Multi-Criteria Decision Making and Applications
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Week 01
Lecture 07

A very good morning, good afternoon, good evening to all the participants and the students. Welcome back to this Multicriteria Decision Making course on the MOOC and my good name is Raghunandan Sengupta from the IME department at IIT Kanpur in India. This is the seventh lecture under this MOOC series of MCDM class and this is the second lecture in the second week. And if you remember we are discussing very few simple definitions for Multicriteria Decision Making, the concept of goals, attributes, objectives and so on and so forth. And I also discussed the last portion of the class about the concept of production possibility frontiers, the concept of how they can give you an idea what will be the best solution. Obviously, there is no ultimate best, so it will be best set of solutions which a person or decision maker can achieve. And he or she can take any combination of that which will basically meet his or her demand depending on what is the collective output the person wants to achieve. The coverage of this seventh lecture would be, we will try to definitely, theory of choice, the simple concept of MCDM axioms. And axioms would be considered as I mentioned more theoretically for each class I do mention what are the important concept we will consider, but the application will soon come up as we proceed. Obviously, reading has to be done by the participants and the students who are taking this course.

And the last part for this seventh lecture if time flows on goes on within that half an hour, we will also consider the Condorcet paradox and the corresponding examples for that. So, coming back to the theory of choice from the western world's point of view, the historical background of rational choice theory from an individual perspective may be traced back to the work of the economist Adam Smith. Though the idea of choice is a decision and the rationality for a nation or the ruler, king, emperor can be found in the works of Kautalya or Chanakya and other works also. So, like the Arthashastra if you read Arthashastra means basically how the country or the kingdom should be run based on the economic factors. Of course, the idea of model choice dilemma has been discussed in Manusmriti and also by Plato, a Greek philosopher.

And Manusmriti as we know basically gives you different detailed text about how the society in India would be run with all different nuances being discussed there. But for us the analysis of this idea is more quantitative in nature and a little bit of qualitative perspective, but definitely not from the moral standpoint of view. Hence, we will desist venturing in the above mentioned ideas and philosophies. I mentioned them because if somebody is interested he or she can read, but I would not be, but it is not possible to discuss from the moral perspective more from the quantitative and the qualitative perspective based on some theories. Now, as I mentioned a decision maker, a person making a choice as an individual or a decision maker when making a decision amongst

different sets of alternative choices would have in front of him or her a set of possibly mutually exclusive that should be important and analyzed. Because mutually non-exclusive concept they are more interesting, but it would not be possible for us to discuss and obviously there would be many assumptions based on which we are going to solve the different ideas and problems for MCDM. So, this is an important fact, mutually exclusive set of alternatives. The alternatives would be denoted by a set capital **A** which is bold and this would have different alternatives. So, alternatives would be if I am buying a car, the car can be as I said the Volkswagen company's car or Maruti's car or Hyundai or Honda or Kia whatever it is. And the number of alternatives or choices would be different defined by i = 1 to M, M as in Mangalore or Mango.

And in many of the cases it can also be denoted by capital X. So, with capital X, X_1 to X_M and this again, as I said capital M is the total number of alternatives or choices the decision maker has only one decision maker we are considering. In order to model the decision maker choice behavior one can use two distinct approaches. One is the idea of preference theory and one is the choice theory, whereby the fundamental difference being that under the former which is preference theory the alternative choices are not observable while for the latter, they are directly observable and analyzed accordingly. So obviously, preference and choice theories based on that will try to deal, but more so from a less intense mathematical point of view quality point of view we will consider the choice are available and they can be discernible by the decision maker. It is not vague, the alternatives I am saying. Under preference theory the decision maker decisions are summarized in his or her preference relationship or list considering the fact that the decision maker makes rational decisions and rationality assumes would hold. We will consider human beings when making the decisions are rational, but in actuality in real sense they are not. Obviously, bringing that into the discussion would not be possible, but there are ways and means in very simple problems where if the choices are irrational or incommensurate based on the logic, there are ways how to rectify that very simple ways. Rational choices are complete. So, they cover all the axioms accordingly and may be defined to mean the process of determining what options are available and then choosing the most preferred one according to some consistent criteria or set of criteria which the decision maker has in front of him or her. For the car example I am repeating these examples time and again, but please bear with me. For the car example it can be the power of the car. What is the horsepower? What is safety features? What is the color of the car? What is the mileage it gives? What is the maintenance cost? What is the luggage space? How many people can sit in the car? So, they can be different ways of trying to analyze and these are the criteria. So, what I mentioned I will come to the concept of the criteria later on.

And number of alternatives which I mentioned as capital M number of criteria which we will define by capital N, N as in Nagpur. M and N need not be same. $M > N$, $M = N$ and $M < N$ also. So, depending on that we will formulate the problem, but the idea would be the same. The property of independence need not hold. So, if you consider the concept of mutually exclusive concept.

So, in general decisions need not be the alternatives need not be independent of each

other. So, they are dependent, but obviously to make our life simple we consider that mutual independence holds. Furthermore, if one looks at the alternatives from the point of view of probability theory , then the alternatives need not be exhaustive. If I want to consider the concept of probability theory where the sum of the probability is 1. So, when I am trying to buy a car it is need not that I am considering the whole set of alternatives which are actually available in the universe.

So, my set considering my budget amount of money what is the what are the sets of car available in India, what is the safety features I may only keep my view point fixed to this 5 or 6 companies which I mentioned again. Consider a Volkswagen, consider a Kia, consider Maruti, consider Honda, consider Hyundai, Tata, but actually if I consider the whole set infinite set it can be BMW, Rolls Royce anything, but those are not to be considered because considering my budget, my safety features requirement and all this or non-availability those alternatives have need not be considered. So, that is why the alternatives need not be exhaustive. So, what are the axioms for MCDM they will be coming up in the problems as we proceed, but we would not be going into the proof for any for this whole course or for the other courses which you have taught you would not go as usual not into the proof only the statements and the implications as we proceed with the examples.

Axiom 1 states the concept of preferences. So, the preferences would be, if I give alternatives A 1 and A 2. or here as in the example they are denoted by script *P* and if this bold **P** has set of alternatives denoted by P_1, P_2. So, P_1, P_2 are only two examples they can be P_3, P_4 and so on and so forth. Then when I am trying to basically follow the concept of preferences under Axiom1, either $P_1 < P_2$ in the level of ranking and this '<' sign is actually we will use in the problem solving simply as '<' and not '≤'. '≤', we will come to that later on. Here we will use '=' or here we will use a > sign, but in the concept of MCDM they are denoted by the '<~' and not '=' and '>'. Any one of these three has to be true. So, the case there is no ambiguity. So, P1 is either < or = or >.

So, car one is either less preferred than car two or the preference for car one and car two are same or car one is better than car two. So, they cannot be any dilemma here. The second Axiom is highlights the idea of natural transitivity which means if there are three cars, and this would obviously be come into the picture different ideas will come up and I will come to that within few minutes. So, if there are three cars consider them as P_1, P_2 and P_3 and if P_1 is less preferred than P_2 as stated here, P_2 is less preferred than P_3 then it is always true that P_1 would be less preferred than P_3. So, they would be a so called ranking system, but in many of the examples you will see that this ranking system may be not followed in the sense that based on different sets of criteria consider if you have a Volkswagen car is preferred than Maruti, Maruti I am denoting by M, Volkswagen by VW and in other case that we have.

So, this is the first one P_1 is better than P_2 where P_1 is Volkswagen P_2 is the Maruti and the second case Maruti is better than Honda. Say for example, this need not be true for the other person this is the decision maker deciding. But it may so turn out, that actually we will think that as per the logic, from 1 and from 2 which is which I have written here

Volkswagen would be better than Honda, but it may not be true it may come out that as the decision maker is making the decision based on different sets of criteria this may come out to be true which is not allowed as per Axiom 2 which highlights the idea of natural transitivity, that means you can rank them. First one is that there is no ambiguity between two decisions and the second one is that if there are more than two decisions to be made, they can definitely be ranked. But one thing should be remembered, the difference of the ranking between, if I consider here, in (P_1, P_2), (P_2, P_3) and (P_1, P_3) here which I am now circling I should use a different color so that this. So, if I consider
$P_1 \leq P_2$ and $P_2 \leq P_3$ and P_1 is obviously based on that we have $P_1 \leq P_3$. The difference is if I draw it on a very simple case on the real line. Consider I am basically ranking them.

So, this is the line I am using a bold marker in order to make things easy and if I consider and on the right hand side is better. So, if I consider P_1 here, and if this is P_1, this is P_2 and this is P_3. So, this is the ranking which I have, but it should be remembered even though in the diagram it may not be very evident, the differences which we will have between P_1 and P_2 which I am not noting by a red line and the differences which we will have between P_2 and P_3 which I am denoting by the black line, these differences need not be same. That means, the difference of the liking disliking between P_1 and P_2 and the difference of liking disliking between P_2 and P_3 need not be the same and that would come out to be true in example.
Say for example, you are ranking the students based on marks consider relative absolute whatever it is. Then the difference between the first and the second student and the difference between the second and the third student need not be the same. Obviously, it would not be the same and we will consider that later on. Axiom 3 says it underscores the objectivity of conditional principles. So, which means that given a set of P values, so P_1, P_2, P_3.

So, let me highlight that. So, we use P_1, P_2, P_3 as usual in Axiom 2, which we consider they belong to script P and $P_1 < P_2$. So, if I consider and ask a third alternative P_3 which I am circling here, and if I consider a convex combination of (P_1, P_3) and (P_2, P_3) separately based on some α. Then we can find out some value of α where the ranking system remains. That means, I am adding on my left hand side if you are looking at the camera and you are looking at towards me. So, obviously P_2 which is higher is on my right hand side, P_1 is on my left hand side.
So, if I add P_3 a second alternative, like I am investing in stocks or I am investing some combination of bonds or if I want to buy a house with some amenities there; garage being there a gated community with a swimming pool, with a garden, the walking space, security whatever. So, if I consider then if I combine those extra benefits as P_3 or an alternative P_3 then some combination based on α of P_1 and P_3 would always be less than P_2 and P_3 because P_2 is better than P_1. So, that combination would always remain true and this alpha would be the important point which we will consider later on which underscores the objectivity of conditional principle under Axiom 3. Axiom 4 gives that if I have the alternatives given P_1, P_2, P_3. So, P_1, P_2, P_3 are considered an example they can be more than 3 also and the ranking system is $P_1 \leq$ to $P_2 \leq P_3$ or $< P_2$ and $P_2 < P_3$.

Then we can find out some combination of α and β and α and β in the earlier Axiom 3 was also between 0 and 1 as it was mentioned in the slide. I did not mention that, but $0<\alpha<1$. Similarly, $0<\alpha, \beta<1$, such that the combination which I will have which can result would be like this. So, some alpha and beta can be done where the combination of (P_1, P_3) the extremes which I take P_1 from my left hand side then P_2 then P_3. So, combination of P_1 and P_3 would be less than the combination considering that would always be less. So, if I have a line, consider the line I denote by this real line I am only doing in the one dimensional thing because numbers would be easy for me to convey easy for us to understand also.

And if I have P_1, P_2, P_3 I denote this as P_1, P_2, P_3, then for some α and some β. So, consider some $0<\alpha<1$, the combination which will have of $(\alpha \times P_1)$, means I am trying to do a convex combination of P_1 and P_3 would always be on the left side that combination would be always on the left side of P_2, for some specific values. And also we can have for some concept of beta. So, some consider some other value it can definitely be on the right hand side. So, if I use β as given as violet color. So, this combination of $[\beta P_1 + (1-\beta)P_3]$, would result in that combination to be on to the right hand side of P_2. So, they would be some ranking systems as that I can have P_2 as so called middle point not the midpoint middle point at somewhere some distance. And by the way this is obviously true that the distance between (P_1, P_2) and (P_2,P_3) is not the same. So, I can have the combination either on the left hand side of P_2 or on to the right hand side of P_2 depending on what α and β I make and this would be evident when we solve simple problems in the areas of portfolio theory and decision making. Continuing the axioms one should remember that this rational axioms help one to analyze the consequences of the preference for the choice behavior and hence two basic assumptions about preference relations also remain true and they are as follows.

As I said the completeness would mean that if we have, and if you remember in the first axiom, I should use the other color, say for example, green in two decisions x is either better than y or y is better than x or both are true in the sense that they are indifferent in say if x is say for example, if I write this: if $2 < 3$ which is fine, but if I write the equation say for example, $2 = 2$. So, obviously we will see that later on in simple constraints which are given I can write, that is $2 > 2$ or $2 < 2$. So, both would basically add up to the case that they are equality and this would be true in the case when we are solving as I said the constraints where equality constraints are there with they can be termed as greater than or less than sign and consider in the problem and simply the boundary of the feasible space can be drawn. What I mean by the feasible space boundary we will see that in the problems.

In the concept of transitivity which came up from Axiom 2 this would also be true in the sense if x is better than y, y is better than z then obviously x is better than z.

So, obviously if I draw the line. So, I will use the coloring consider the line as x is the line. So, you have better on to the right hand side. So, this is x is better than y and y is better than z. So, obviously x is better than z. Again I am mentioning time and again, so, forgive me if there is repetition. The differences of x, y and y, z the breadth or the distance

between them based on the concept that we are able to utilize quantitative length to differentiate that would not be equal, they can be equal, but need not be equal.

Of course, if $\geq$is rational then greater than is both irreflexive and transitive and $\neq$or = here, $\cong$, is reflexive, transitive and symmetric and both greater than and almost equal to as transitive and we will see that in the examples as we proceed. For the second approach which is the choice theory. So, there were two things preference and choice. For the second approach, which is the choice theory, the choice behavior is represented by a choice structure. So, they would be budget set and they would be choice rule.

Choice rule means the rule based on which you are taking the decision which would be a selection of the budget set what is the total amount of money which I have or what is the budget or the requirements which you make and what are the choices which can be fulfilled by that budget. So, obviously if my budget is to buy a car is 15 lakhs, but my choice set need not in that case, I am not able to consider a car like, say for example a very high ended car which is very costly and that would not be under the choice set. If my budget set is there I want to buy a house maximum about 50 lakhs inclusive of all registration, down payment everything then obviously I would not go into buying a high end bungalow. The choice rule would say that this set of alternatives would be out of my reach based on my budget set. Hence the idea is basically, given a budget, I have some set of feasible choices which can be made and based on that an important assumption for this choice theory is the weak axiom of revealed preferences.

In some way this concept of weak axiom of characteristics influences the choices and is akin to that concept of rationality assumption in preference theory. In the preference theory we are considering that human beings are rational and this concept when we have the budget and the choices and if weak axiom of revealed preferences, so, the preferences would be given forth not in the strict sense in the weak sense, so, that would have an implication that rationality assumptions under preference would hold. So, this is the first one. So, preference theory as I will mention and use a dark colour. So, was the first one and this choice theory if you remember was the second idea. So, they are different, but in some case based on the third bullet point which I am highlighting and with the star they would mean that in the weak sense, revealed preference would basically give me the idea of rationality under the first which is the preference theory. At this point if one is curious to ask the question whether there is any equivalence between the idea of choice theory which is the weak axiom characteristics which I am highlighting and the preference theory which was rationality assumption then the answer is yes there is I am not going to the proof. Without going to the mathematics one can summarize the idea as follows.

If a deficient maker has a rational preference ordering then does it mean the question is that is mean that his or her decision from choice theory perspective satisfy weak axiom characteristics? So, we are we have basically the preference theory which we will denote by PT and another is choice theory CT. So, if in the first start point if you see, this one, first one I will mark with 'I', has a rational preference ordering which, does it mean the decision from choice theory perspective satisfy weak axioms? The answer is yes.

So, this way implication would be yes. On the other hand if the decision maker from choice theory perspective on the right hand side if it satisfies the weak axioms characteristics, does it imply rational preference theory is maintained? The answer is maybe, which means this is, if I going from preference theory to choice theory it is a sure one way street while coming back may not or be possible. Say for example, I am considering two localities. So, to give this example two localities consider in Delhi. So, consider them I am just denoting as GK1 area and GK2. So, one can move from GK1 to GK2? Yes, always, 24 hours a day, but moving from GK2 to GK1 may only be possible in the morning from 9 o'clock to, say for example, evening 5o'clock or the time after 5o'clock in the evening till morning 8:59 it is not possible.

So, consider the movement are not immediate in the sense both the implications always are not applied here. That means one way the brown street goes yes always 24 hour $\times$ 7 while the coming back blue one which I have said here may not be possible. The choice structure of the budget sets and choices satisfies the weak axiom of revealed preference where given x and y are part of B1 which is the budget, then for some combinations of x and y in B2 the second budget set, it would be true that both of them are individually belongs to the budget set B2. So, B1, B2 $\in$ B (total budget). This simply means that if x is chosen even with y being there for selection then they cannot be any budget set containing both an x and y where y would be preferred over x. So, I have taken x, y is there, but the other way round, that I am overwhelming the decision of y in place of x may not be true.

So, x and y are taken as two decisions. So, this seventh lecture has been a little bit more intense with concepts. We will try to use those in the examples as we proceed and also this seventh lecture has been little bit more than 30 minutes, but I am sure it will fulfill the criteria and if you read it you will definitely understand the ideas and then implications later on, once we solve very simple problems. Have a nice day and thank you very much. Thank you. .