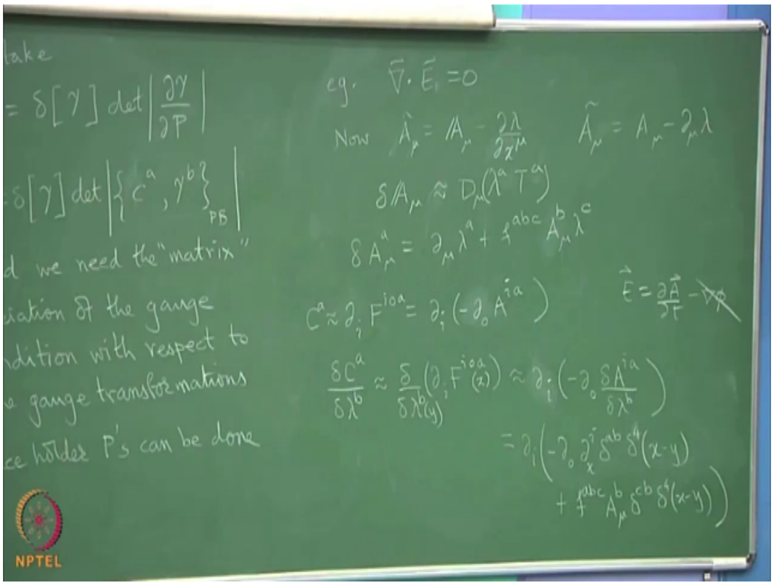
Path Integral and Functional Methods in Quantum Field Theory

Prof. Urjit A. Yajnik
Department of Physics
Indian Institute of Technology, Bombay

Lecture - 30
Gauge Fixing and Faddeev Popov Ghosts – II

(Refer Slide Time: 00:23)



So, since we have set $A^0 = 0$, note that $\partial_i F^{i0a} = \partial_i(-\partial_0 A^{ia})$ and let us put in the gauge index. So, the electric field is $\vec{E} = \frac{\partial \vec{A}}{\partial t}$.

And therefore this is the gauge condition ok; this is C. So, varying C with respect to the gauge condition means that you have to calculate this ok. So, that is same as

$$\frac{\delta C^a}{\delta \lambda^b} = \frac{\delta}{\delta \lambda^b(y)}(\partial_i F^{i0a}(x)) = \partial_i(-\partial_0 \frac{\delta A^{ia}}{\delta \lambda^b})$$

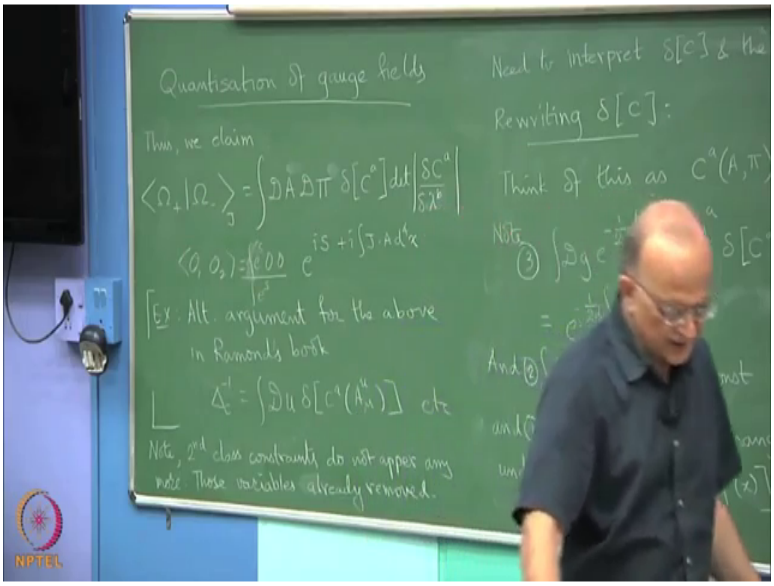
$$= \partial_i(-\partial_0 \partial_x^i \delta^{ab} \delta^4(x-y) + f^{abc} A_\mu^b \delta^{cb} \delta^4(x-y))$$

So, this matrix; this is a gigantic matrix ok. It is not just these labels, but also they are coordinate labels.

So, I went through this in great detail because that is that crystallizes what all this fancy abstract reasoning. So, in practice of course, any good physicist would have done all suffered all this first and then said I have a clean picture, then you say I am Dirac, the constraints in there are first class constraints and second class constraints and there is canonical structure on space of constraints etcetera. But this is what the grungy calculation boils down to.

So, this is the determinant you will need to insert in the path integral to make sense of it. Now this was observed by Feynman in his own way without suffering all this, but suffering other things by in practice looking at what the diagrams needed etcetera and that by in trying to impose gauge condition, you have to remove some of the diagrams.

(Refer Slide Time: 05:41)



So, we now claim W[J] yes so right. In these are all star if you like only the real ones remain. So, these are the full set because we have putting

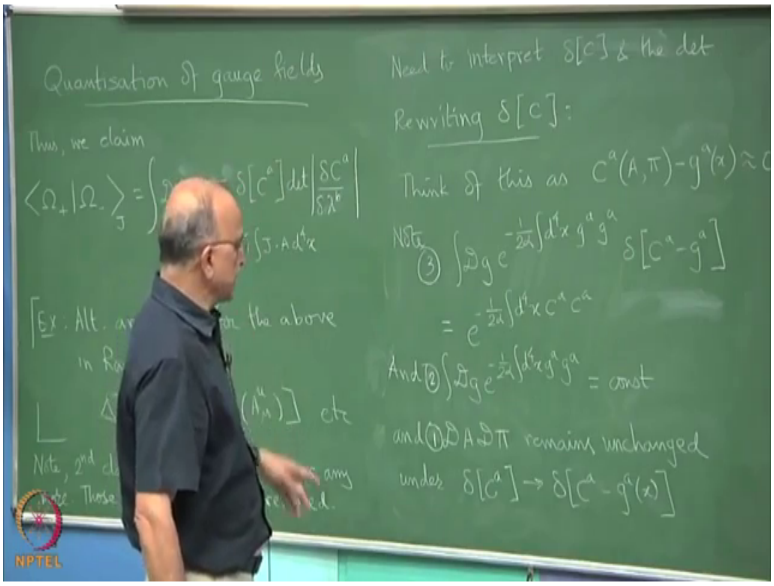
$$\langle \Omega_{+} | \Omega_{-} \rangle_{J} = \int DA\, D\pi\, \delta[C^{a}]\, det \left| \frac{\delta C^{a}}{\delta \lambda^{b}} \right| e^{iS + i\int J.A\, d^{4}x}$$

So, now, we are stuck with this delta function and there is another series of tricks which then gets you to Fadde'ev-Popov answer. One of the tricks is that, that is simpler trick is to rewrite. So, yeah simpler trick is to rewrite.

There is I am remembering here that there is exercise; please somebody note alternative argument for the above in Ramond's book which is actually quite popular with lot of people. It introduces some formal thing called etc ok.

So, one should go through this to see there is another formal elegant argument which also leads to the same answer. It is the same in spirit, but done a bit differently. So, we deal with this $\delta[C^a]$ in a slightly trivial way and it is simpler of the two problems.

(Refer Slide Time: 08:57)



One is that think of so, interpreting or rewriting $\delta[C^a]$. So, here we claim that think of this as $C^a[A,\pi]-g^a(x)\approx 0$. In Dirac's language when you convert the C, $\boldsymbol{\gamma}$ language to the canonical language, the second class constraints which can be explicitly eliminated will not appear and this will remain only on the first class constraints ok. So, the A^0 and $\boldsymbol{\pi}^0$ are not required. So, we can also say that so, all this is only dealing with the first class constraints; once that cannot be trivially is set to 0 or solve for. In fact, if you will read the slightly lengthy discussion in Ramond's book, he does so for the Abelian case, you can get by without doing these technical things because you actually just invert the gauge condition div A equal to 0. In the Coulomb gauge, you just get an awkward looking long expression which is end of the story; no determinant to be solved for you explicitly solve and stuff it in.

So, then you only have two variables to integrate over. So, here the more superfluous as one the second class one does not even appear anymore and it is the second one that is one is trying to do. And by the way this argument which is the alternative and basically says that we want to get rid off superfluous integrations so, our gauge copies of A. So, effectively you have to factor out the volume of the gauge group except there it is function of x. So, the gauge valued u of x gauge transformation.

So, it is that which is being removed. For that you do not the superfluous as one can be assumed to be already gone and then you remove the gauge volume for the remaining gauge conditions. What we note is that look at this formal integration,

$$\int Dg\, e^{-\frac{1}{2\alpha}\int d^4x\, g^a g^a \delta[C^a - g^a]} = e^{-\frac{1}{2\alpha}\int d^4x\, C^a C^a}$$

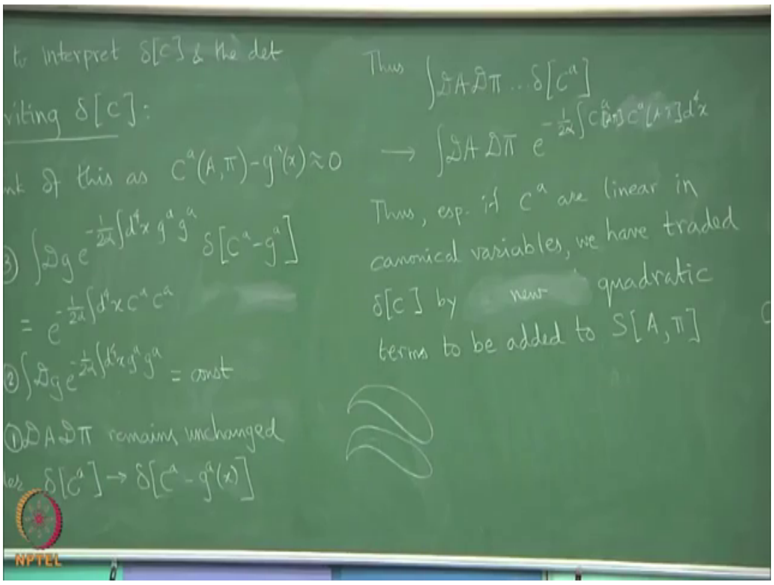
So, it becomes this, but we note that this whole expression which is the Gaussian integral. So, is that best giving you some answer which is not dependent on canonical variables so, this right.

So, if you look at this answer is some Gaussian right. It is determinant of let say word, I mean multiple of $\sqrt{\alpha}$ at infinite number of points in space and time, so, it is some constant. So, we take this expression which is a constant and multiplied to this where we shift the C^a by this constant without affecting the Jacobian of the this integral. So, the big functional integral is going to maybe accumulate some Jacobian again which is not going to depend on the canonical variables. So, I went in reverse.

So, think of this as firstly, you do this transformation. So, this is 1, this is 2 and one this and. So, if you a believe these two that this is nothing, but some constant similarly shifting C^a actually does not cost you anything in this functional integral, then you first shift the C^a. Do the number two step first, then multiply on the left by this ok. Now you do g integral. So, this is step 3. You do it in this order 1, 2 and 3 in reverse actually.

Yes. So, note that shifting this by some constant some c-number function not a canonical variable did not cost anything. Also we multiplied by and overall Gaussian which is not going to change canonical variable answer, then we basically note that after doing that manipulation, I can carry out the g integral and replace my $\delta[C^a]$ by this Gaussian version of it.

(Refer Slide Time: 17:55)



Why is that good? Because instead of having to interpret some formal delta function over some condition in the phase space. We have replace the phase space expressions in the same line as the action functional and if the C^a are typically linear in the variables, then we just have added some quadratic terms in the effective action ok.

So, by ok; so, that is not so bad. So, it just makes it life more convenient. So, let me remind you of the other main philosophy why all this very sleight of hand works. As I was saying in the very beginning, all of these are fortunately not any numbers you are calculating. This is the what I try to emphasis the path integral or the functional approach is never a very useful tool for doing quantum mechanics or chemistry.

It is primarily a tool for deriving identities and relationships between Green's functions. And so, it is symbolic dependence on the canonical variables that is all that matters. It is not meant to actually give you any numbers. So, if you have infinite factors multiplying it, it does not really matter eventually you will vary and you are also going to divide by you know when you take expectation value of something it will be equal to insertion of this, but divided by the thing without an insertions; so all the infinite volumes are going to cancel out. So, if it over all changes any numerically the number, we do not care about it.

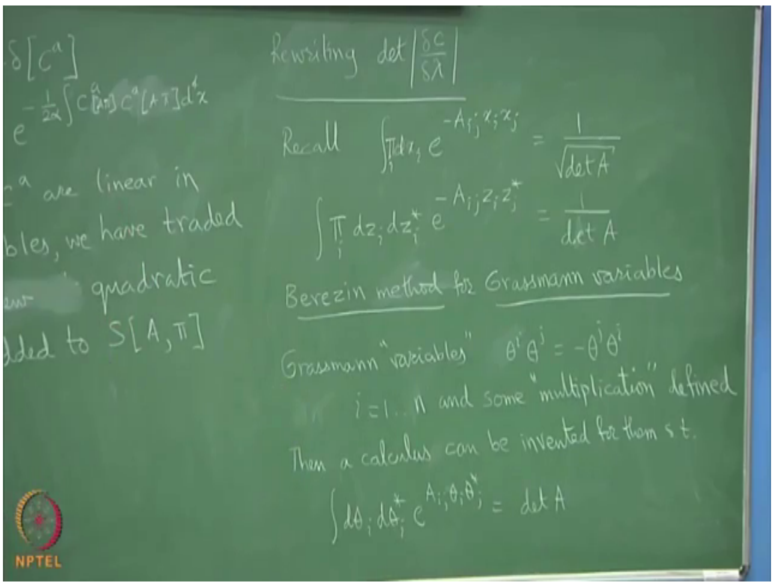
It will in this stack of things, I was drawing instead of being this surface it will become someone other surface that is how it will do.

The identities that you will derive by where say you are calculating some operators by

$\langle O_1 O_2 \rangle = \frac{\int e^S O_1 O_2}{\int e^S}$; these things will not change; those relationships will not change. It is actually meant to derive relationships between operators or observables in I mean identities between eventual observables even Green's functions are not directly use as observables, you convert it S-matrix in the end, but those things are not going to be affected by these manipulations. So, the path integral itself the functional integral itself does not actually calculate anything, but it shows that the dependence on the canonical variables is essentially Gaussian exponential or exponential of whatever the expression is, but that is all it really captures.

So, these are in the general as it is called generating functional. It is basically a generating functional of various identities and not itself any value of any numerical value to be calculated. So, this will be replaced by this which is nice we got thread of one delta function, the other one has a really bizarre story and which is where there Russians come in and we will only start today and complete it next time.

(Refer Slide Time: 24:11)



So, rewriting $det\left|\frac{\delta C}{\delta \lambda}\right|$. So, here clever people observed that I mean they only good thing, we can do here is Gaussian integrals. This is what; however, illegal we have all accepted as friends that we can do Gaussian integrals. So, what Gaussian integral can give you a determinant. So, of course, you are very clever so, you know that determinant can be got by doing. So, if I had a multivariate Gaussian

$$\int \prod_i dx_i e^{-A_{ij}x_i x_j} = \frac{1}{\sqrt{det A}}$$

So, we got a determinant, but hack it is a square root and in the denominator. So, we say ok, we can do a little better that is effectively secretly bringing in two variables.

$$\int \prod_i dz_i dz_i^* e^{-A_{ij}z_i z_j^*} = \frac{1}{det A}$$

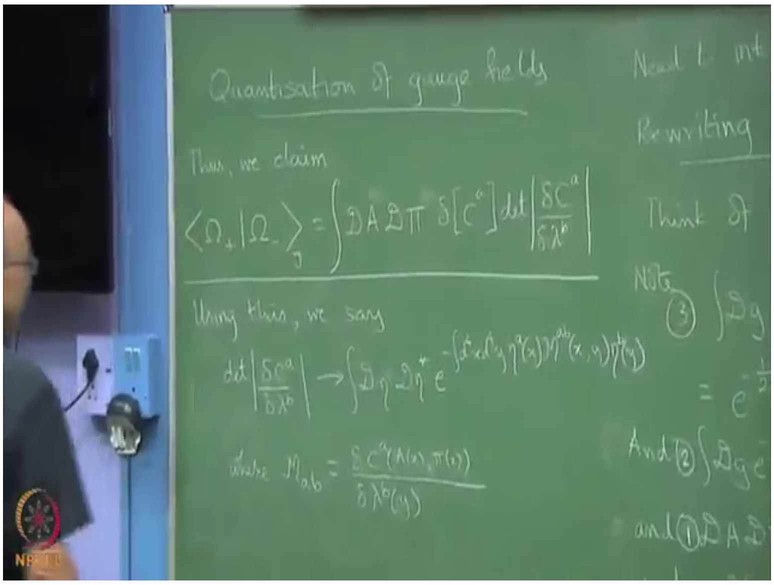
but now this silly thing is in the denominator, but we wanted in the numerator. So, there was this man Berezin. So, Grassmann variables are the things that are anticommuting whatever the θ's are $\theta^i \theta^j = -\theta^j \theta^i$, they are not numbers ok. So, and there is some multiplication among them. So, there is some kind of an algebra, some kind of variables is called θ, some number of them i = 1,…,n and some multiplication defined over them, a binary operation.

We make up the we invent a calculus for this. So, let me first tell you the answer, then a then a calculus can be invented. Sets that integral

$$\int \prod_i d\theta_i d\theta_i^* e^{-A_{ij}\theta_i \theta_j^*} = det A$$

So, this is what we will check next time, but let me start it off. So, but anyway just to also jump to the answer.

(Refer Slide Time: 29:37)



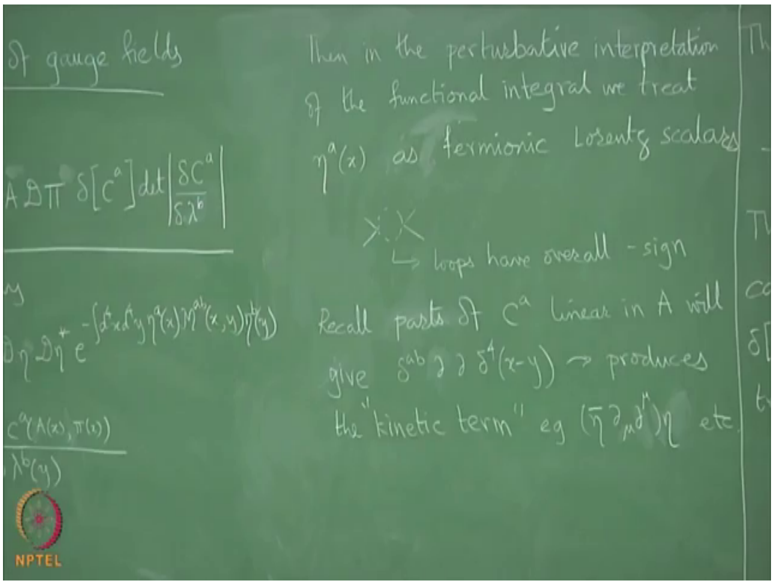
Then we can

$$det\left|\frac{\delta C^a}{\delta \lambda^b}\right| \rightarrow \int D\eta D\eta^* e^{-\int d^4x d^4y \eta^a(x) M^{ab}(x,y) \eta^b y}$$

with $M_{ab} = \frac{\delta C^a}{\delta \lambda^b}$ as we worked out in that example. It is a complicated expression usually right, but it can be found. So, it will become a function of x and y and we will has some delta functions in a and b. So, this is what happens. So, you replace in then that determinant also by something which again as you can imagine. If the C's are essentially linear in the canonical variables, then this variation will at best produce because the covariant derivative is after all sort of quadratic in the A's.

It will with some residual A this M_{ab} will be again at best quadratic in A ok. It will be something functional of A, but new variables $\boldsymbol{\eta}^a$ and $\boldsymbol{\eta}^b$ which are complex anticommuting variables over which we have to do the integral. The advantages so, we will see the mathematics of that next time, but the advantage of this is that we then interpret the $\boldsymbol{\eta}$ as if they are anticommuting complex scalars.

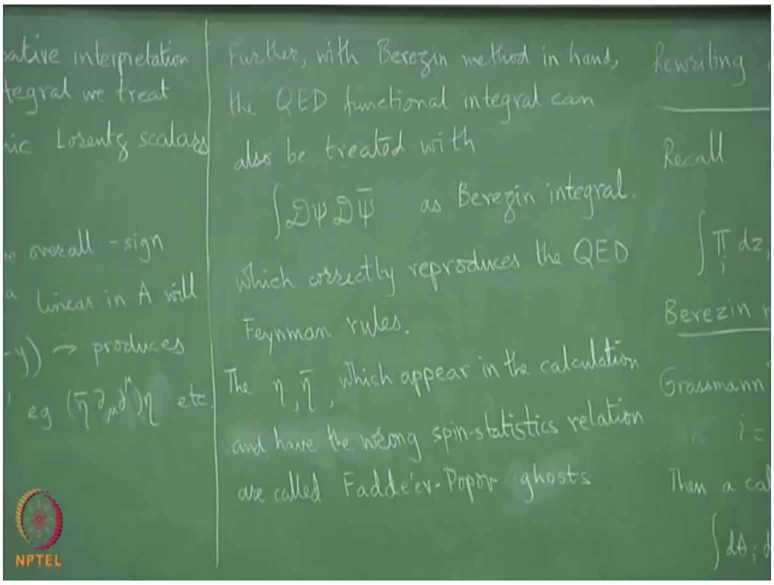
(Refer Slide Time: 32:49)



Well, I complex is not so crucial. So, fermionic Lorentz scalars which basically means that there are loops; loops have overall minus sign. And there kinetic terms so to speak gets decided from here because recall that $\delta^{ab}\partial\partial\delta^4(x-y)$.

So, which produces the kinetic terms; usually looking exactly like it is a direct spinner you know $(\bar{\eta}\partial_\mu\partial^\mu)\eta$; it like a scalar of course, so, not fermionic. So, it is; so, it will give some scalar like kinetic term to them, but they have to be treated as fermions because there anticommuting and the good thing is that your genuine fermions electrons etcetera have to be dealt with also by Berezin integral only.

(Refer Slide Time: 36:25)



So, and if you do it then you actually recover QED Feynman rules. Of course, nobody actually bothers to do QED by that method because it is easy to quantize mode directly, but if you did then, it will give you the correct answer. The $\eta, \bar{\eta}$ are called ghosts. Well appear in the calculation and have the wrong statistics are called Fadde'ev-Popov ghosts. Let me write ghost properly, otherwise ghosts may not be happy with me.