Multi-Criteria Decision Making and Applications
Prof. Raghu Nandan Sengupta
Industrial Engineering and Management Department
Indian Institute of Technology, Kanpur
Week 02
Lecture 09

A very good morning, good afternoon and good evening to all the participants and the students for this course multi criteria decision making, which is under the MOOC series and my good name is Raghunandan Sengupta from the IME department at IIT Kanpur in India. As you know we have already completed 8 lectures that means we are in the second week and today will be in the ninth lecture, which is the second but last one for the second week. As we were considering in the several lectures for the second week, we have covered definitions some concepts of utility and we will consider that utility as I told you would be a little bit bigger part, we will consider the subsequent examples for utility and the different concepts of safety first principle later on and all these things. What would be the coverage under this set of lecture, which is the ninth one. We will continue discussing utility theory with examples, consider what is expected value of utility, then different concepts of lotteries and then going to rational choice properties of utility functions and so on and so forth. So, the slides which is being covered here coverage is basically the plan whatever you cover that would be exactly mentioned, when you access the different type of slides which will be put up accordingly.

And obviously the other parts of risk aversion, marginal utility and all these things will come up not today, later classes. Now, consider this example and this is a match which is being played by anything, can be cricket, football, kabaddi, hockey anything and based on the wins or the losses of the draws, points are assigned to the teams. And let us consider the team x and y are there and all of the teams all the set of teams who are there playing from the tournament have played the same set of number of matches. So, for our example we are taking only two, it can be expanded also.

So, team x has wins of 40 number, draws 20 number, losses 10. So, if you count them the total value which is important to be noted it comes out to be 70. So, if I consider team number y, which is there in that same tournament same type of game it has basically won 45, drawn 5 and lost 20. So, also if I consider the total number is 70; it can be any other number, but important point is 7 for each. Now, consider that you see for yourself that there is one set of criteria based on which the ranking is to be done. Consider that, as I will highlight, as case 1 and another one is the second set of criteria based on criteria based on which ranking would be done is case 2.

For case 1, consider the points are given like this, which I am highlighting in red, which is, if the team wins it secures two points a draw one point and a loss 0 point. Similarly, if I consider for case 2, the corresponding points for winning draw and a loss are given as 5, 1, and 0. So, if I want to find out the ranking of x and y based on win, loss and draw for case 1 point system and then for case 2 point system, let us consider it accordingly. Now, in case 1, so if I find out, so let me do it for case 1. So, the corresponding values are given in the last, so I will just do the calculations. So, it is for team 1, it is 40 under case 1, it is 40 into 2 plus 20 into 1 plus 10 into 0. So, obviously 0 point is for all the losses. So, if I consider the score is 80 total for win, 20 for draw, 0 being for a loss. Total score is 100 which is for A, under case 1. If I consider B under case 1, so the scoring is given by $45 \times 2 + 5 \times 1 + 20 \times 0$. It is 90 + 5 plus 0, which is 95. So, if I consider this issue for case 1 for team x and y, so my, it should be this is y and this is x for both for case 1.

So, if I consider as mentioned here, team A secures 100 which I have calculated, team B secures 95 which we have calculated, which means A would be ranked better than B. So, A if A and B are the contenders for first and second position. A becomes the winner B the second position holder. Now, consider the case when we are considering both team A and team B or x and y whatever based on case 2 outcomes. So, I will use for team x case 2. The point system is 40×5 because 5 is for win then 20 into $1 + 10 \times 0$. So, this becomes 200 + 20 which is 240 and if I consider the corresponding value for y the points, I will write it here, the next slide would give you the summary. So, it will be $45 \times 5 + 5 \times 1 + 20 \times 0$. So, 225 + 5 = 235. So, 235 is the point secured for y under scheme 2 case 2 and if I consider case 2 for x team the first one the point comes up to 240. So, now you see initially it was given that the points secured were 195. Now, it is 240. Let me double check the calculations. 225 + 5 is, sorry my mistake. So, this would be 225 plus 5 is 230. So, this is 230. So, in the initial case if you see, A was 100 B was 95. So, A was higher in the ranking. If I consider case 2, A is now 230 while B is 240. So, A is 230 and B is basically 240. Did I do any calculation mistake let me check. This is basically coming out to be 40 was fine this was 40 was 200 my mistake. 200 + 20 was 220. 230 and 220. So, this is right. I just miscalculated, my apologies. So, for case 1, A is 100, B is 95. A is ranked higher. This is important to note, while in case 2, A is 200 and B is 230 which means now B is ranked higher than A: just the reverse, which means depending on the scoring system even with the same number of outcomes the result may be different. Now, according to simple concept of utility on a general nomenclature we should have basically the concept of expected value of utility which is given by this. So, expected value of utility. So, many of the examples you will see, this would be denoted by E[U(W)]. Expected value of utility is given by the utility functions multiplied by the probability and in this case what we will have is basically, you will have the utility, multiplied by the ratio of the number of outcomes which supports N(W). So, consider that W is the wealth. So, wealth can be W_1, W_2, W_3, so on and so forth. So, the corresponding values would be given by N which is the number of outcomes for wealth 1; outcome divided by the total summation of all the outcomes based on the values of W it can take. So, W is the wealth as mentioned, U(W) is the utility function which is a function of the wealth which we will consider later on probability is basically Pr is the probability of the utility outcome based on the numbers and N(W) is the number outcomes for a particular value utility and obviously $\sum N$ is basically the whole set of total outcomes happening for all the different wealth values which can occur. Now let us consider this concept in further details.

So, what we mean by risky choice and lotteries. So, these concepts of risky choice or alternatives or terms and lotteries whereby for simplicity for discussion we denote a simple lottery by the case where there are outcomes with different probabilities and probabilities are denoted by $p_1,,,,,,p_n$ such that sum of all the probabilities is 1 and obviously all the probabilities ≥ 0 and definitely probability ≤ 1. So, p here, pi's are the corresponding probabilities for each outcome. On the other hand consider this is a compound lottery which is denoted by the corresponding L_1 to L_k. So, what we mean is that for each outcome for simple lottery they would can be expressed as multiple outputs for each arm for the simple lottery. So, this if you denote as L_1 to L_k and n and k are different, L_1 and N_k are the outcomes for the corresponding compound lottery, such that their corresponding probabilities are given by α_j's. So, j, I denote from 1,,,,,k and i, I denote from 1,,,,,n and this each arm is basically given by the corresponding values of the simple lottery and we will discuss that with an example. So, compound lottery can be deduced to a simple one, a simple lottery can be made into compound one. Now, we will consider that in details by adding an empirical unrestricted assumption to the rational choice we can represent it as one which maximizes a real valued utility function. So, we will consider that in such a way that the ranking would be based on the concept of utility functions alone.

Obviously, they can be other methods also, but we will consider the concept of utility function. It will be invariant for any strictly increasing transformation, it will be called an ordinal ranking system. So, any transformation of the utility function and if the ranking system is invariant based on increasing transformation, the ranking would be termed accordingly and we will consider problems here or the ideas here. While cardinal properties would not be preserved in the sense the numerical values associated with the alternatives in x and the magnitude of indifference of the utility functions between them alternatives may change based on the cardinal properties which we will consider again as the ideas. So, consider the lottery example which we have seen.

Suppose we have a compound lottery and here what I want to emphasize. So, the compound one has three arms or three so called outputs donated by the vectors (1 0 0), (0 1 0) and (0 0 1) with the corresponding probabilities given which was alphas as half one-fourth and one-fourth. So, if you check the sum of $\frac{1}{2} + \frac{1}{4} + \frac{1}{4}$ comes out to be 1. So, (1 0 0) corresponds to $\frac{1}{2}$, (0 1 0) to $\frac{1}{4}$, (0 0 1) to $\frac{1}{4}$. Then the corresponding simple lottery would be given by $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$. So, it is just not the simple transformation of the probabilities here what we had into the corresponding simple lottery. So, what is the step of calculation? The step of calculations would be corresponding like this. So, when I mean by $\frac{1}{2}$ it means the $\frac{1}{2}$ which we have for the simple lottery it is basically the corresponding multiplication of the first arm for the compound lottery with its corresponding probability which is $1 \times \frac{1}{2}$, then the value is $0 \times \frac{1}{4}$, third value is $0 \times \frac{1}{4}$. So, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$ are the probabilities α's and (1 0 0) are the outcomes for the first arm. So, the value comes out to be $\frac{1}{2}$. So, I will mark and this is the $\frac{1}{2}$ which you have as a first term for the simple lottery.

If I consider the corresponding second arm I am using arm in order to make things simple for us to understand is the second output for the compound lottery with the same corresponding α values of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$ and the arm values being (0 1 0). So, 0, I am using a different color to differentiate that $0 \times \frac{1}{2} + 1 \times \frac{1}{4} + 0$ into one-fourth will give me $\frac{1}{4}$ which is basically the second value which you see for the simple lottery. And similarly if I have the corresponding third output for the compound lottery which is (0 0 1). So, (0 0 1) multiplied with its corresponding values of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$, gives us this $0 \times \frac{1}{4} + 0 \times \frac{1}{4} + 1 \times \frac{1}{4}$ gives me $\frac{1}{4}$. So, the values of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$, basically constitutes the corresponding idea that we have that simple lottery which is the transformation of the compound lottery.
Also one can visualize a different compound lottery. Now you are thinking that which means the compound lottery is fixed there is only one. It is not that. We can have a different compound lottery and the corresponding compound lottery is structured differently. Now the structure which I mean is that in the first compound lottery there were three outputs or three arms and in the second compound lottery there are two arms with the corresponding outputs and the probabilities, but that should all lead us to the simple lotteries again. So, the simple lottery if you remember, I am here highlighting it with this yellow color which was $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$. So, let us see whether you are able to get with the different compound lottery.

That the compound lottery I will use the same coloring scheme like red one was for the first arm then green was for the second arm for the first case where there were three so called arms for the compound lottery and the color blue was used for the third arm I will use red and green here. So, we can visualize the corresponding arm which was (1 0 0) now is $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$, with probability as $\frac{1}{2}$ and $\frac{1}{2}$ correspondingly and similarly you will basically have the corresponding values as denoted. So, the simple lottery in this case would be $\frac{1}{2} \times \frac{1}{2}$ which gives me $\frac{1}{2}$; first term; second value would be $\frac{1}{4} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{2}$, so, these values are going together, in the first case these values are going together. So, if it is $\frac{1}{2} \times \frac{1}{2}$ which is the probabilities α_1 and α_2 + $\frac{1}{2} \times \frac{1}{4}$. So, that will give me the value of one-fourth which is the second arm and if I consider the third output combination I am using the color yellow in order to differentiate again the values are $\frac{1}{4}$, $\frac{1}{4}$ with probabilities $\frac{1}{2}$ and $\frac{1}{2}$. So, $\frac{1}{4} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{2}$ gives me $\frac{1}{4}$. So, this is the third value. So, if you see the corresponding simple lottery in both the cases are same, but the corresponding values for the compound lotteries are different and here is what I will show that with the diagram.

So, for the first case of the first compound lottery I did mention there were three arms. So, I will highlight as I was doing with the colors. So, this is the first arm (1 0 0) with the corresponding value of $\frac{1}{2}$ if you remember the probabilities α_1 and α_2. So, this, we will denote by α_1 for the first case. Similarly, if I take the second output and use the mark green (0 1 0) with the probability $\frac{1}{4}$ this is basically α_2 and these were basically values where if I consider $L_{1,1}$ comma 1, I am denoting for the first compound lottery.

So, this I should use red color as I was using. This will be green which is $L_{2,1}$, second arm for the first lottery and if we use continue using the same concept of blue color for the third one this will be $L_{3,1}$ third arm for the first lottery values of $\frac{1}{4}$ which is α_3. So, when I multiply the corresponding values accordingly. So, $1 \times \frac{1}{2} + 0$ into $\frac{1}{4} + 0 \times \frac{1}{4}$ the value comes out to be $\frac{1}{2}$ which is for the case of the simple lottery. Here it is. When I multiply $0 \times \frac{1}{2} + 1 \times \frac{1}{4} + 0 \times \frac{1}{4}$, I have the second part of the simple lottery $\frac{1}{4}$. The calculations we have already given and if I multiply consider the case of the third arm for the first lottery $0 \times \frac{1}{4} + 0 \times \frac{1}{4}$, with $1 \times \frac{1}{4}$ is $\frac{1}{4}$. So, this is basically the value which I would have for the simple lottery in the first case. Now if I consider the second compound lottery which has two arms not three arms, the values I will mark again in the same corresponding even though I use the different colors in the last slide but I will use the same concept of the coloring for the first arm second arm so on and so forth. First arm being red second being green and so on and so forth. So, if I use the first so this is half which is basically α_1, the corresponding is the arm for the compound lottery and I will denote it by $L_{1,2}$ when 1 first suffix is basically the arm 2 is the compound. And if I use the green color, this is α_2, and you recollect $\alpha_1 + \alpha_2\ \alpha_3$ for the first case was 1 similarly $\alpha_1\ \alpha_2$ for the second case is also 1. This is basically $L_{2,2}$ and if I multiply the corresponding values I will get similarly the same simple lottery as it was given. So, they can be different combinations of lotteries. So, here I consider an example continuation. So, now the corresponding values of the arms for the compound lottery are given as below. The first one which I am marking in red, the second one I am marking in green and the third one I am marking in blue. So, it is (1 0 0) and the arms are $\frac{1}{4}, \frac{3}{8}, \frac{3}{8}, \frac{1}{4}, \frac{3}{8}, \frac{3}{8}$, with the corresponding probability values as given which are highlighted in yellow $\frac{1}{3}$, $\frac{1}{3}, \frac{1}{3}$. If I multiply the corresponding, this is in second example, corresponding another compound lottery is also given where the corresponding arm is $\frac{1}{2}, \frac{1}{2}$, 0 and another one is basically $\frac{1}{2}$, 0, 1 with values probabilities being $\frac{1}{2}$ and $\frac{1}{2}$. So, $\frac{1}{2}$ and $\frac{1}{2}$ here is basically as I mentioned I will just denote it as α_1, α_2, and here $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ for the first case, α_1, α_2, α_3.

Now, where are they denoted? In the first case, this is basically for the first compound lottery in the schematic diagram which you see. This is α_1 and this is $L_{1,1}$ for the first compound lottery 1 for the first term 1 for the compound lottery. So, I multiply the values as given here which can do it. Similarly, this is the second one $L_{2,1}$ and the third one is $L_{3,1}$ and if I go to the compound one, the corresponding is α_1, α_2. Again sums are same and that results in the simple lottery as given here. So, with this I will end this ninth lecture. This is the last but one lecture for the second week and continue discussing more about utility function later on. Thank you very much have a nice day.