

Multi-Criteria Decision Making and Applications
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Lecture 60

Now, welcome back my dear friends, a very good morning, good afternoon and good evening to all of you for whichever part of the world you are listening to this lecture and whichever time frame it is or time of the day when you are listening. As you know this is the last class for multi criteria decision making the course under NPTEL MOOC, it has been spread over 12 weeks with 60 lectures and we are in the 60th lecture. We have covered many different topics in multi criteria decision making first part main chunk was multi objective, then utility theory, then nonparametric methods, then the last one would be an example. This would only give you a feel that how the problems would be solved. If you recap we have discussed about linear programming, integer programming, quadratic programming, then consider multi objective cases of variance of them, utility theory and all its differentiates very simple concept I did not go into the proofs. Then you have considered multivariate utility theory again as I keep saying very briefly, then nonparametric methods, electre method, epsilon electre, topsis, vikor, AHP, DAA the last two lectures were in decision tree diagram.

And only decision tree diagram if you remember I did mention that the diagrams was quite detailed. So, I just covered it using the word document not the PPT slide obviously everything will be shared like for all lectures all the slides have been shared each week you get it. Now, is to based on that and the lectures which we here you take the assignment which is ten questions for each week and then the end time. So, this is the last lecture of multi credit decision making and we would be considered a problem here and different without going to the details I discuss the problem.

Details means not going to the detail solutions. This example in multi objective decision making and the background is about portfolio optimization. Even though if we remember we have discussed problems of choosing a house and apartment in different cities depending on the price, the safety features, the location near to the bus stop or transportation facilities being nearby. How close are the school, how close are the hospitals, how close are the market or the markets, whether it is a gated community, whether your relatives are there, whether your friends are there, what is the price, how many bedroom apartment is there. Then we consider the concept of idea of trying to choose an education institute based on rigor, the placement facilities, the internship facilities, the cost, how far it is from the home and all these things.

Then we considered the idea of trying to buy a car depending on safety features, power, cost, maintenance, resale value, luggage space, the style factor, all these things. There were other different problems also for production problems were there, the problems of decision tree and this is of trying to build up a factory to sell a product or build up a factory to sell a moped. Here the problem is on the financial perspective. So, the problem is that you have a, you want to formulate a portfolio, portfolio basket of different assets

and we have initially a wealth W to invest in n number of assets. What is needed to be found out is basically the total percentage of the total amount which you can invest in this n number of assets.

The problem could also be looked at from the point of view of what is the total quantum of number of assets you are going to buy or what is the total value of the money you are going to invest in trying to buy those number of assets. The idea remains the same, why? I will just very briefly mention it. Consider you have total amount is say for example, A . You have total amount of A , you have total amount and the assets are marked 1 to n . So, the numbers you buy in these assets are N_1 to N_n , the prices are S_1 to S_n which means W_1 till W_n which is the proportions would be found out accordingly. Consider A amount has been totally invested in all these assets starting from 1 to n , no amount of money is left which means W_1 which I will mention here would be equal to $n_1 \times S_1 \sum N_i S_i = A$.

Similarly, you can find out W_1, \dots, W_N . So, W_i 's are the weights of the proportions of total wealth invested in the i th asset. V is the variance covariance matrix for the returns for the N asset, what is return? If you remember we have found out the concept of total return and rate of return. We are taking the rate of return and we discuss, we did discuss about the ideas that utility theory that if the quadratic utility function was applicable and the rate of return would be normal and vice versa. M is the vector of returns of N assets, N is the number of scripts or assets we have only considered the scripts.

Return* or r^* , risk star or σ^{2*} are the respective predefined values of the returns set by the investor from the portfolio and the second value being predefined value of the risk set by the investor from the portfolio. So, technically r^* , I will denote I will use a different color consider blue. So, I will use r_p^* is this value of return and this value of risk would be given by $\sigma^2 p^*$ where p is not probability p is the portfolio. So, the objective can be either minimize the risk or maximize the return in that case it has single objective functions. Now, I combine them when I combine them you minimize the risk maximize the return subject to continuations being return for that portfolio has to be greater than r^* because that is what the investor wants and the second constraint being risk being $\leq \sigma^2 p^*$.

Obviously, that has to be return has to be more than distributed value and risk has to be less than the stipulated value \leq or \geq . Obviously, $\sum \text{weights} = 1$, weights are between 0 and 1. Here is one important point to remember people who do finance remember there is the concept of short selling in which case weights < 0 , also which you are not considering here. So, here to go into the detail of the nomenclature N total number of stocks 1 to n_t . The time period, why time period is important I will come to that. W_i 's are the weights. So, in the vector format it is w_1, \dots, w_n each of the stocks can be invested within a particular bound the minimum bound is w_i mean maximum is w_i max. So, if the minimum, the maximum are between 0 and 1, obviously, all of the values of w_i would be between 0 and 1.

If I say the second stock cannot have an investment of more than 30%, in that case, w_2 max value is 20% or it is 0.2. Similarly, if I say the fifth stock cannot have anything less

than minimum value is required is 50%, in that case the value of the minimum w_i would be for that particular stock would be 0.5. r_i is the random variable represent the rate of return for the i th stock which I have said and r_p^* is return threshold required by the investor. r_{it} is the realization of random variable r_i during time period t σ is the standard deviation of the i th stock ρ_{ij} is the correlation $\sigma^2 p^*$ is the threshold value which I have mentioned SD is the standard deviation A is the matrix of returns ρ_i returns and B is basically the minimum portfolio which is to be required. I am not going into the details of the these, but I will just highlight the important points. Remember that in this multi objective problems it can be that you want to buy some stock at each equal interval of time and sell also stocks. So, obviously, buying and selling will entail some cost because you have to help get the help of the broker consider that. So, obviously, the some percentage amount has to be spent both for buying and selling those values which I consider is basically $W +$ being an N vector of stock purchased and $W-, W+$ and $W-$ are vector. Plus means purchased, minus being sold and W_0 again the vector is basically the initial stock value Q, which is basically denoted as V and when we are doing the actual understanding of the general formulation the problem is basically the variance covariance matrix.

Why variance covariance matrix the principle diagonal are the variances and off diagonal element are the variance covariance values for i and j . Now the first problem I will only discuss the problem again I said not going to the solution technique only discuss the results. The first problem is I will use two different colors for the minimization red maximization green. For the maximization problem it is maximizing the return, return is already $r_i \times w_i$ which I have already mentioned r_i is the return for the i th stock w_i is the weight corresponding to the i th stock $-p$, transpose p , this is the because you have to multiply row vectors and column vectors and considering that compatibility of row and column multiplication is there in matrix that is why it is known as p^T or p transpose, $p^T \times w+$, which is $w+$ is which are being purchased. So, obviously, there is a price for that and $w-$ are the those are sold that is also price for that price means for the broker.

Similarly, so you want to maximize this and from the minimization one you want to minimize the quadratic function. Quadratic function is basically if you if you go into it is $\sum \sum w_i w_j, \sum_i j$, w_i and w_j are the weights $\sum_i j$ is the covariance values for the i th and j . What are the constraints? So, I will let me start from the simplest one some of the weights I should use a different color. So, let me use the violet color some of the weights is one we know the weights are between minimum and maximum we have assumed the fourth one if I am going from the last assumption of the constraint this is like a rate flow equation. w basically which you started with $w+$ which you bought $w-$, you sold with a $+$ and $-$ sign corresponding to the inflow and outflow this basically is the initial value. The variance covariance matrix is $\leq \sigma^2 p^*$ as I have mentioned based on the investors set risk value and $\sum_i r_i w_i$ being $> r^* p$ is also the return value set by the investor.

So, one is and the first one is basically the constraint based on the value values of the individual weights. So, here a is basically matrix based on the weights parameters w 's are the weights corresponding to the amount which you want to invest in the N number of

assets B is basically the right hand side they can be formulated accordingly. So, main being the second one the third one the fifth one and fourth and first have been added in order to bring more realistic flavor to the problem. Now, the stocks taken are BSE, DAX, Dow Jones, Hong Kong, STI Singapore for 3 year time period from 2012 to 2015, January of 2012 to 2015 January and we take windows now if you remember I mentioned the time frame. So, what why we are taking windows is basically to bootstrap means that when you are tying your laces of the boots.

So, you basically keep tying them from the lowest level and then tighten up. So, the shoe which you are or the shoes which you are wearing are really tight around your cuff muscles and your ankles. So, bootstrapping is a method of sampling which has different varieties of way of trying to handle it I am not going to the details that would fall under the ambit of statistics and simulation. So, in order to do bootstrapping I consider the total set of length of the time frame for the stocks is 3 years and I break that into windows. So, this is the total time frame I use the violet color here and each length of the window which I take I will use the blue color and there are and I keep shifting the windows I is shifted, but there is a overlap this overlap depending on the number of days I here I am marking as light blue.

So, this O is the overlap L is the length of the window and total length of time is t . So, generally the length is taken as about 1 week or 2 is maximum and the overlaps are taken to about 3 to 4 days. So, you keep repeating the overlap and based on the average value what you want to use the standard deviation of those windows you find out such number of times of the windows have been formulated and then bootstrap to find out the average value. So, as I say there is a whole bunch of studies. Here the window size is 150s signifies we have taken in 150 days we signifies 6 months seasonality in the trend as I mentioned it was 7 days or 14 days here we have taken 150 we repeat 25 number of times and merge it together in order to start the bootstrap, bootstrap numbers are taken as 500, 1000, 2000 and it can be extended.

So, when we do the rebalancing with the transaction cost. So, the first problem the risk return profile as it should be. So, you remember there was a two functions one was I am trying to maximize the return based on the selling and buying cost and one was the risk variance. So, when I plot it if you remember if you have done simple finance even portfolio analysis the graph the risk return is like this efficient frontier this is exactly what we are getting and use the yellow color. So, not to some like make the blue dots vanish.

So, this is the set of colors obviously, it should be a if you do more number of series simulation it will be a smoother curve, but considering the time frame for each simulation we took number of simulations accordingly and this is the portfolio risk and return for BSE if you remember we are taken Hong Kong, Singapore, Dow Jones accordingly. The rebalancing part based on the weights as they keep changing for whole set of data this is how the rebalancing is done for the first problem. If you see I do not have the coloring scheme here, but if you see the dark blue color slowly vanish becomes less and less as you increase the combinations of risk and return and the yellow one

which is basically would slowly start dominating as you increase the risk and return. Red also comes into picture brown is almost stable and so on and so forth. So, if you had the coloring scheme of the stocks you can easily find out which stock is being slowly removed which stock is coming into the picture depending on the combinations of the risk and return for the first problem.

So, if I have the portfolio risk and return for the rebalancing transaction cost. So, this curve which you look actually if you remember the pareto optimal frontiers the one which you have done and if you remember I had highlighted that depending on maximization I will use four different colors both by the way this is objective one O_1 is objective one, O_2 is objective two I am trying to plot this is a two dimensional one it can be done for three dimensional and so on and so forth. If both are being maximized we will try to basically move to the top right hand corner the first quadrant top both are being minimized you will move in the left bottom corner in the third quadrant. Similarly, one being maximized another being minimized the combinations of the points should point in this I should use the not the green color my apologies this we have already done. So, you can basically find it out.

Now, considering the max of return minimum risk. So, rebalancing is not there we consider the return the second one mean return being greater than r_p^* , risk $< \sigma^2_p$, sum of the weights is one and here I want the return average return of the weights is increasing return should be such it is greater than one. So, again without going to the details this is the effect consider the Hong Kong I am not considering the B S E. So, I am trying to basically utilize the model for all the in scripts, but giving a feel of the diagram. So, then return to risk again similarly like the risk return profile.

Here is the asset weights distribution it looks quite jagged, but this is what we got. So, basically this is the part where one of the stock really dominates and gets over went with the other one which is shown in and creamish yellow. Then the risk return profile considering the pareto optimal frontier is exactly this if you see this is f_1 , f_2 at the two objective functions we have considered. Now, we come to the max return over threshold value.

So, the minimum. So, all are by objective this is the minimization problem I should mark in red this is the maximum problem I should mark in green again the constraints are I will use a different color violet greater than r_p^* . So, here risk is not being brought into the picture $\leq \sigma^2_p$ is not there in the picture some of the weights is one weights are between 0 between minimum maximum and the returns considering is increasing r such is ≥ 1 . That means, if r_i s are negative. So, it will be 1- that. So, it may turn out to be on the downward here.

Here I do it for the DAX with the German again the risk return profile without the scatter plots on the lower side the general trend looks like risk return profile the minimum threshold return for the next one considering the weight distribution looks like this I am not explaining I am just showing you the results. And again the f_1 , f_2 which is the objective function for the third problem considering the pareto front is this.

Obviously, you can have different Pareto front, but I am not going to all those details. And so, with this problem we had considered three different problems with rebalancing simple problem and at the max deviation part. So, in the risk was always the same the standard deviation square in all the three problems and we used all these stocks using BSE, DAX, Hansing, Singapore, Dow Jones, German and we showed their working for a time frame of three years and how we took the overlapping time windows use bootstrap that has also shown.

The next problem we considered is basically another type of problem where we are going to consider the conditional value of risk and not variance. And here is one of the problem formulation which you have already discussed in multi objective case. The nomenclature and the variables remains the same r_{ji} is the loss return we are taking the loss return. Now, what is the loss return if you if you can recollect r_i which is basically the in the numerator we have the return we have the value of 2 day minus the value of 1 day before divided by the value of 1 day before. So, that can be positive and negative depending on the numerator value.

So, return would be the loss distribution will be the case if you take only the negative return positive return would give you the profit distribution. So, that is why the loss return distribution r_{ji} the loss distribution for the j th stock for the i th day \bar{r}_{ji} is the nominal value of the loss return σ_j , sigma value, I am not reading suffix is the vector of variances x_j 's are the weights which was w_n is the number of stocks t is the time period α , β_1 , β_2 are the corresponding values for the value at risk which is a very useful measure in finance. And β_1 , β_2 are the corresponding constraints probability value will take γ δ_L , δ_L is basically the loss distribution would be for the \bar{c} value and λ_1 , λ_2 are the weights which will assign to the \bar{c} and the r_p . If you remember r_p means the return of the portfolio if you remember in one of the multi duty case we did consider λ_1 , λ_2 and based on that we solve the problem. Considering so these this nomenclature from 11 to 19 which are shown in slide are based on the problem type solution methodology we take.

So, this is considered which is not part of the course I will just mention the name is known as robust optimization I am not going to consider that for further discussion. So, the first problem is here we minimize two objectives take weights of λ_1 to \bar{c} which is one risk and λ_2 to return. Now, why it is a - sign because we are minimizing. So, if you maximize it will be plus and minus. So, we are minimizing the risk and obviously trying to consider the return also.

The constraints are weights add up to 1 weights are excess λ values add up to 1 because they are two λ values the weights can be between minimum maximum, but here they are unbounded. The first case if I consider first case means the first constraint if I consider without the probability that means β_1 is not there and second one β_2 is not there. This means the conditional value of risk is less than some stipulated value and the second part here which I am highlighting using the red color here is basically for the case when the returns are considered. So, returns were greater than some stipulated value similarly risk were less than the stipulated value in the first problem.

Here risk is conditional value of risk. So, obviously risk has to be less than the stipulated value. So, these are the chance constraint uncertain parameters. So, which I will show, but not discuss in details in problem. So, these are the chance constraint the first and the second one with the probabilities are and the uncertain parameters are given. What are the uncertain parameter? Uncertain parameters are the returns because they are dependent on the past data.

The data is taken from the Australian, Brazil, France which is CAC, DAX, Germany, FTSE, England from Canada, HAN SENG, Hong Kong, Argentina, Nikkei, Japan, NSE India, NYSE USA and Singapore. We took it from 2000 to 2010 I am not going to the details which is total of 2840 days and the summary statistics and the Moore's rubbing concept were considered. These are the summary statistics which means mean, median, standard deviation, kurtosis, skewness, medium, minimum, maximum value for all the different indices and again the values can be calculated very simply. These are the density plots, these are the extreme value distributions depending of the loss distribution which was considered here.

This is for the DAX only. Similarly, you can have the negative distribution, loss distribution for all the other indices which I have not shown, I am just showing one sample. These are the data description and the Q-Q plots, theoretical standard normal with the empirical distribution being done and then again the same concept of shifting window. So, the windows which I am marking in red are L and the O part which was overlapping this is considered as blue which is S, total time frame is T, same concept. So, we take three window size, one day shifting, two window size, one day shifting, one window size, one day shifting. Actually there if you remember you have taken 150 days.

So, it can vary, but generally depending on the problem we would suggest for finance to be about two week window with three or four days overlapping. This is shifting given, shifts. These are the data statistics for used in model 1 and model 2, model 1 and model 2 which I have shown based on that for all the different steps and these are the 3D scatter plots. Here why 3D? Because if you are looking you have now \bar{C} value which is a risk, the bar value which is also a risk and return which I will use a different color green is positive. So, you want to increase return, decrease bar and decrease \bar{C} bar.

So, if I look at from this direction, so obviously I would be considering from this left hand side. So, consider the room where I am giving the lecture, if somebody is seeing from the left hand side. So, and the axis which is coming towards me from your side is x which is going on to the right is y and the one which is going top is z. If somebody is looking from the left hand side obviously it will be x and z being understood because y you cannot see is going to the depth from that left hand side. If I am looking from this side I will be only I would not be noticing x, I will be noticing y and z and y.

Similarly, in different direction as you look you will have the field accordingly. So, these are the \bar{C} return and bar plots and I have taken different levels of probabilities which I mentioned on the top. As I said I would not go into the details depending on different values of betas the values are given accordingly. So, I will just go through it

and they give you a very good feel that how multi objective case can be analyzed for different problems, but this problem I had solved. So, it is much easier and with confidence I can share the results here.

So, with this I will end the whole set of series of 60 lectures for multi criteria decision making. I am sure it would be helpful to for many of the students for any queries for people who are doing this course please get in touch with the TAs and for any queries if you have for further studies or any other details you want to understand you can most welcome write to me over my email and you can find my email in my home page. Thank you very much and best of luck to all of you. Thank you.