Path Integral and Functional Methods in Quantum Field Theory

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Lecture – 16
Effective Potential – IV

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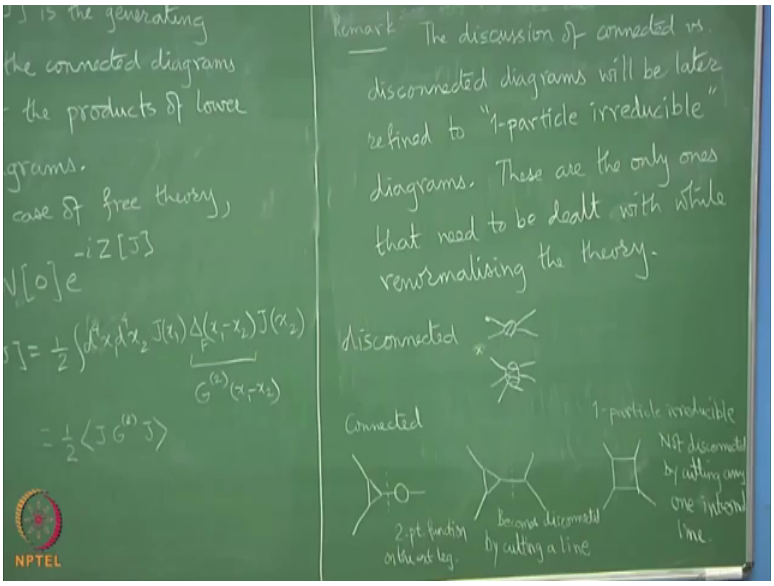

So, and I can tell you right away what this one particle irreducible is. So, we already saw that disconnected simply means two diagrams do not talk to each other right, I can have, but I can simultaneously or something else going on here ok. So, this is and so, there is a multiplication between the two, but they are disconnected. Connected diagrams can be of three types and for arguments say come taking from some book, but this is of course, connected because everything is connected to everything.

Then, we have this is no specific theory I am writing, but I am just drawing some graphs like graph theory, these are all connected, but the point is we say this only has a 2-point function on the external leg. That just gets reinterpreted as the if you that just gets reabsorbed into the mass of the particle.

We just say the correct propagator such that these things are all absorbed into the. So, whatever higher order effects generate things like this where there are only two external legs, we just reabsorbed it into the propagator and we do not consider it as new physics.

This thing is said there it can be cut into two ok. So, becomes disconnected by cutting one line. So, we do not like this we say that is too simple ok, but this last one is what we call one particle irreducible, one particle meaning one line, something that cannot be made disconnected by cutting one line is call 1-particle irreducible any one internal line.

So, for this reason it is useful to keep track of what is connected disconnected and what is one particle irreducible. And it also goes to the core of this locality and cluster decomposition. So, if physics is local ultimately, then you want that if you start with the larger Green's function end point function.

And if I separate first 5 and the next 7 particles and send them far away it should just become product of a 5-point and a 7-point Green's function. That property is assumed of so, it can be proved that so, long as you write your Lagrangian as local products of fields and derivatives.

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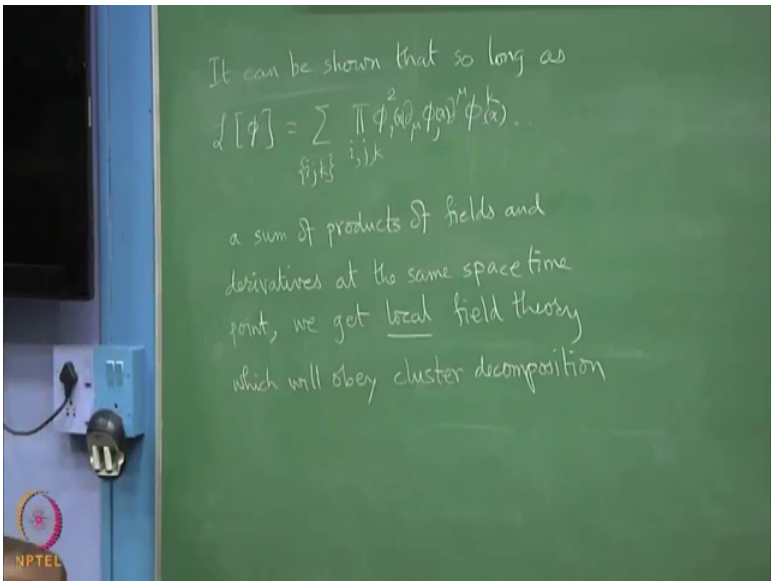

And just to not stress you with unbalanced Lorentz indices something like this ok, products of is just a generic example of product this as one field and two derivatives, but you could have powers of you know you just make a square if you like. So, that it is no charges are hanging. But, the point is so, long as this is a product of at the same space time point ok.

So, I will not try to write the definition of cluster decomposition because to be precise when I used to write a lot, but you got the idea, that if I separate out particles then they become products of individuals scatterings.

So, this kind of locality of the theory again goes back to the possibility that higher order Green's functions contain products of lower order once, but they are just nuisance, their because they follow because after all it is a local field theory. And the only non-trivial thing that we have to worry about while renormalizing is to worry about the one particle irreducible diagrams that to there will not be too many of them.

If there are too many of them then you are sunk in the sense that you have a large number of unrenormalizable terms and the theory reduces in it is ability to predict. At present we have theory of pions and nucleons in terms of what is called a chiral Lagrangian. And this chiral field theory is you have to include all possible higher order terms consistent with symmetries of the theory. And each coefficient then gets re normalized if you do higher order scattering. So, you have to just determine each coefficient by doing scattering at the required energy. And so, it is predictivity goes because you cannot predict what the coefficients are.

So, if the theory has only finite number of one particle irreducible diagrams needing renormalization, then you have a renormalizable theory. And then you have lot of predictive power like the Standard Model has with which gives us confidence to build a machine like LHC.

So, those are the advantages of this and that is why this kind of formalism is developed. So, now, you may ask. So, last time we introduced what I is called effective potential and that is what we are going towards now or effective action.

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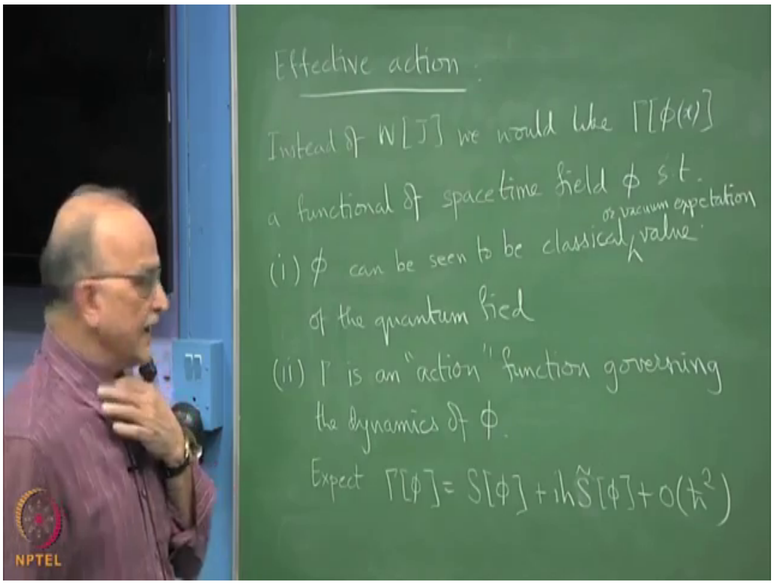

So, far we got a functional which is the functional or some auxiliary thing J. So, what do we deal with what are we dealing with? So, instead of W[J] we would like $\Gamma[\phi]$ a functional of spacetime field ϕ which can have such that we get the dual interpretation, by which I mean expectation value right, vacuum expectation value of the quantum field ϕ. So, we want to construct an energy from that Γ is interpretation of governing the dynamics of ϕ.

So, in particular we expect to recover S of ordinary ϕ, the ordinary action of the ordinary ϕ as the lowest order the zeroth order approximation in

$$\Gamma[\phi]=S[\phi]+i\hbar\widetilde{S}[\phi]+O(\hbar^2)$$

So, this is what you would expect. If, there is no quantum if there are no quantum corrections, then the field theory should just remain classical field theory you we will see.

So, if you do the quantum action principle for the that is this Γ for our free field theory, we will simply recover the same equations of motion for the correctly define classical field, which can now being interpreted as vacuum expectation value of the quantum field ok. So, we are in search of such a formalism.

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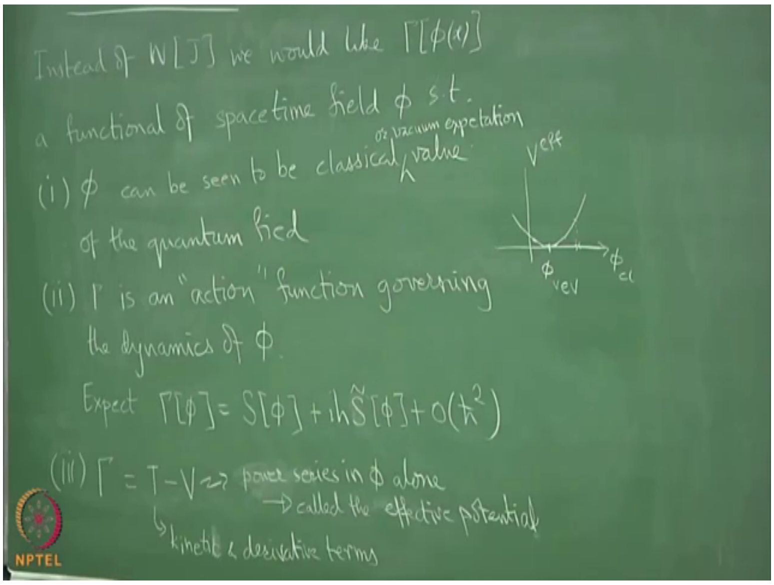

And of course, **Γ** will have **Γ** will be of the form T - V, well T is kinetic term, but I should say kinetic and derivative terms and V a polynomial well power series in **ϕ** alone not the derivatives. So, that is what we called the potential and this is what we call the effective potential.

So, this is called effective action and this is called effective potential. Because, if you deal with a so, after you construct this. So, these are a exactly like I have been trying to say in thermodynamics you have some gas and whatever complicated dynamics it has, the variables that you see for the gas as a whole are pressure, volume, temperature etcetera. Volume is an extensive parameter. If, you change if you add something then the volume also increases. Pressure is only local; it does not if you have more gas the pressure at that point does not change. So, those are intensive versus extensive variables. And you can construct free energy of the system and ask what is it is dependence on volume.

So, this **ϕ**, you ask for it ask for the dependence of field on well this is an intensive variable, you ask for it is dependence on this particular collective variable **ϕ**. And it tells you the answer if you switch of the derivatives then it tells you the energy it has. And it can be then thought of as a question of suppose I constrained my **ϕ** to a acquire a part it need not even be a vacuum, I could just take. So, typical plot of V^{eff} is something like this ok. What this means is this is the minimum this will be actually the web vacuum expectation value, because that is where V is 0.

So, we take this is how it is done right you have whoever is seems simple examples of symmetry breaking and so on. You vary with respect to ϕ and wherever you find minimum that is the vacuum expectation value. You may ask what are the what is the meaning of this part of the curve, well the answer is suppose I constrain my ϕ to of this value somehow I arrange it, because after all I have that external current J in hand.

So, I could crank up the J in such a way that ϕ gets this value what will be the energy of the system for that value of ϕ. So, we trade the auxiliary variable J for the actual physical variable ϕ, then we can ask such questions. And then we can constraint the value of ϕ to be where we like or we can make it both space and time dependent then we will learn action for it and we can ask if my ϕ has this particular space time be aware what is the value of the action and so on. So, that will be the meaning of that.

So, the question is how do we go from this description W[J] to $\Gamma[\phi]$. And it is a very interesting and elegant mathematical connection, because it uses the Legendre transform what we observe is that sorry the 2-point function that we had is something that has 2 ϕ's in it $G^{(2)}$ is expectation value of 2 ϕ's. So, somehow if we vary this G with respect to ϕ_1 which should have information corresponding to 1 ϕ left ok.

So, I will now try to follow the calculation I have. So, we go back to this free theory example. So, we have to motivate firstly, what this ϕ is?

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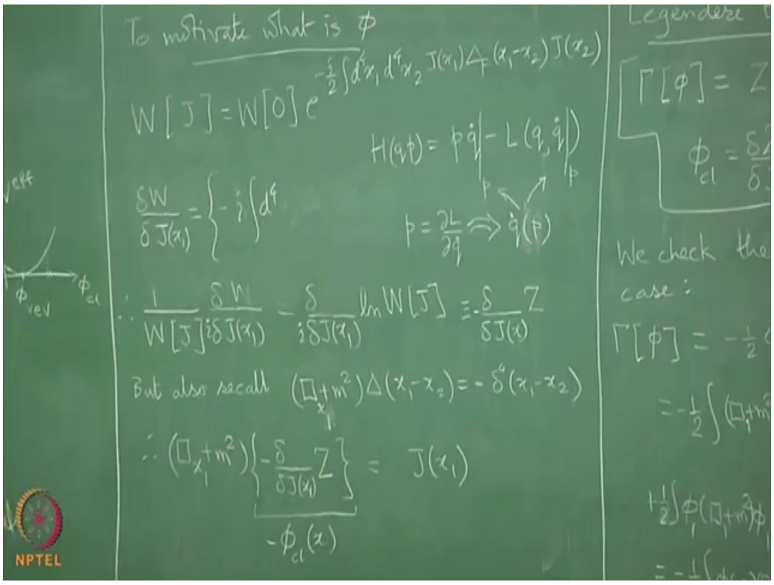

So, we take this

$$W[J] = W[0]\, e^{-\frac{i}{2}\int d^4x_1 d^4x_2 J(x_1)\Delta_F(x_1-x_2)J(x_2)}$$

Now, we see that if I do

$$\frac{\delta W}{\delta J(x_1)} = \left[-i\int d^4x_2 \Delta_F(x_1-x_2)J(x_2)\right]W[J]$$

We had that little annoyance there. So, now, we divide out by W[0]. So, now we say therefore, one over

$$\frac{1}{W[J]}\frac{\delta W}{i\delta J(x_1)} = \frac{\delta}{i\delta J(x_1)}\ln W[J] \equiv -\frac{\delta}{i\delta J(x_1)}Z$$

but now we note that $(\Box_{x1} + m^2)\,\Delta(x_1-x_2) = -\delta^4(x_1-x_2)$

So, if I this thing is a function of x_1 it is a functional of a function of x_1. So, if I hit this now which is I mean this refined one the one without the W. So, if I therefore, $(\Box_{x1} + m^2)\{-\delta/\delta J(x_1) Z\} = J(x_1)$ So, I am going to now announce that this expression is what I call ϕ_{cl}, because indeed that is the one that was going to obey this Klein-Gordon equation.

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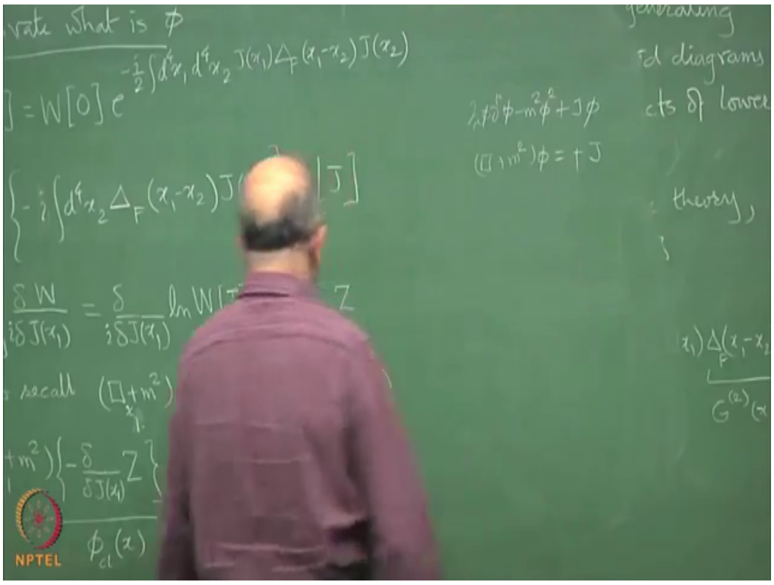

The from the original action which was KG action, what is the equation of motion we get well we get ($\square + m^2)\phi = J$ I guess.

So, when I have this is exactly what should be called ϕ_{cl}. So, ϕ_{cl} is nothing, but variation of that log of the W with respect to the external current. So, now, what we do is we declare this ϕ to be the real the actual variable in terms of which I want to deal with the theory and I want to throw away the J.

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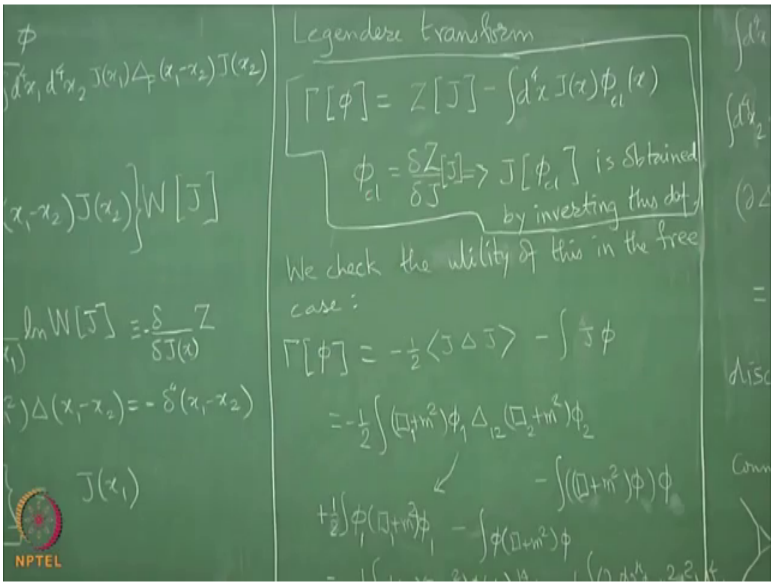

So, what we do is we do a Legendre transform. So, we define

$$\Gamma[\phi]=Z[J]-\int d^4x\, J(x)\phi_{cl}(x)$$

. Now, here we have to so, $\phi_{cl}=\frac{\delta Z[J]}{\delta J}$.

So, I want this definition. And the meaning of this expression is that, this , $\Gamma[\phi]$ but you see here J's. The point is wherever you see J you have to invert this equation. So, this implies that $J[\phi]$ ok. So, we have an expression ϕ_{cl} equal to which will come out to even expression in J, Z is the some functional of J. So, if I vary with respect to J I will have some functional of J. I have to invert this and express J in terms of ϕ_{cl} and then plug that J everywhere.

So, this J and this J will be this J substituted there that is when Γ will be defined in it is correct domain ϕ_{cl} right. This is the usual Legendre transform trick, but that is what we are now going to propose.

So, now we can see we can actually prove that, we recover the effective action, we will recover by systematically plugging this thing we will exactly get what we expect, we will get just S because there are no quantum corrections and it is in 2 steps. So, in the present case we check the utility of this in the free case we find $\Gamma[\phi]$.

So, first have to write

$$\Gamma[\phi] = -1/2 \langle J\Delta J\rangle - \int J\phi = -1/2 \int(\Box_1 + m^2)\phi_1\, \Delta_{12}\, (\Box_2 + m^2)\phi_2 - \int((\Box + m^2)\phi)\phi$$

So, if we make this act on the Δ, we will get a δ function. Because, there are this is second derivative, if we transfer both derivatives in succession the sign will not change and it will act on this, but that will produce a $\delta^{(4)}$ function. And that will set the argument the same for both. So, what I am saying is that this can be reduced to being simply $\Gamma[\phi] = 1/2 \int\phi_1(\Box_1+ m^2)\phi_1 - \int\phi_1(\Box_1+ m^2)\phi_1$

That we do it twice because it is a second order derivative, so, that sign does not change. And survival will recover exactly half battery sign aside from assign we will just recover the classical action. So, now, let us just see what we are seeing over there I think all of you got the idea, but let us write it out.

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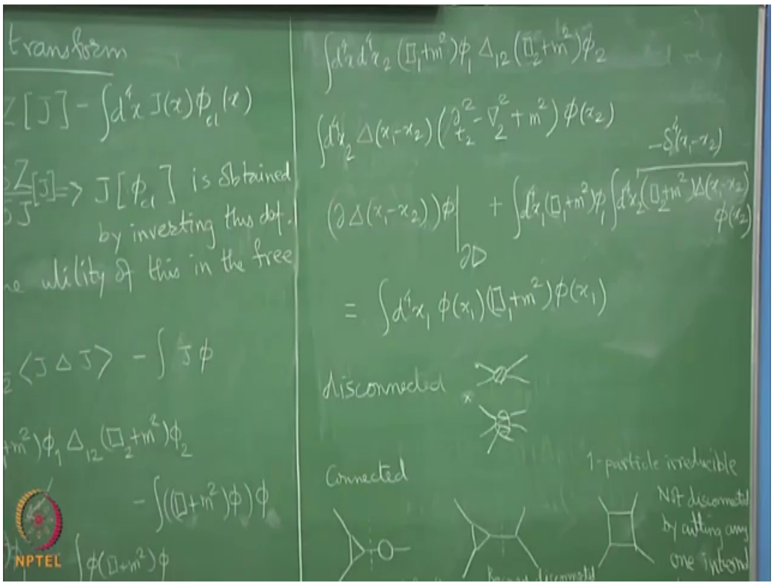

So, what we are claiming is that if I have integral $\int d^4 x_1\, d^4 x_2\, (\Box_1 + m^2)\,\phi_1 \Delta_{12}\, (\Box_2 + m^2)\,\phi_2$.

So, $\int d^4 x_2\;\; \Delta(x_1 - x_2)\;\; (\partial^2_{t2} - \nabla^2_2 + m^2)\,\phi(x_2) \;=\; \int \partial\Delta(x_1 - x_2)\,\phi \;+ \int d^4 x_1\, (\Box_1 + m^2)\,\phi(x_1) \int d^4 x_2\, (\Box_2 + m^2)\,\Delta(x_1 - x_2)\,\phi(x_2)$

So, then using up the x_2 integral it just became $\int d^4 x_1\, \phi(x_1)\, (\Box_1 + m^2)\,\phi(x_1)$ and you can then do a partial transfer of a derivative. So, the minus sign should eventually go. So, this is correct. So, right so, this manipulation all leads to this basically.

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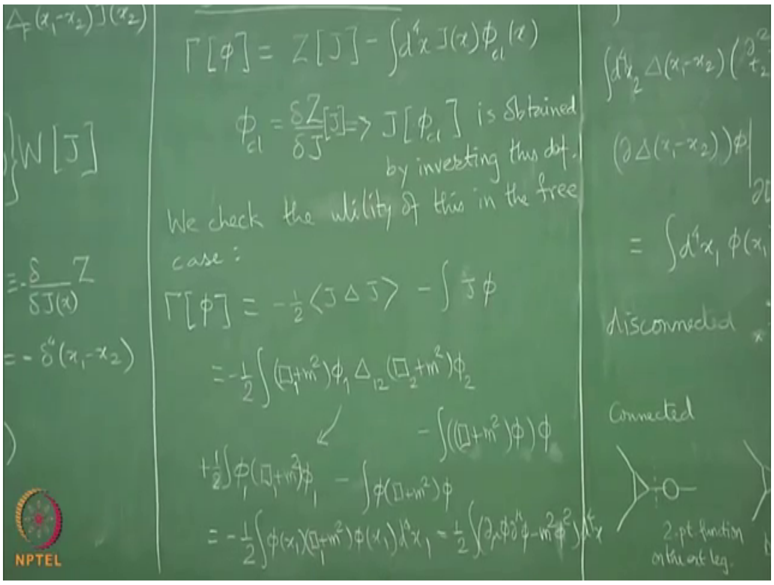

And therefore, we are left with minus a half of this integral. But, now I transfer one derivative not both to get my usual form with a plus sign, but that would change the relative sign, $\frac{1}{2}\int\left(\partial_{\mu}\phi\,\partial^{\mu}\phi-m^2\phi^2\right)d^4x$, but which is exactly my classical action. So, modulo some slight of hand I may have done with minus signs this is the basic concept involved ok. So, this is the idea of effective action and we can actually carry out the manipulation sensor that for free field theory such a abstract defined process actually gives you back your original action.

So, this is the grand definition with all these legal text attach to it, because without that it does not make sense lot of people end up writing Hamiltonian as function of velocities. So, you have to remember that whenever you would say $H=p\dot{q}-L$ well how does it work well it works, because you have to say $\dot{q}$ is evaluated at p's and this $\dot{q}$ by inverting in the definition $p=\frac{dL}{d\dot{q}}$.

So, this as to be transferred to read q dot s function of p in it has to be inverted to give $\dot{q}$ s function of p and then substituted here, that is when it becomes Hamiltonian, until then it is not Hamiltonian function ok. So, by inverting when you put it back you get the domain of dependence of $\boldsymbol{\Gamma}$ correctly, domain of dependence of $\boldsymbol{\Gamma}$ is not J. So, you have to replace J by it is dependence on $\boldsymbol{\phi}_{cl}$, then you recover everything right.