

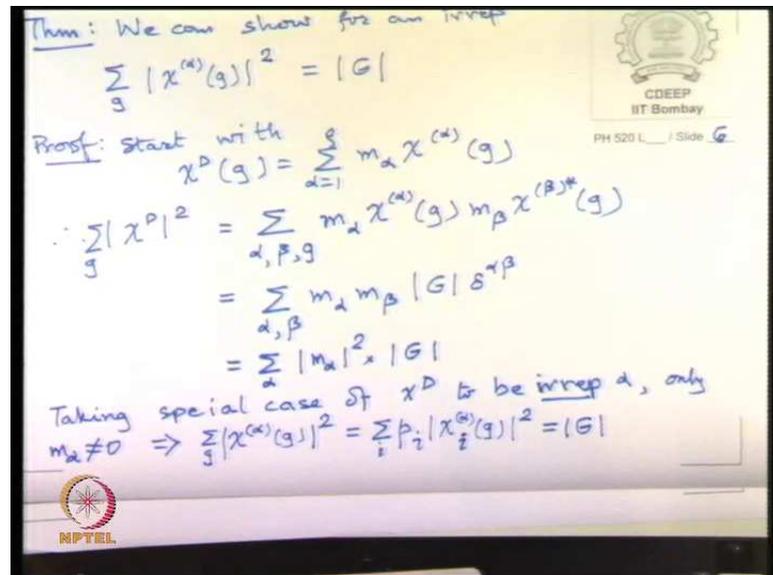
Theory of Group for Physics Applications
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Lecture - 22
Orthogonality for Characters - II

So, there is the other relation one can derive from this is that we can show the theorem

that for an irreducible representation, $\sum_g |\chi^{(\alpha)}(g)|^2 = |G|$.

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Now, proof. We had this relation earlier that $\chi^{(D)}(g) = \sum_{\alpha=1}^p m_{\alpha} \chi^{(\alpha)}(g)$. So, when we want to do mod square, we have to write this sum and we have to write square star of this sum multiplying it we just put different summation increases in α and β in the two. So, we will get $\sum_g |\chi^D|^2 = \sum_{\alpha, \beta, g} m_{\alpha} \chi^{(\alpha)}(g) m_{\beta} \chi^{(\beta)*}(g)$. But now we can exhaust the g sum by just taking focusing on the $\chi^{\alpha} \chi^{(\beta)*}$ that produces for us $\delta^{\alpha\beta}$.

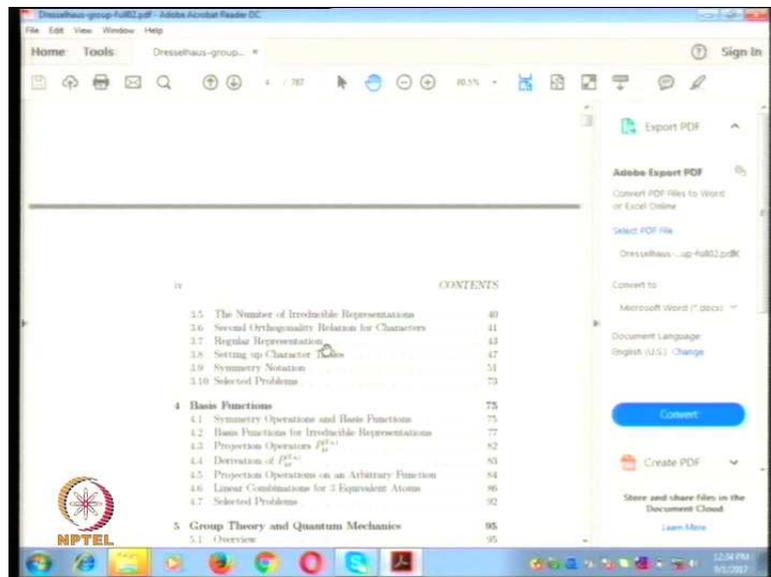
So, that collapse is this sum to just being some over $|m_{\alpha}|^2$ and then there is constant factor which is order of the group, but now you specialize χ^D to any favorite representation you have. Then be irrep we basically get that because then there is only

one m_{α} which is nonzero. So, that implies this, $\sum_g |\chi^{(\alpha)}(g)|^2 = |G|$.

So, those are all very interesting restrictions because eventually you want to know all these characters for any representation because you can analyze the small vibrational modes of a molecule by its symmetry group, knowing the symmetry group we can classify the vibrations in to different conjugacy classes and you want to know the characters. So, this puts restrictions on what kind of character table you can have ok. So, I think this is how much we are going to do in terms of formal stuff.

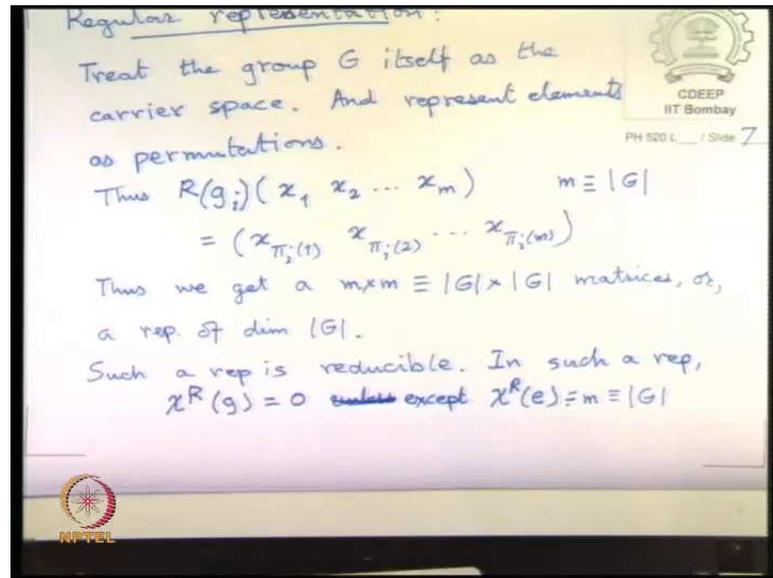
The next thing is to introduce what is called a regular representation because that is an important construct and for this I wanted to refer to this text book of a Dresselhaus.

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So, the idea of a regular representation is that you treat the group element themselves as the carrier space.

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And represent elements as permutations depending on how left action of that element permutes the rest of the group elements. So, this is the action of group one itself left action and we know that left action will re-shuffle the order of the elements. So, it is essentially a permutation and anywhere we know any finite group is subgroup of the permutation group. So, we are not surprised that it is permutation.

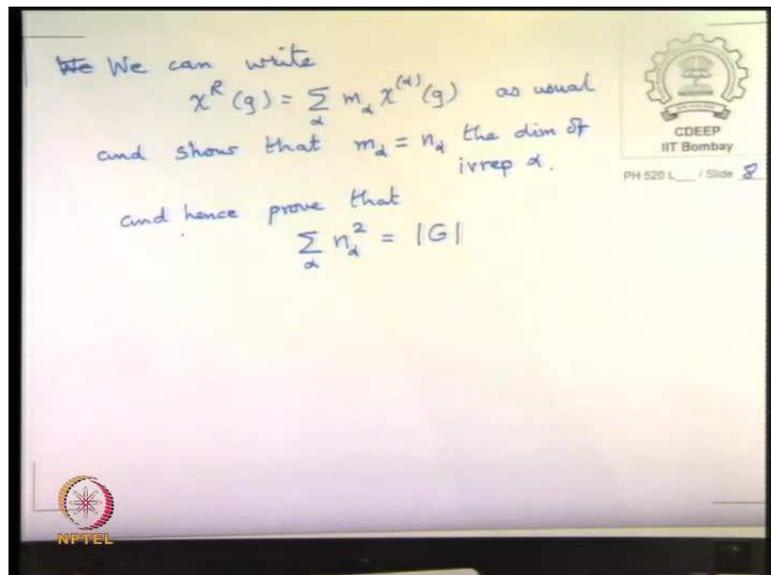
So, we represent them as permutations and. So, we make the rule that g which is the action of some element g_i on, right. So, we treat this has essentially components. So, representation R , is the matrix representation of g_i acting on components x_1, x_2, \dots, x_m where actually m is the order of the group. So, there are as many components as there are group elements. So, essentially once you treat G itself as the carrier space it is m or $|G|$ dimensional vector space and the x_i are the components in it. So, we defined this operation as been equal to $(x_{\pi_i(1)}^x, x_{\pi_i(2)}^x, \dots, x_{\pi_i(m)}^x)$. So, we permute the elements, we permute the components such that component number one goes to whatever is the fate of group element g_1 was under left action of g_i that is roughly what we are saying.

And what happens in such a representation is that thus we get $m \times m$ well which is same as size by $|G| \times |G|$ represent matrices or a representation of dimension G , we do not write the $|G| \times |G|$ right and it is a matrix of representation of dimension G means the corresponding matrices are $|G| \times |G|$. So, this representation is somewhat a plotted, it is highly reducible.

In particular you will find that the character of every single element is 0 in such rep except identity. So, we write this $\chi^R(g) = 0$ except $\chi^R(e) = m$, the identity element is always be represented by the identity matrix in linear algebra. So, that cannot be the trace can be 0 and it has to exactly tell you the dimension of the representation, but the dimension is itself G .

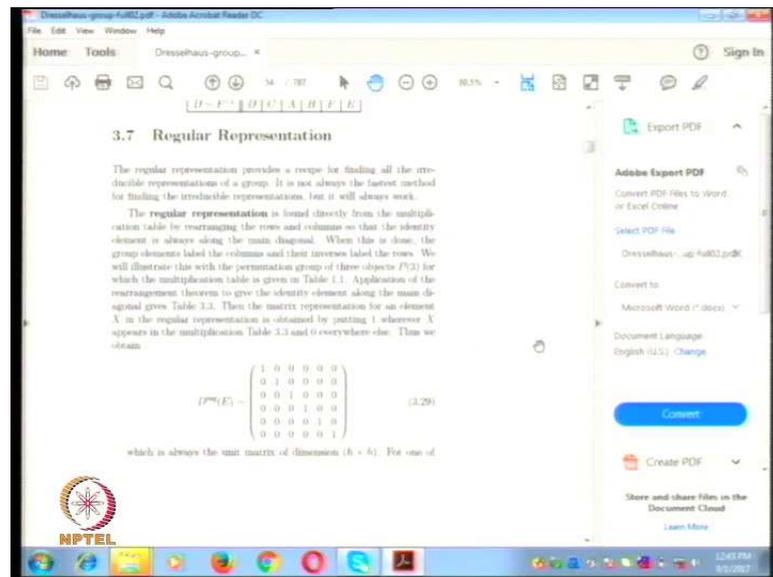
Now, this represent this can be also used to prove an important result that in this reducible representation any particular irrep is contained exactly as many times as the dimension of that representation. Actually I should have gone on to that it is, I hope it is not becoming all to abstract because this book as a nice example, ok. So, let me do the following I say this and then we do the example and postpone the proof.

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So, we can write $\chi^R(g) = \sum_{\alpha} m_{\alpha} \chi^{(\alpha)}(g)$ as before and find that $m_{\alpha} = n_{\alpha}$ the dimension of irrep α and hence we can prove that $\sum_{\alpha} n_{\alpha}^2 = |G|$.

(Refer Slide Time: 14:25)



Now, this book as a nice proposal how to find the regular representation; to find it is usually little cumbersome because technically you have to start with you have to fill out this table where you have to check you take a particular element g start acting with it on all the other elements and then see where its end each off the elements and then you have to fill out this matrix.

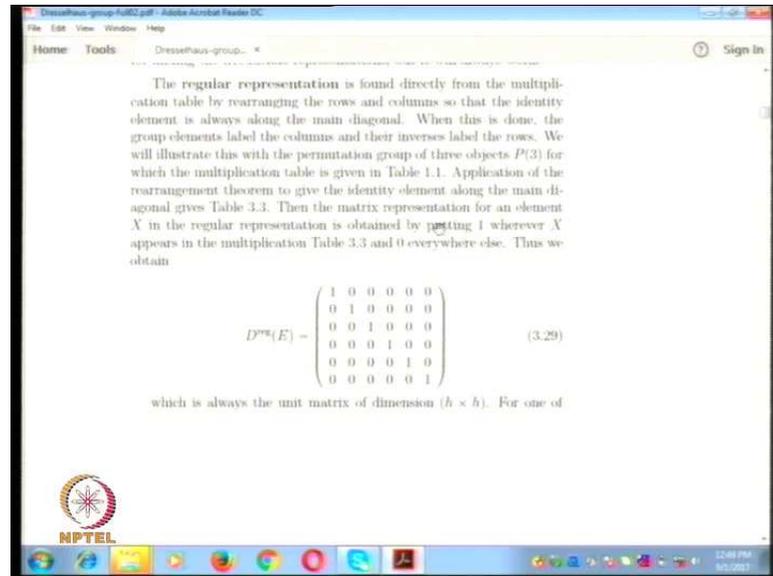
So, this book suggests a recipe for finding the regular representation. Regular representation is found directly from multiplication table by re-arranging the rows and columns. So, that the identity element is always along the diagonal, ok. So, you know that you do this write out multiplication table identity can occur wherever the inverse occurs, but now you are re arrange the tables such that the rows and columns so that identity comes exactly along the diagonal.

When this is done the group elements label the columns and their inverses will able the rows and we will illustrate this with permutation group of 3 objects which is S_3 , for which multiplication table is given in 1.1. Application of the re-arrangement theorem to give identity element a along the main diagonal gives table 3.3. So, this is the table 3.3. Here, this is just the equilateral triangle its rotation and it flips and E is identity, A and B are the 120 and 240 rotations.

So, C D E are the flipped rotations where you would change the signs of either one top row or left column. So, these are one conjugacy class and these are one conjugacy class

and this is E is conjugacy class by itself. So, first of all they are pointing out that this is the multiplication table. So, they have re-arranged the multiplication table, so that E is along the identity right that along the diagonal.

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Then the statement says after you do this re-arrangement what you have to do is then the matrix representation for an element X in the regular representation is obtained by putting one wherever X appears in this multiplication table.

So, if you want say A, the basic rotation first then you put 1 here and you put 1 here, you put 1 here, 1 here, 1 here and that is all, ok. So, that will be the matrix for representing A and so on. So, for any given element like to produce C you put 1 here, you put 1 here, you put 1 here wherever you C is the C. First thing you can see write away is that the character of any arbitrary group element will be 0 because you never get it on the diagonal, only the identity element will be on the diagonal. So, they are writing for example, for A in regular representation it will look like this 1 and 1 there and like we have identified the rows and columns; so by construction only the $D^{\text{reg}}(E)$ has nonzero trace.

(Refer Slide Time: 18:30)

44 CHAPTER 3. CHARACTER OF A REPRESENTATION

the other elements in the regular representation we obtain

$$D^{\text{reg}}(A) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad (3.30)$$

and so on. By construction, only $D^{\text{reg}}(E)$ has a non-zero trace!

We now show that the regular representation is indeed a representation. This means that the regular representation obeys the multiplication table (either Table 1.1 or 3.3). Let us for example show

$$D^{\text{reg}}(BC) = D^{\text{reg}}(B)D^{\text{reg}}(C). \quad (3.31)$$

It is customary to denote the matrix elements of the regular representation directly from the definition $D^{\text{reg}}(X)_{A_i^t, A_j}$, where A_i^t labels the rows and A_j labels the columns using the notation

$$D^{\text{reg}}(X)_{A_i^t, A_j} = \begin{cases} 1 & \text{if } A_i^t A_j = X \\ 0 & \text{otherwise.} \end{cases} \quad (3.32)$$

Now, we show that the regular representation is indeed that trying to check that is not too much of a surprise one can check that. I wanted to get to the detailed example.

So, they are actually going through the proof of the fact that this actually is the homomorphism and that it will work as a representation. So, then the theorem that regular representation contains each irreducible representation a number of times equal to the dimensionality of the representation, this theorem I stated. But what I wanted to show was this recipe of how to generate the regular representation for a particular group.

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$$\begin{cases} 0 & \text{otherwise} \end{cases}$$

Therefore D^{reg} is, in fact, a representation of the group A_1, \dots, A_k , completing the proof.

The following theorem allows us to find all the irreducible representations from the regular representation.

Theorem: The regular representation contains each irreducible representation a number of times equal to the dimensionality of the representation. (For the group $P(3)$, this theorem says that D^{reg} contains $D^{(1,1)}$ once, $D^{(2)}$ once and $D^{(3)}$ twice so that the regular representation of $P(3)$ would be of dimensionality 6.)

Proof: Since D^{reg} is a reducible representation, we can write for the characters (see Eq. 3.16)

$$\chi^{\text{reg}}(C_k) = \sum_{\Gamma} a_{\Gamma} \chi^{(\Gamma)}(C_k) \quad (3.38)$$

where \sum_{Γ} is the sum over the irreducible representations and the a_{Γ} coefficients have been shown to be unique (Eq. 3.20) and given by

$$a_{\Gamma} = \frac{1}{h} \sum_{\Gamma'} N_{\Gamma'} \left[\chi^{(\Gamma')} (C_k) \right]^* \chi^{\text{reg}}(C_k). \quad (3.39)$$

We note that $N_E = 1$ for the identity element, which is in a class by itself. But by construction $\chi^{\text{reg}}(C_k) = 0$ unless $C_k = E$ in which case $\chi^{\text{reg}}(E) = h$. Therefore $a_{\Gamma} = \chi^{(\Gamma)}(E) = \ell_{\Gamma}$, where $\chi^{(\Gamma)}$ is the trace of an ℓ_{Γ} dimensional unit matrix, thereby completing the proof.

So, using this you can use the trick of rearranging the table until you get everything on the diagonal that re-arranges of course, the important thing to remember is that it is no longer the multiplication table it is a rearrange table, where you can see the first entries here the entry is E A B C F D whereas, here is it E A B C and then D F. So, the rows and columns are not now labeled by exactly same things, rows are labeled by inverses and columns are labeled by the elements, but that puts E along the diagonal.

So, I think we will stop here. Today we have proved quite a lot of things, and using these results but we will continue next time.