel and I will calculate all the forces and these accelerations for this small parcel of fluid ok. What do I mean when I say its small? Its small in the sense that it is smaller than the scales, it is smaller than scales at which the velocity changes appreciably; at which the velocity change is appreciably, right.

So, for example, if you have a velocity variation which looks like this I do not know, you take a small enough segment such that in this segment you can say that the velocity is something constant, ok. So, whatever is the scale of variation of velocity for this fluid that you are looking at, you choose a small box of fluid, a small parcel such that within that box you can assign a velocity to that parcel, ok. So, you can assign a velocity; however, you cannot make it too small because remember we are doing a continuum description.

So, we do not want to go so, small that you have only a few molecules in there because that is the description of molecular dynamics, ok. So, you have to have a macroscopic number of fluid molecules; however, it must be small enough that it is smaller than the scales at which the velocity changes. So, I take such a fluid element and let us say that this fluid element is at some position x, y, z at some time t; at some time t. Let us take a simple 1D case like I said I will do it I will make as many simplifications as I can. So, let us take a simple 1D case, I say that this velocity which in general is a function of this r and t.

I will say that this is simply a function of x and t and is directed in the x direction ok. So, the fluid this is my x direction in this moves like this. So, I have this fluid parcel which was at x, y, z at time t, after some time delta t it is moved to x plus delta x, after some time delta t, it is moved to x plus delta x at time t plus delta t, ok. So, I want to write down is this velocity at this new position x plus delta x at time t plus delta t and I do that by doing a Taylor series expansion around this v x t. So, this is v x t plus delta x del v del x plus delta t del v del t, I will just write terms of the first order, ok.

So, the velocity after at this new position after a time interval delta t I write in this Taylor series expansion. This delta x is the distance it is moved in this time delta t, right. What is this distance delta x therefore? It is this velocity with which it is moving times delta t, right. There is the distance it is moved del v del x plus delta t del v del t, right and here is v of x of t, ok. Therefore, the change in velocity delta v which is this subject minus this v x, t is therefore, del v del t plus v times del v del x with an overall delta t outside ok, I take this delta t and this delta t common. So, I have a del v del t plus v times del v del x.

And, if I divide this delta v by delta t in the limit the delta t is very small that is nothing, but my acceleration, right. So, my acceleration I can write down my acceleration as del v del t plus v times del v del x, ok. So, this captures the explicit time dependence, this captures the explicit time dependence of the velocity. This captures the change due to the parcel moving itself, due to the parcel itself moving right. So, this is the spatial dependence is the, this is called the convective term, this is called the convective term right.

So, this parcel itself is moved to a new position and the change due to this new position is captured in this convective term, the explicit time dependent change is captured in this del v del t term. So, the total change the rate of the total rate of change of this velocity which is the acceleration has this term plus this convective term the del v del t plus v del v del x, ok. If you were to generalize this so, I took this for the simple case of a velocity which looks like this. If you were to generalize this to any v r t any general v r t, then what you can show is that what you will get is this explicit time dependence term, but then something like actually this.

And, this if you write in vector notation is nothing, but del v del t; del v del t plus v dot del operating on the velocity itself ok. So, the first term captures the explicit time dependence of the velocity, this captures the change due to the fact that this fluid parcel is changing its position as it moves in time, alright. So, that is my acceleration. So, the total time derivative let me write it somewhere.

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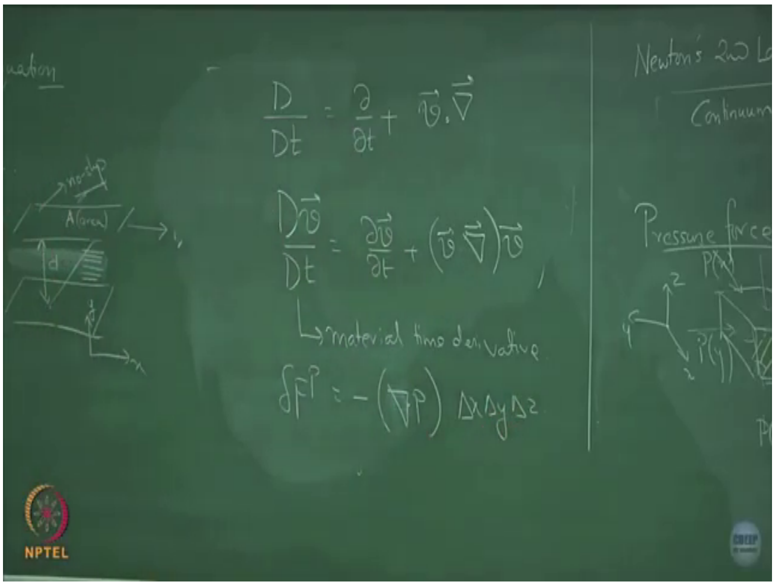

So, the total time derivative which is often written with this capital T is this operator del t plus v dot del ok. So, if you operate it on the velocity del v it is this dv dt is del v del t plus v dot del operating, v ok. So, this is my acceleration term, ok. So, I have this fluid element, the acceleration in terms of this velocity vector I can write down the acceleration term in this fashion.

So, this is call this is also called the material time derivative, this is called the material time derivative ok; that is one side of the Newton's laws. So, once I have the acceleration I can put it over here, now what I need to do is I need to write down the forces that are acting on this fluid already, alright.

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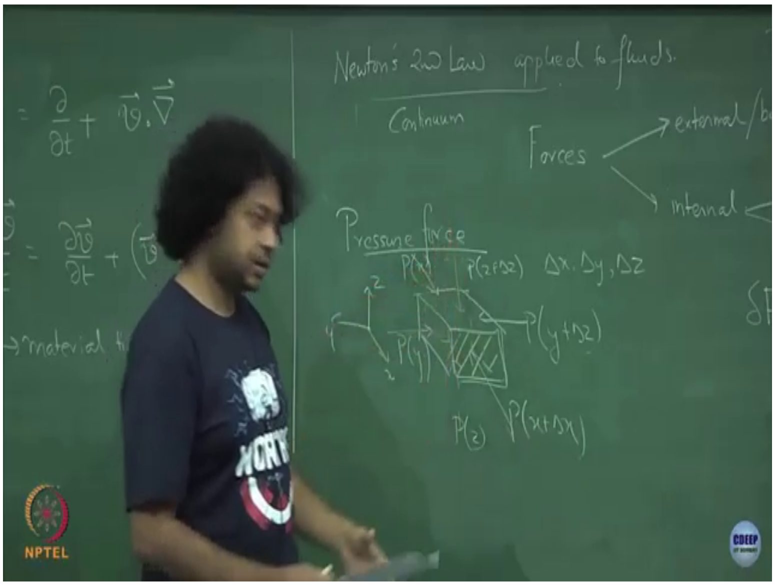

So, forces I can write it could be of two types: it could be external or body forces which act on the fluid as a whole for example, gravity; for example, gravity or it could be internal or internal which arise due to interactions with other fluid elements. So, in these internal forces you could write a pressure force which arise due to pressure gradients across the fluid or you could do and you could write a viscous force which arises due to viscous friction between these different layers of the fluid, ok.

So, these two forces are what I will try to write down, the pressure force and the viscous force. Once, I have written down expressions for these forces, I will put them together with this acceleration term and I will write down f equal to m e, ok. So, let me first do this pressure force and again I will do it as a in the simplest of cases. So, let me draw a cube, ok. So, the here is my fluid element that I have taken, let us say it has dimensions delta x, delta y, delta z, ok. So, there is the same fluid parcel for which I calculated the acceleration.

Now, I will calculate the force on this, let me draw my axis let us say this is x, this is y and this is z, ok. So, the pressure force will arise due to pressure variations in the fluid and it will act inward and normal to these phases. So, for example, like this and then like this on that back face like this and this and so on for the top and the bottom, ok. So, each of these each of the 6 faces of this cuboid that I have taken of sides delta x delta y delta z it will face a pressure force. So, this will be; let us say this will be on this face this will be P of x on this face will be P of x plus delta x and this is my y. So, this will be P of y this is P of y plus delta y and similarly P of z; P of z plus delta z, ok.

So, each face will experience some sort of a pressure that will give rise to a net force on this fluid parcel. So, let us first let us consider this faces let us say that lie along the y z plane; so, that back face and this front face, alright. So, this face over here and this back face over there. Note that the pressure force sort of acts inward. So, these two forces, these the vector associated with these two forces act in the opposite direction.

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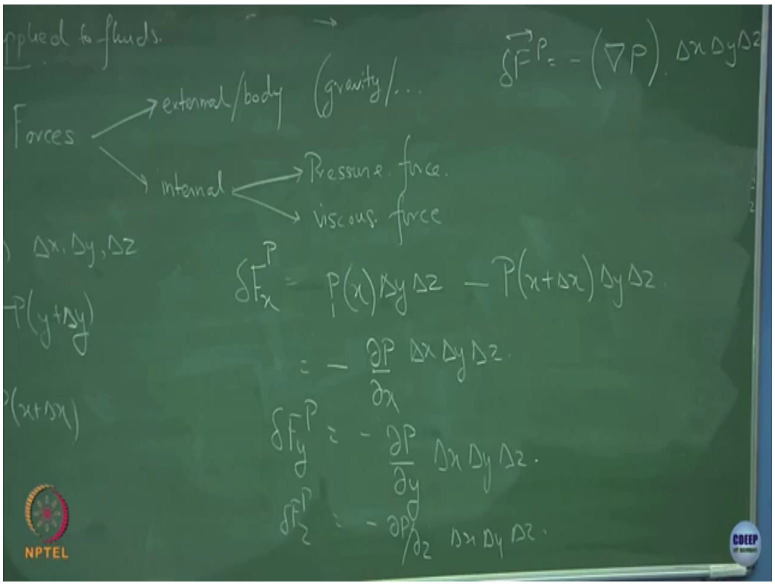

So, I can write the total pressure force along this x axis which will be a combination of the forces on these two faces. So, the pressure force along the x axis will be because of this face over here which is P of x times the area which is delta y delta z plus the force due to the on this face which is going to be minus of P x plus delta x delta y delta z, ok. Pressure is force per unit area so, force is pressure into area the area of each of this is delta y delta z, the pressure on that is P of x, the pressure on this front face is minus v of x plus delta x.

So, what is this? This is nothing, but minus of del P del x right which is P x minus P x plus delta x divided by delta x. So, which means if I multiply and divide by delta x what I get is this, right. So, minus del P del x into delta x delta y delta z right that is a lot of the force along this x axis. I can do the same thing for this force along the y axis and the force along the z axis, right. So, I can write the force along the y axis which will be a sum of this P y and this P y plus where I have written y plus delta z sorry y plus delta y on this face, ok.

And, again they are going to be opposite; so, again this will be minus del P del y and the area of these faces are delta x delta z and you will get a delta y because of this derivative. So, this will come out to be again delta x, delta y, delta z right and similarly in the z direction it will be minus del P del z, minus del P del z right. So, if I write it in vector form; so, the pressure force then is what? Is minus gradient of P right which is which has these components del P del x along x axis, del P del y along y axis will be del z into delta x, delta y, delta z, ok.

So, that is the force due to this pressure variation across the fluid. So, let me also write that. So, delta F P is minus gradient of the pressure into this volume of this fluid parcel delta x, delta y, delta z, ok. So, that is the pressure force and now this last part is this viscous force, ok.

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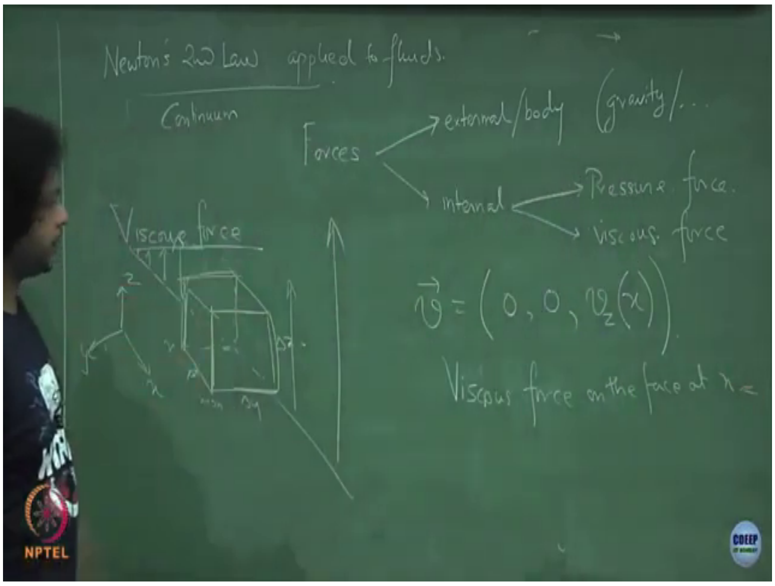

And, remember viscous forces arise because of this viscous friction. So, as you have layers of fluid which are flowing past one another here one layer like this, one layer like that, right, they will exert a friction on each they will exert a friction relative one will exert a frictional force on the other and that will give rise to this viscous force. If they are moving with different flow velocities then this gradient in the velocity will give rise to this viscous force, alright. Let me again draw while draw about that, anyway let me again draw that fluid parcel, ok.

Let us say, let me continue this; let us say that is my x axis. So, let us say that is my x axis, this is my y axis and that is my z axis and remember that these sides are delta x, delta y and delta z. So, that is the cube that I have chosen, ok. So, instead of deriving the full form is a little complicated; so, again I will take a, I will use the simple argument which is to say that I will say that let me assume that the velocity field is of this form, ok. The velocity field only has a component along the z direction, there are no x and y components.

So, it only has a component on the along the z direction and that component depends only on the x position, ok. What does that mean? It means that if I were to plot the velocity field, it will only have components along this z direction right and the height of this which is the module which is the magnitude of this will depend on the position along the x axis. So, let us say I take something like this, as you move along the x axis, the component gets larger and larger, ok. So, its smallest here larger, larger, larger we are moving along the x axis it is larger. This is a very cooked up sort of scenario, but still it makes a life a little easy and one gets to the same equation.

So, I take a I take a velocity field like this ok. So, and I will try to calculate the viscous forces; remember the viscous forces will arise due to derivatives of this change in the velocity right; del v I have rubbed it out. How does the velocity change as you go from one layer to another which means that the only derivative that will survive is this delta v delta x right, because I have taken these other components to be 0 which means that will only be non-zero for faces along this y z plane, it will only be non-zero for this face and for that face. So, it will only be non-zero for this face over here and this back face over there, ok.

So, if I take this face let us say this is my x and this is my x plus delta x. So, if I take this face at x and I can write down the force, let me write down the force see this is the viscous force on the face at x, ok so, on this face over here.

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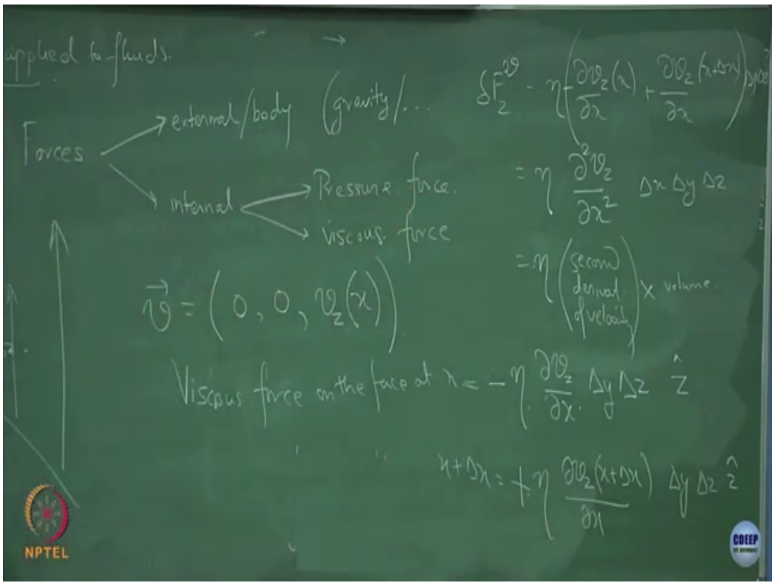

What is that going to be proportional to? That is going to be proportional to this gradient of the velocity which means del v z del x right, there will be a constant of proportionality remember eta. So, eta times the gradient that was the force per unit area. So, therefore, the force will be this into the area which is delta y delta z right, this is along the z axis. Similarly, you can write down the force on the face at x plus delta x this front face. So, this is the same at x plus delta x that will again have this viscosity.

It will have a del v z at x plus delta x and del x right and the area of the face is the same because I have taken a cube so, z hat. There is going to be relative minus sign for one of these; so, this inside is greater. So, if I have taken a velocity field like this at this face at this back face, the velocity just outside is smaller than the velocity just inside right which means that this frictional force is going to try to bring down the velocity because it is a frictional force. Similarly, for this front face, the velocity just inside is smaller than the velocity just outside which means it is going to try to increase the velocity. So, there will be this one will have a minus sign, this one will have a plus sign, ok.

So, the total force therefore, the net force I can then write as over here let me rub all of this delta F z ok, this is let me write z here; this is the viscous force is going to be the sum of these two. So, minus eta del v z del x at x plus del v z at x plus delta x del x times delta y delta z, ok. What is this? So, if I this no, the minus sign is not overall this is minus that is plus right which means that if I would multiply and divide by delta x, what I get is a eta the second derivative del 2 v z del x 2 right, times now delta x, delta y, delta z, ok.

So, this viscous force then I have taken this very simple form, but what it tells me is that this viscous force is proportional to the viscosity which is fine. There is a second derivative of the velocity; there is a second derivative of the velocity into the volume of this parcel, into the volume of this parcel ok. This is for the simple case; I can now try to generalize it through any arbitrary case.

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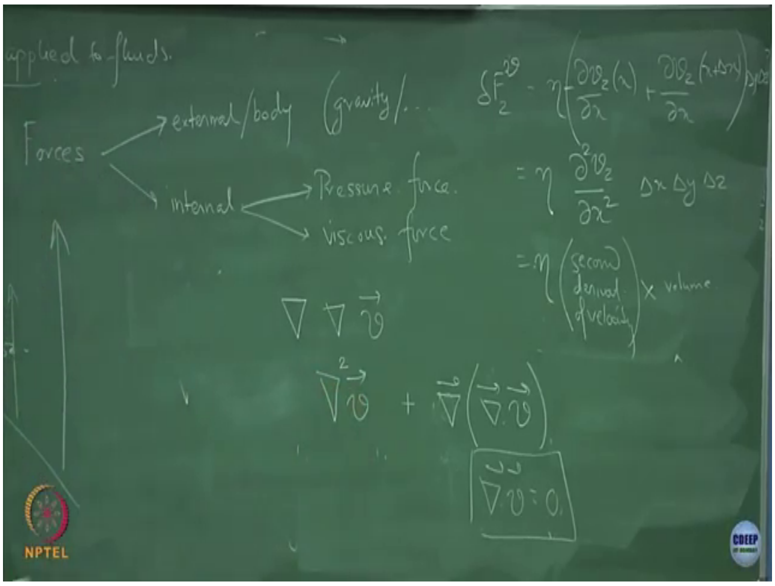

And you can show that; so, the general derivative is of course, del right and I have my velocity which is v. So, if I need to take the second derivative of the velocity it means that I need to have 2 del operators which will act on this velocity field, right. And, you can show in this elementary vector calculus that the only two linearly independent combinations of 2 dels and v are one is this Laplacian of v, ok and, the second one is gradient of the divergence of v. These are the only 2 linearly independent combinations that you can have of 2 del operators and a v, ok.

So, if you generalize this expression that we derived for this very simple velocity field, it will be in general a combination of a term that looks like this and a term that looks like that, right. Now, what sort of a fluid will not have this term, anyone?

Student: (Refer Time: 38:41).

Incompressible fluid right, if your fluid is incompressible then an incompressibility condition means the divergence of v is 0.

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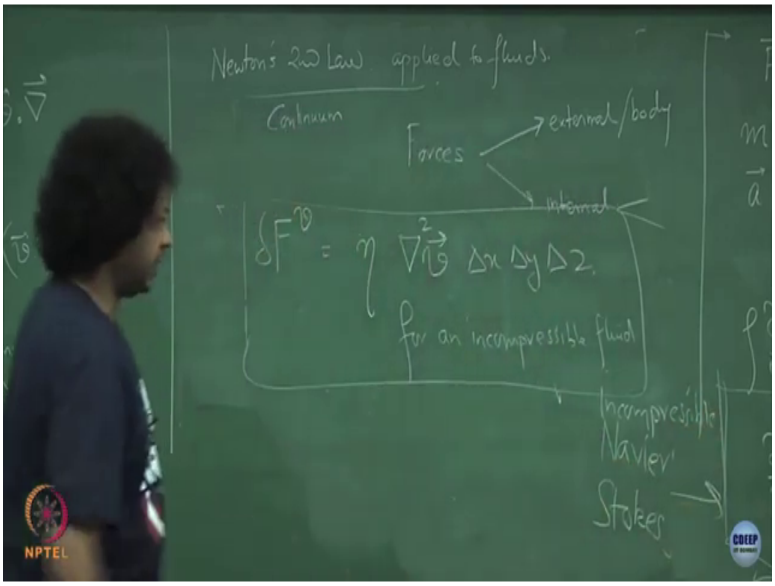

So, if you talk of an incompressible fluid then the only term that will survive is this vector Laplacian of the velocity field which means that this viscous force; which means this viscous force F v will be the viscosity, the Laplacian of the velocity field times delta x, delta y, delta z for an incompressible fluid, for an incompressible fluid, ok.

For compressible fluids you will have a correction term which will have something which is proportional to this since (Refer Time: 39:43). Since everything is on this parcel let me write the delta, alright. So, I have everything together now; I have the acceleration term, I have the special force, I have this viscous force and whatever external body force that I might have. So, you can put all of this together in this F equal to ma and see therefore, what I get.

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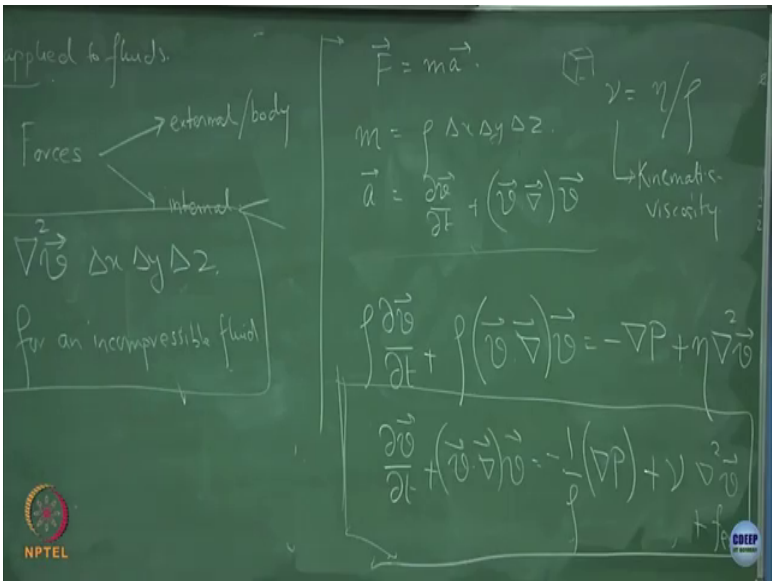

So, what I want to write is F equal to ma for this parcel of fluid right; remember, this parcel that I have taken is this cuboid, very nice cuboid of delta x, delta y, delta z ok. So, therefore, what is the mass of this parcel that is the density of the fluid times the volume delta x, delta y, delta z, right. The acceleration I have found out is this, right. So, the acceleration is del v del t plus v dot del operating on v that is ma and the forces are this plus this.

So, if I write everything together note that when I do m into a I have this volume of this parcel coming in, this volume appears in all the terms. It appears in the ma term, it appears in the viscous force, it appears in the pressure force therefore, that will cancel out right as it should. You should not get a different equation depending on whether you have taken a smaller parcel or a larger parcel, right. So, all these delta x delta y delta z's will drop out. So, what I will get for this on this ma side is rho times del v del t plus rho times v dot del v right that is my ma.

And, then on this side I will get a minus gradient of P plus the viscosity times the Laplacian of v, right. So, that is my force equal to mass into acceleration for a fluid parcel of this of some arbitrary volume. Now, generally you just take this rho to the other side. So, you write del v del t plus v dot del v is equal to minus 1 by rho gradient of the pressure plus eta by rho Laplacian of v. And, this eta by rho which is the ratio of the viscosity to the density of the fluid is often written as mu which is called the kinematic viscosity.

So, nu is equal to the viscosity divided by the density and it is called the kinematic viscosity, ok. Then what we have this equation over here plus of course, if you have any external body force you write that as well. So, if you were to have for example, gravity it would write rho into delta x delta y delta z into g right mg. So, this delta x delta y would cancel you would get rho g over here. So, whatever you get you write sum f external depending on whatever external body force you have. So, this equation is the famous Navier-Stokes equation ok, I did a very loose derivation like I said.

But, there are more formal ways to derive it in all its full glory in without assuming this cuboid and so on and so forth. You can do it for an arbitrary fluid parcel and you come to the same equation the Navier-Stokes equation. So, this is my Navier-Stokes, this is well let me be more correct; this is for an incompressible Navier-Stokes; this is for an incompressible. This is what the canonical Navier-Stokes is, but of course, if you have a compressible fluid then you will have this additional term which will be gradient of divergence of v over there, something like that, alright.