Multi-Criteria Decision Making and Applications

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Week 12

Lecture 57

Now, welcome back my dear friends and students, a very good morning, good afternoon, good evening to all of you at whichever time you are listening and hearing this lecture. The course title is multi criteria decision making which is run on the NPTEL MOOC series and my good name is Raghunandan Sengupta from the IME department at IIT Kanpur. And as you know this course is spread over 12 weeks which is 60 lectures each week we have 5 lectures each lecture being for half an hour and we are in the last week which is the 12th week. In the last lecture which we started for the 12th week we started the discussion about the data and development analysis and rather than solve problems I did discuss in details about the basic concepts of DA. So, this is the 57th lecture and the idea as you know for this whole course is multi criteria decision making, we have covered multi objective decision making details, multi attribute utility theory as I mentioned rules. Then on the multi attribute decision making we have covered electre, epsilon electre, topsis, Vikor AHP we are now into DA.

We did spend for electre, epsilon electre, Vikor, topsis, AHP a lot of time and the DA part is the last part which is left for the multi attribute or different type of non parametric methods. The coverage is continuation of data involvement analysis. So, if you remember we were discussing about decreasing return to scale the concept of input output as bundles along the X-axis, Y-axis, or Y-axis X-axis depending on the way you want to depict and the examples when it will be increasing return to scale, decreasing return to scale or constant return to scale we will be discussing once more. And the point of decreasing return to scale I did mention and the same example can be brought forward and discussed for the increasing return to scale and the constant return to scale also.

The example was basically of production the quantum and the revenues depending on the input quantity of input and I consider the quantity of input as the number of levers. So, here the increasing return to scale the diagram which we have already considered in the last class the last slide in the last class was where input was measured along the X-axis output along the Y-axis. Here we have just reversed the diagram considering input and output are being considered along the Y-axis and X-axis respectively. Here also the DMU which is inefficient can reach efficiency which is in the frontier either moving vertically up going horizontally to the left or moving in the radial direction and do remember the green one which is the radial direction in all the diagrams touches or meets the curve in such a way that at that point the angle is 90° the angle I mean this one. Even though in this diagram it may not be apparent it is 90°, but it should be 90°, and as I mentioned that the DMU inefficient can become efficient when it reaches the frontier based on the concept of input oriented model output oriented model and this what we will highlight in the further slides and the discussion for this lecture.

In the constant return to scale as the diagram shows which is very apparent again we are measuring input along the X axis output along the Y axis the DMU which is inefficient which I am just marking using the red color it can reach efficiency either going vertically up moving horizontally to the left or going in the radial direction. Radial what I am using in order to signify the way it is moving to reach the efficient frontier and do remember the angle which I am now drawing in the red color is 90°, even though it may have not been drawn accordingly, but that would be 90°. Again the concept of DMU_1 efficient in blue color it is output oriented why output oriented again I am mentioning because the inputs are fixed output is increasing. The DMU_1 efficient input oriented which is pink in color why it is input oriented because here the output is fixed input is decreasing input would not increase it will decrease remember that because you want to basically increase the efficiency. And the variable to scale can be a combination of the input output idea or more to discuss in details would be try to combine that increasing return to scale and decreased return to scale.

The ideas of increasing return to scale and decreasing return to scale I have discussed in quite details in the 56th lecture and also in the 57th lecture which is the one which is going on. So, I would not go into much detail discussion about graph here. So, input again here even though input is along the y axis output is along the x axis, but if you have the input along the x axis and output along the y axis you can similarly draw conclusions with the graphs where it is variable return to scale considering a combination of increasing and decreasing return to scale. Now comes the formulation and do remember I will very briefly tell about the formulation and move logically in this direction. Now if you remember in the output oriented model and the input oriented model I always highlighted in which direction it was moving it was either moving vertically up or horizontally left.

When it was moving vertically up and if the input was being measured along the x axis then input was fixed and if it was moving horizontally to the left and again if it input was being measured along the X-axis their outputs were fixed. So, based on that we will try to analyze how the models can be made and how the analysis can be drawn. In the output oriented model the model formulational like this I will first mention the model and as I am going along mentioning the model I will tell you the reason why it is being formulated like that. Now whenever you want to understand a DMU which or a set of DMUs, you will try to compare the efficiency and if I use the word efficiency the word which will immediately come in your mind is that how can we measure efficiency. Efficiency can be measured by taking the ratio of the total amount of output divided by the total amount of input.

So, if it is amount of money which you are investing and the output which you are getting is goods which you want to sell obviously, it would be the cost of the goods. Cost means the selling price of the goods which you are selling in the market divided by the input cost of the goods and that will give you the rate of the price increase which you have may have basically gained. Now when I consider the DMU it is basically just a system a machine. So, in that machine if I consider efficiency again the efficiency would be considered in a very simple way like a ratio of the bundle of output divided with the bundle of input divided that means, input is in the denominator and output is in the numerator. Now let us see when it is an output oriented model how we are trying to basically analyze.

Now once again let us go back to the diagram because again I will come back to the slide number 7. Now if I consider the output oriented model for the continuous return to scale I will mark it once with the different colors say for example, green this is the output oriented model as it is written in the suffix OP output oriented model. So, in the output oriented model what is fixed input is fixed. If I go back to this diagram which is increasing return to scale, but here input is along the Y-axis and output is along the X-axis the output oriented model again I am highlighting is where the input is basically fixed and output is changing. That means, in the output oriented model all your focus is on the output keeping the input fixed and similarly for the input oriented model it will be just the reverse we will revisit that idea later on.

So, in the output oriented model coming back to the slide from where we left if you see the ratio forget about the max, max part max is basically maximization. If you pay attention to the ratio in the numerator what you have is the summation of all the outputs which are there and in the denominator you have the summation of all the inputs which is basically the efficiency. So, for each DMU you want to maximize the efficiency that is why the optimization model is basically maximizing the ratio of the output bundle divided by the input bundle. Now in this case what would be the different type of constraints and the constraints are also very logical. So, when you are trying to basically maximize the efficiency of one DMU obviously you would always assume and rightly so that the efficiency of none of the other DMUs including itself would ever exceed one which means the ratio of the output with the input for all the other DMUs including that one has to be less than equal to one and this is what is given.

Now this is the first constraint I am not come to the fact what are the suffixes still now. Similarly you have the kth one for the kth DMU. Now here capital K is taken so you are considering the efficiency of the first k is equal to one then k is equal to two then it goes down to k = K. So, what is small k =1, is this one what k = K is this one and in between you have small k = K -1, which I have not written and where are they being highlighted based on the fact that k =1,…..,k = K. Now if you see the numerator and the denominator in the such thus constraints you have $[V_{j1} \times Y_{j1}]/ [U_{i1} \times X_{i1}]$, where V and Y are related to the output and U and X are related to the input that is why it is a ratio of the output bundle to the input bundle and now basically you sum them up for all the amount of outputs to inputs and what is the number of outputs you have capital N in number what is the total number of input which is capital M in number and if you see the summation in the numerator is from j = 1,…,N and summation in the denominator is basically from i = 1,…., M.

And here the second suffix after V after Y after in after U and after X all are one that means for small k = 1, similarly if I go to the last constraint again the summation in the numerator and the denominator are respectively from j =1,….,N in the numerator and i =1,…., M in the denominator and in the bundle of outputs you have the variables and the scaling factor the variables is Y scaling factor is V in the numerator and in the denominator the variable is X and the scaling factor is U and the suffix if you see is j capital K then also j capital K and similarly in the denominator i capital K i capital K. So, these are all the constraints. So, there are capital K number of constraints and what do we have in the objective function objective function is basically the ratio of the output to the input which basically gives you the efficiency of the kth DMU. Now in that case it means the optimization problem are capital K in number each for one of the DMUs. So, as it is mentioned $i = 1,\ldots,M$ $j = 1,\ldots,N$ $k = 1,\ldots,K$.

Now immediately the problem formulation would make you pause and think. So, I have been mentioning is in easy optimization problem, but this is as you see in the problem formulation this is not linear which is true. So, I want to basically bring linearity into the optimization problem and similarly ensure linearity in the constraint also. So, can it be done the answer is yes it can be done. How it can be done? If you remember I did mention that the efficiency of any DMU including the kth, 1/ kth one can never exceed one max it can be one which means if you pay attention to the objective function maximization can occur by ensuring that I keep the value in the denominator which is input fixed and basically keep increasing the numerator with the maximum possible value and this is what we are going to do.

So, if you see the output oriented model for a case where you outputs for DMU as $k = 1,..,3$ which I am marking in blue. So, capital K is 3, j which is the number of outputs is 4 which I am again marking in blue double tick and the input which is denoted by the nomenclature $i = 1,\ldots,5$ which I am marking in triple tick. So, if I consider the problem formulation for the first DMU and here I am marking the suffix in red color in order to differentiate that you want to maximize the bundle of output to input for the first DMU based on the idea that the efficiency for the first DMU which I will now use a different color first DMU ≤ 1, max can be 1. Similarly, the idea of the second DMU also being ≤ 1 is which I am marking in blue 1 blue that means in the constraint and the idea the constraint for the third DMU is also ≤ 1, is marked in violet. Similarly the idea can be replicated for DMU 2, DMU 3 and in that case the problem formulation would only remain the same only the suffix of the coloring scheme which I have put as 1 would change corresponding to 2 for the second DMU to 3 for the third DMU and this is what is happening.

If I consider the output oriented model you have again same thing $i = 1,\ldots,5$, $j = 1,\ldots,4$, $k = 1,\ldots,3$ the optimization model for the second DMU just notice the suffix 2 which is marked in red which is for the second DMU. Similarly the first tick mark which is for the constraint is for the first DMU, the second tick mark which I am using in the green color is specifically for the second DMU and the third tick mark which I am using the blue one is for the third DMU. Similarly for the third DMU problem formulation you have again the same idea $i = 1,\ldots,5$, $j = 1,\ldots,4$, $k = 1,\ldots,3$, the objective function is for the third DMU and the constraints are exactly the same only pay attention that the number 3 which is $k = 3$ is being highlighted to make the things much more apparent. Now when I come to the output oriented optimization model now all these three optimization model for DMU 1, 2, 3 were non-linear now I want to make them linear. And if you remember what I did mention few minutes back I would try to ensure the denominator is kept fixed and we try to maximize the numerator that means output and that is what I meant that the input would be kept fixed output would increase to the maximum level.

So, if I remove the denominator, but removing it does not mean that I am totally removing it from the system as such it is not being removed it is being transferred here I will use a color red that the denominator was there and for the bundle of inputs as it is an input fixed model it is being transferred here. See here the summation of the bundle of inputs is being fixed as 1 such that the maximization problem is not a simple linear problem maximization the bundle of outputs. And similarly the constraints have also been very nicely changed rather being non-linear they had been converted into linear part how if you notice one of the constraints which was I will write it down in black color. So, it was bundle of I am not using the suffix anymore because that would be apparent for each and every problem formulation which I do it is V Y and summation of U X that was ≤ 1 remember. So, in that case what we do is that it is summation of V Y summation of U X is taken on to the right hand side again brought on to the left hand side it becomes $- \sum U X \leq 0$.

So, here you see the non-linear constraints each of them can be converted into linear constraints and that what we have done. If I consider the first constraint the first constraint which I will highlight using the green color is this, this is related to the first DMU. Similarly, they would be for our second DMU third DMU fourth DMU. So, this one is for k is equal to small k is equal to 1 and the one which I am now marking in violet is for small k is equal to capital K. And the one which I have marked in red is the extra constraint which we mean brought from the objective function on to the constraints in order to ensure that is an output oriented model.

So, now you have a very simple linear programming maximization needs to be solved for each DMU and the constraints have just increased by 1 for each of the separate optimization problem. But remember that considering they were capital K number of DMUs initially all the optimization problem had capital K number of constraints. The extra one which is capital K + 1 th one would keep changing depending on what DMU or which number of DMU optimization problem you are solving. So, if it is the first one the first K number capital K number of constraints are same the capital K + 1 th one would be the denominator part corresponding to the first DMU which is being transformed and being made into a linear programming. For the second optimization problem which is related to DMU 2 the constraint capital K would all remain the same the capital K + 1 th one would basically be the denominator which is coming from the optimization problem related to the DMU number 2.

And we will continue basically changing the K th + 1 constraint accordingly based on the fact that what is the DMU we are concentrating on in order to solve the optimization problem. So, once we do that without going to the details because I have already explained and the slides would be there with you. So, you will want to maximize a linear programming and remember the same problem which we are considered $i = 1,\ldots\ldots\ldots,5$, $j = 1,\ldots,4$, $k = 1,\ldots,3$. And here the corresponding parts are related to the first, second and third. Similarly, when i for the second DMU it is for i = 1,…,5, j = 1,…,4, k = 1,..,3 this is the maximization corresponding to DMU 2 as you can find out from the suffix which is ret 2 given ret 2 means the DMU 2.

And the constraints are as usual as was the case for the first optimization problem first linear optimization problem, but only the highlighting factor is the second DMU respective the denominator which is being transformed to the object constraint part. Finally for the third DMU again i = 1,…,5, j = 1,…,4, k = 1,…,3 you maximize the output bundle for the third DMU and the constraints are based on the first, second and the third part has been highlighted using the ret 3. So, it would basically give you the idea that third DMU is being optimized. Now coming to the input oriented model, the idea on the input oriented model if you have understood the idea for the output oriented model is just the reverse. Now in the output oriented model the idea was keep the out input fixed concentrate only on the output increasing obviously, you would not decrease the output you will increase the output.

In the input oriented model the idea is keep the output fixed and keep decreasing the input. Now this is what we are doing. So, if you want to basically decrease the input what will happen is that rather than maximization you will want to basically minimize, minimize the inverse of the efficiency. Now what is what was the efficiency in the first case when it was an output oriented model there you found out the ratio of the bundle of outputs divided by the bundle of inputs. Now in this case rather than maximizing you will basically try to minimize the inverse that means, minimize the ratio of the bundle of inputs to bundle of outputs.

In the similar way in the maximization problem what we did was basically maximize the numerator keep the numerator fixed at 1. In this case input oriented model you will minimize the numerator keeping the denominator fixed because this is now an input oriented model you want to decrease the input and that is what the formulation is. You want to basically minimize as I said and here the ratio is basically bundle of inputs because you see U and X in the numerator V and Y in the denominator the summation of Uand X is basically from i = 1,…,M summation of V and Y is basically from j = 1,..,n and small k is basically the kth number of DMU. Now the constraints were in the initial case of the output oriented model were efficiency cannot be > 1 they were basically ≤ 1. Now if you are taking the inverse so obviously, the inverse of the efficiency would be ≥1.

So, if you see it here the bundle of inputs divided by bundle of outputs for each and every d Mu has been written such that they are all greater than 1. Initially it was what if I write it in the output oriented model it was I am not using the suffix I am only using V and Y and U and X. So, for the output oriented model they were all ≤ 1. Now as I find out the inverse so obviously, it has to be less than greater than equal to 1 of course, just do a multiplication you can find it out. So, also we would now have similar problem k number capital K number of DMU problem solution, minimization type all the constraints are basically of ≥ 1 and here also i = 1,…,M, j = 1,…,N, K = 1,…,k.

And here again for the same problem formulation where i = 1,…,5, j = 1,…,4, k = 1,..,3 the problem formulations are given. Only pay attention when you get the slides is for which DMU the problem formulation is being done here it is basically being done for DMU = 1 that is why the suffix 1 has been numbered accordingly to highlight the importance of which DMU optimization is being done it is being highlighted by the red color. And here also remember the constraints are all ≥ 1. Similarly in slide number 17 which you see now it is basically minimization or the bundle of inputs divided by the bundle of outputs because obviously, it has to be minimization and it is for the DMU number 2 the constraints are given as usual and they are same for DMU 1, 2, 3 being ≥1 for each of the case and here input numbers is 5 output number is 4 number of d Mu's is 3. Finally for the third DMU input oriented model you have input numbers are 5 output numbers 4 DMU 3 you try to minimize the objective function which is the ratio of input bundle by the output bundle for the third DMU and the constraints are exactly the same.

Now all this all this three problem formulations for DMU 1, 2, 3 are non-linear in nature you want to convert into linear in nature what we do is that fix the denominator. Now the denominator is not related to input it is basically related to the output because input oriented model we want to minimize minimize the input keeping the output as the fixed level. So, output is fixed. So, you want to minimize and the output factor which was there in the denominator is now no more in the objective function it has been brought into the constraint as you can see bundle of outputs = 1. So, we ensure for the kth DMU which we are trying to optimize it is equal to 1 and it is a minimization problem not the maximization as in the output oriented model.

And here if you remember in the output oriented model all the constraints were of what type it was basically ∑ V Y, ∑U X ≤ 1. Now they are the reverse. So, obviously in this case they are now I will use a different color they are ∑U X which is the bundle of input divided by ∑ V Y, which is bundle of outputs is ≥ 1. So, when I basically convert into linear format they are all ≥ 0. So, now you have all linear constraints and 1 = 1 the one which is coming from the denominator of the minimization objective function and the objective function is basically simple a minimization problem.

So, when I utilize that for DMU 1 2 and 3 the problem formulations I like this I would not repeat it because once you go to the slides it will become evident. Only remember i = 1,…,5, j = 1,…,4, DMU = 1,…,3 and here the highlighted color is basically related to the DMU. So, this is for first DMU, this is for second DMU and this is for the third DMU. So, with this I will end the second lecture for the last week which is related to the DA and before I end the lecture you all of you may be thinking that I have been talking about the formulation what is the solution. Solution we have for if you remember I mean I started the DA class I did mention that ideas on optimization of simple optimization problem of linear programming which are a prerequisite would all have been solved by you.

So, just take one of those problem solution or simplex method and try to utilize that to get the variables and solve the problems accordingly. Obviously, there is a huge branch of study in D A, but considering the course is basically in multi criteria decision making covering all the things may not be possible, but I thought I will just give you the ideas of D A in brief accordingly. Thank you very much and have a nice day. Thank you.