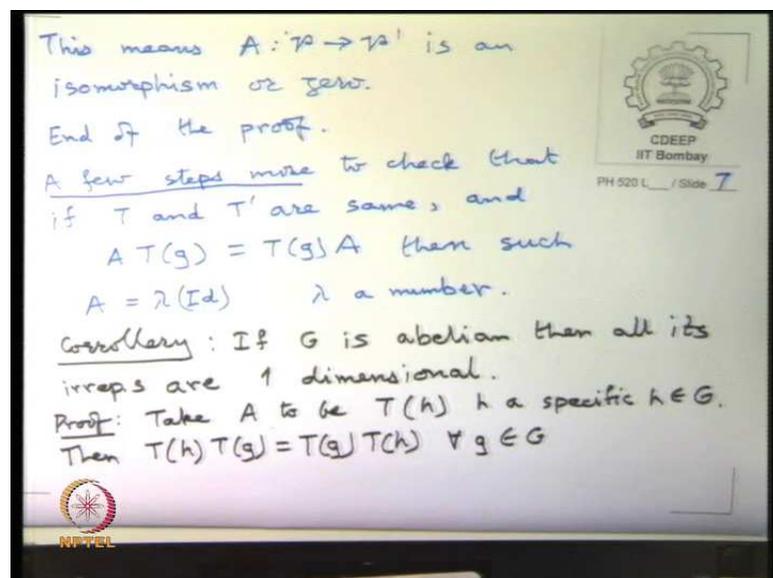


Theory of Group for Physics Applications
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Lecture – 20
Schur's Lemma & Orthogonality Theorem – II

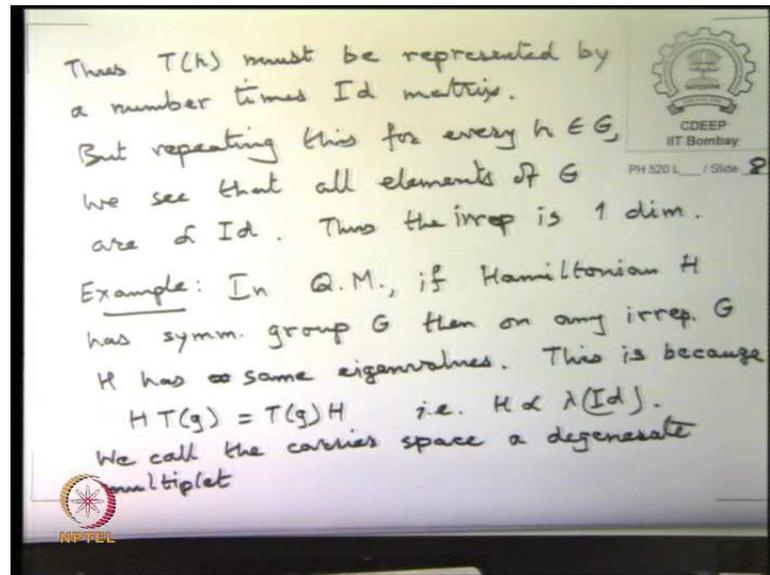
Now, one of the immediate consequences of this is that if you have an Abelian group then all its irreducible representations will have to be one-dimensional.

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This can be seen from this preceding statement because take a specific element h and take its representation because the group is Abelian $T(h)$ will commute with any $T(g)$ for all g ; that means, $T(g)$ has to be proportional to identity up to a constant. But I can play this game with every single element h of the group G .

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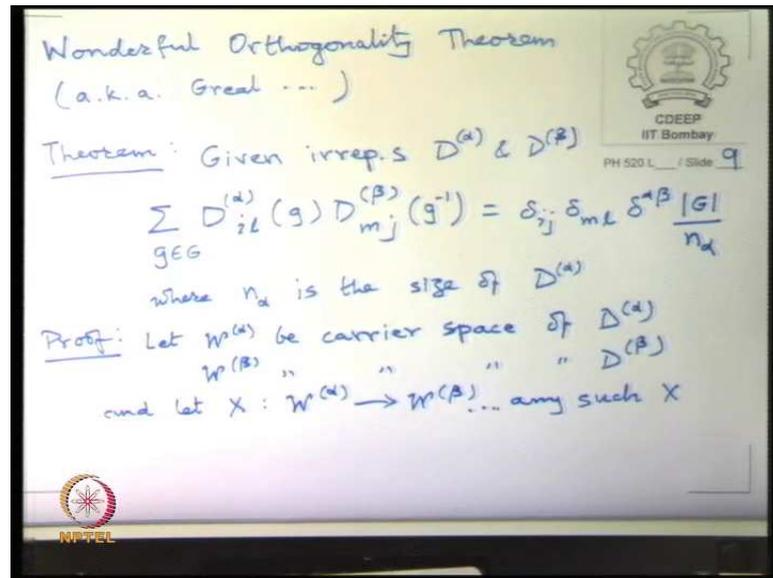
Repeating this for every $h \in G$, we see that all elements of G are proportional to identity; which is then the identity matrix whatever size it is $n \times n$ is irrelevant, it is essentially the numbers that are representing those elements.

The irrep is 1 dimensional. So, that is one nice theorem that immediately follows. In case of physics we also know that there are symmetries of the Hamiltonian, in quantum mechanics if Hamiltonian H has symmetric group G , then the representation for on any representation of irrep of G , H has common Eigen values. This is because $HT(g)=T(g)H$ because H has to commute with elements of G , G is symmetry of H .

So H has to commute with T elements. So, this means that $H \propto$ (Identity) over that whole multiplet. So, this is we usually call degenerate multiplet. So those are sort of the immediate consequences of Schur's lemma ok. So much for Schur's lemma, we next wanted to prove the Great orthogonality theorem. Actually the proof is not very long and although it is also technical it is somewhat less of abstraction it is the constructive proof like the proof we had about finding a unitary representation.

So, let me see if I can attempt this because we do want to finish of the formal parts. So, this proof of GOT or WOT is quite nice and it is similar in nature to that existence of finite dimensional unitary representations. So, let me call this the Wonderful Orthogonality Theorem also known as Great orthogonality theorem.

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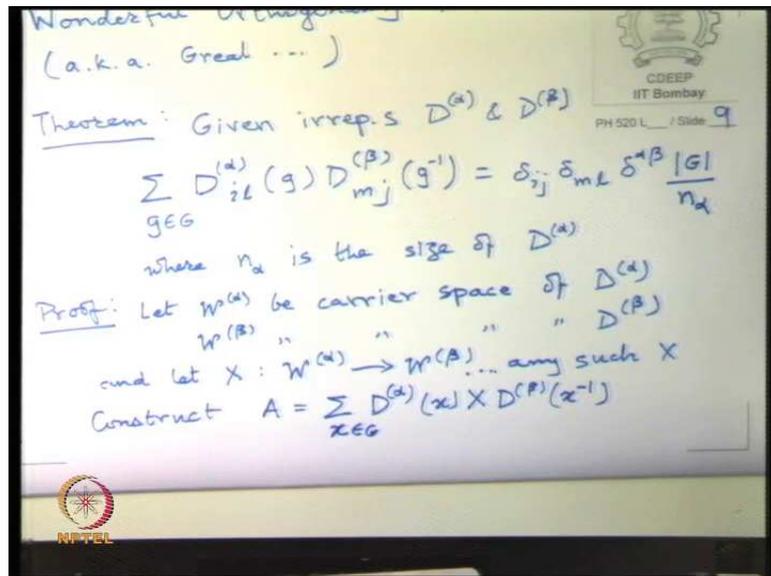
Dresselhaus book the one that I said I have given link today. It ascribes this to a mathematician called Van Vleck. So, I do not know you can call it what you like some people just call it the orthogonality theorem, then it becomes taught I guess ok. So, in here we want to show that we want to show is result of this type given irreps $D^{(\alpha)}$ and $D^{(\beta)}$. Now watch carefully what we are writing, $D^{(\alpha)}_{il}(g)D^{(\beta)}_{mj}(g^{-1}) = \delta_{ij} \delta_{ml} \delta^{\alpha\beta} |G|/n_\alpha$. So, you take matrices from representation $D^{(\alpha)}$ and on the right you multiply by matrices from representation $D^{(\beta)}$ some other representation and then sum over all g .

Note that here the sizes of D do not matter because I have kept in fact, all the i, l, m, j are not summed over. So, I take a particular representation and I take another representation, I pick any one element il from one of them and mj from the other, but now I sum over all the group elements. So, the summation is only over g . The claim is that this will result in a bunch of Kronecker deltas, δ_{ij} , so, all $g \in G$ where n_α is the size of the representation and $|G|$ is order of G .

So, this is the orthogonality theorem that if you take representation matrices from two different representations you are going to get 0 effectively. If you sum over and you pick some specific elements and you sum over all g you will just get 0, unless $\delta_{ij} = \delta_{ml} \delta^{\alpha\beta}$. So, there is a δ_{ij} unless the representations are the same and in that case you are forced to choose $i=j$.

So, $i=j$ and $m=l$ only then I will get a non-zero answer and that answer will be equal to the order of G divided by the size of the representation. So, it is some kind of an orthogonality relation only if various things coincide then the vector magnitude of the vector turns out to be this $|G|/n$, otherwise it is all those things. So, this essentially treats these elements of the representation to be vectors of size G , the order of G because this summation runs over g means that if you think of this as an inner product then we the number of code components the number of components is equal to the order of G ok.

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So, to prove this one takes a clever construction. So we take any arbitrary $X : (\) \rightarrow (\)$

and then construct $A = \sum_{x \in G} D^{(\alpha)}(x) X D^{(\beta)}(x^{-1})$. So, we have so far not used X for elements of g , but we will just call it X , this sum is over g and since X is this $(\) \rightarrow (\)$ kind of map it can act on size matrices.

So, if this is a rectangular matrix which links on this side to representation this side to representation, but now we sum over all g and construct such an A . Now what we will use do is cleverly use Schur's lemma using this as the A that commutes with and .

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and let $X: W \rightarrow W^1 \dots$ along some

Construct $A = \sum_{x \in G} D^{(\alpha)}(x) X D^{(\beta)}(x^{-1})$

Consider

$$D^{(\alpha)}(g)A = \sum_{x \in G} D^{(\alpha)}(g)D^{(\alpha)}(x) X D^{(\beta)}(x^{-1})$$

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Now

$$D^{(\alpha)}(g)D^{(\alpha)}(x) = D^{(\alpha)}(gx)$$

$$D^{(\beta)}(x^{-1})D^{(\beta)}(g^{-1}) = D^{(\beta)}(x^{-1}g^{-1}) = D^{(\beta)}((gx)^{-1})$$

Thus

$$\begin{aligned} \text{RHS} &= \sum_{gx} D^{(\alpha)}(gx) X D^{(\beta)}((gx)^{-1}) D^{(\beta)}(g) \\ &= A D^{(\beta)}(g) \end{aligned}$$

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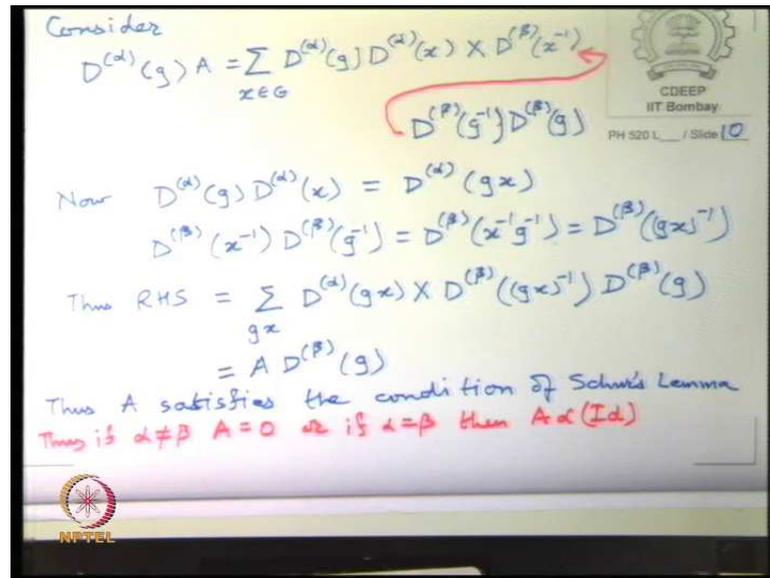
So now, consider having such an A , I take this $D^{(\alpha)}(g)$ and bring it under these write inside the summation and that is why we had kept this summation index x and now just look at the expression for A which I am reproducing here now.

But now and I have $D^{(\alpha)}(x^{-1})$. Now I multiply the right hand side by $D^{(\alpha)}(g^{-1})D^{(\alpha)}(g)$ this is identity right it is just taking gg^{-1} product and I apply to this side. So, g is a constant it is some specific element where x is what is being summed over. So, if I hit by $D^{(\alpha)}(g)$ on the left I can take this inside this summation, but additionally from the right hand side I hit by $D^{(\alpha)}(g^{-1})D^{(\alpha)}(g)$ where g is the same.

Now, you can see what has happened. So, now, $D^{(\alpha)}(g)D^{(\alpha)}(x)$ as D is a representation it has to convert it into $D^{(\alpha)}(gx)$ and $D^{(\beta)}(x^{-1})D^{(\beta)}(g^{-1})$ has to produce $D^{(\beta)}(x^{-1}g^{-1})$ which is same as $(D^{(\beta)}(gx))^{-1}$ right and now we can think of the summation to be over gx and we are left with this trailing $D^{(\beta)}(g)$.

But this is nothing, but our good old A is just that we have shifted the summation by multiplication by g and I think this is where they invoke this so called rearrangement theorem. It is essentially like having rearranged all the g elements in another order. So, this is same as $AD^{(\beta)}(g)$. So you see what has been done, $D^{(\alpha)}(g)A$ has been converted into $AD^{(\beta)}(g)$. So, we can apply Schur's lemma to this A .

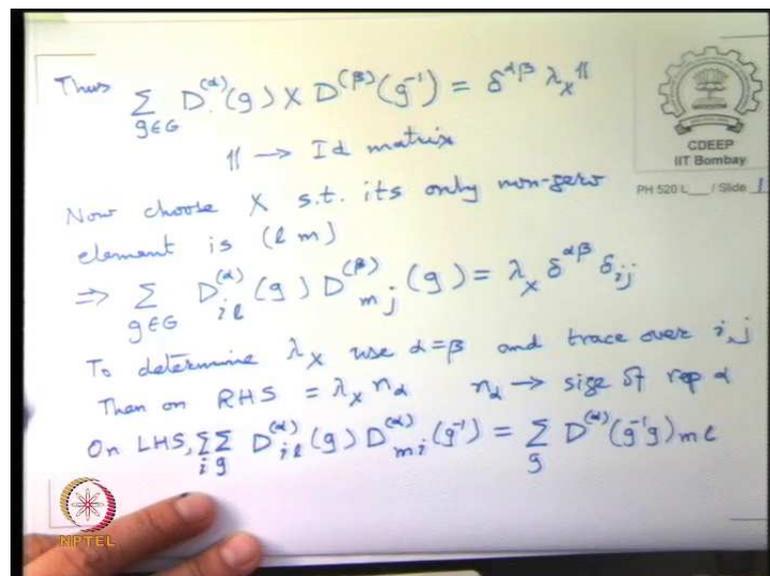
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And I suspect that this is why it is called lemma because it was been used in this theorem, although by itself it looks quite a powerful theorem.

So, now by Schur's lemma A would be 0 unless $\alpha = \beta$, then A would be proportional to identity.

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So, we can write this statement as this has to be firstly proportional to $\delta_{\alpha\beta}$ because otherwise it will be 0; if $\alpha \neq \beta$ it is to be 0 and if it is non-zero then it has to be

proportional to some I (identity matrix) and this may depend on the choice of X , what X is we will decide what this is, other than it is identity.

And most often I write this it is called black board style or something this with size of the matrix. So, it is like $D^{(g)}$ or something like that. So, it is $D^{(g)}$, anyway you understand what this means. Now what we do is that we choose X ; such that it is it is only non-zero element is (lm) . So, firstly we have already established $D^{(g)} = I$. So, all these are now square matrices and in that representation in that size matrix in X set everything to 0 except one particular element (lm) .

So, that will mean that I can now drop this X and just replace. So, I write out the matrix multiplication explicitly, but this as only non-zero at l and m . So, this implies that sum over all $g \in G$, now $D^{(g)}$ and I take some i , but X allows me to have non-zero l element and then $D^{(g)}$, there well this is also going to be $D^{(g)}$, but $D^{(g)}_{mj}$ would then be equal to this undetermined x and because this is identity matrix; it means that it has to be it has to set $i=j$.

Finally we need to determine this x . So, to do that firstly, of course we use the fact that $D^{(g)} = I$ and sum over trace over i, j . Then on the right hand side I simply get $x \cdot n$. So, I basically get summation over this Kronecker delta whose size is the size of the representation n .

So, n is the number of diagonal elements in the identity matrix that is equal to size of the representation. Whereas, on the left hand side we get $\sum_g D^{(g)}_{il}$ and we have agreed now to take the same representation. So, now I am setting $i=j$. So, there is summation over this i as well but this is same as; so so far I wrote this out as these are just entries of the matrices, but once I am doing a sum of these numbers to the left of these because it is a row column it is a matrix multiplication. So, I am getting basically the $l m$ element of $g^{-1}g$.

But $g^{-1}g$ is the identity. So this is just it will be represented by identity matrix. So, it will be proportional to δ_{ml} and the summation over g will just give me size of the group G .

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$$\sum_{g \in G} D^{(\alpha)}(g) X D^{(\beta)}(g^{-1}) = \delta \cdot \lambda_X$$

$$\parallel \rightarrow \text{Id matrix}$$

Now choose X s.t. its only non-zero element is $(l m)$

$$\Rightarrow \sum_{g \in G} D^{(\alpha)}_{il}(g) D^{(\beta)}_{mj}(g^{-1}) = \lambda_X \delta^{\alpha\beta} \delta_{ij}$$

To determine λ_X use $\alpha = \beta$ and trace over i, j

Then on RHS = $\lambda_X n_\alpha$ $n_\alpha \rightarrow$ size of rep α

On LHS, $\sum_i \sum_j D^{(\alpha)}_{il}(g) D^{(\alpha)}_{mi}(g^{-1}) = \sum_g D^{(\alpha)}(g^{-1})_{mc}$

$$= \sum_g D^{(\alpha)}(e) \rightsquigarrow = \sum_g \delta_{mc} = |G| \delta_{mc}$$

What it says is sum over all the g or that is sum add you have as many terms as there are elements in g but each term is just the identity matrix. So, it is just counting the number of elements in g ; this summation has collapsed because this is just equal to if you want it

is equal to $\sum_g D^{(\alpha)}(e)$.

But you have to do this for every g you have to count this for every g . So, that produces G . So, let me put in quote mark this is what it boils down to. So, it just counts the size of G and $m=1$ otherwise this will be 0 because this is representing the identity matrix. So, we got this on this side and that ends the theorem because now we put together these left hand side and right hand side.

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our choice X s.t. its element is $(l m)$
 $\chi^{(\alpha)}(g) \chi^{(\beta)}(g^{-1}) = \lambda_x \delta^{\alpha\beta} \delta_{ij}$

Thus $\delta_{me} |G| = \lambda_x n_a$
 or $\lambda_x = \delta_{me} \frac{|G|}{n_a}$

$\Rightarrow \sum_{g \in G} D_{ik}^{(\alpha)}(g) D_{mj}^{(\beta)}(g^{-1}) = \delta^{\alpha\beta} \delta_{ij} \delta_{me} \frac{|G|}{n_a}$

$\sum_{g \in G} D_{ik}^{(\alpha)}(g) D_{jm}^{(\beta)*}(g) = \frac{|G|}{n_a} \delta^{\alpha\beta} \delta_{ij} \delta_{me}$
 in unitary representation

You can see it right; equal to $\chi_x n$ or $\chi_x = \frac{|G|}{n} \delta_{ij}$; so, putting it back and that is the G.O.T. whatever you want to call it, g^{-1} . But we quickly can improve on it a little bit by observing that we will always be using unitary representations. So, $g^{-1} = g^*$. So, for unitary reps this becomes equal to; so, then it becomes just summation over g except that I have to change the order in from mj because $U^\dagger = U^{-1}$. So, I will use unitary rep and use it in this form. That is why I made the mistake of not putting g^{-1} there.

So, then it actually becomes much nicer just summation over g , you do not have to refer to the g^{-1} s; sum over all g . This is the orthogonality theorem. So, this will be used to prove things such as the number of irreducible representations is the same as the number of independent conjugacy classes. You can imagine the conjugacy classes in a group is infinite. There are only so many conjugacy classes and now we will be able to check that the number of irreducible representations can be more than that it is exactly the same as that. So, that list of irreps stops, it is also finite and that is how character tables are usually represented there just row and columns are just irrep versus classes ok. So, we will stop here today.