

Multi-Criteria Decision Making and Applications
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Week 11
Lecture 55

Welcome back my dear friends, a very good morning, good afternoon, good evening to all of you all the students at any point of time wherever you are listening and hearing the set of lectures. The title of the course as you all know is Multicriteria Decision Making and this is another NPTEL MOOC series and my good name is Raghunandan Sengupta from the IME department at IIT Kanpur in India. And as you can recollect remember the main portions which we were discussing for quite a set of weeks was basically about non parametric multicriteria decision making. I will repeat few of the things which you have already done in details. Electre method, epsilon-electre, topsis, Y-core and then we are in AHP method. We have discussed some of the AHP methods and here in the class which is the last lecture of the 11th week we will consider the remaining portions of AHP.

Few of the slides would be repetition with respect to the 54th class, please bear with me. The main course name or the sub topics or topics whatever you say is multicriteria decision making. We have covered whole lot of multi objective decision making, utility theory, some concepts of multi attribute decision making which you are doing, multi attribute utility theory also just the basics. And we are in the 55th class out of the 60 lecture series or class series.

The coverage is the continuation of analytical hierarchy process AHP and I have drawn some diagrams in details to explain that in the 54th class. Now, if you remember we were discussing about buying a car and all the three examples we have been discussing about brands of cars or the different type of colleges one would like to select and third one problem was about buying an apartment. We will stick to the buying the a car from a set of brands and if you remember for buying a car there are many different criteria to be analyzed. Review of them I will repeat this one is the horse power of the engine, the safety features, the looks of the car, the cost of the car, maintenance cost also, resale value, then the luggage space, number of passengers which you can sit. And based on that we are considering a very simple set of criteria and I will mark them accordingly here.

The first one is style we had discussed, but I will just briefly again rerun those important concepts cost and fuel economy and these are all criteria for the first level. First level means the hierarchy we are considering also along the row we have style cost and economy and as I mentioned the principle diagonal which are marking in red color is all one because we are considering style to style, cost to cost, fuel economy to economy fuel economy. And the of the diagonal element which I will now mark in green which I do that in order

to highlight the different concepts using different colors. So, cost to style cost is twice and obviously, the reciprocal would be for style. So, cost is more important for the same person decision maker if I consider cost to fuel economy the ratio is basically $4 : 1/4$.

So, again I am marking in green and the last one if I consider fuel economy to style the ratio is $1/3 : 3$. So, these values of $3 : 1/3$, $2 : 1/2$, $4 : 1/4$, all depend on the decision maker. So, obviously, if I am taking the decision to buy a car my set of points may be different from your or say for example, in the same family if two people are deciding to give some weightages about all these qualities of buying a car like say for example, husband and wife both have a say in trying to buying the car they may have different notions of how the scores are given. Now, remember one thing which I will draw in order to. So, your main decision which I will mark in a dark red box is basically the decision and under that you have the different criteria sub criteria and hierarchy.

So, I will mark the first level in green color. So, consider this is 1, this is 2, this is 3, then the next one I will mark in blue in order to differentiate for the first one there are 2, second one there is 1 and third one again there are 2 and these first hierarchy and the second hierarchy are interconnected. So, I will draw them here is the list of the first connection for the first hierarchy, second hierarchy and then the next one I will draw and finally, there is the alternatives. Alternatives you know and mark in violet consider there are 3, consider for this example. So, they are connected from the second level accordingly.

So, again this point is connected and finally, the third. So, I will mark the alternatives this is a repetition plus please bear with me this is A1 alternative 1, A2 alternative 2, A3 alternative 3 and if you remember the criteria are marked by C1, C2, C3 and if I consider the next level I will mark in blue color they are correspondingly C11, second one is C 12, third one is C 21. So, obviously there is no C 22, but still I will mark C 21, the next one under C 3 is C 31, the last one is C 32. So, when I am considering why did I draw all these things when I am considering the comparison of the first hierarchy which is basically style cost and fuel economy I am comparing the criteria which are shown in green. And obviously, I can continue doing so for the second hierarchy and continue accordingly then I will consider the alternatives with best with the criteria also.

So, this snapshot or this slide number 4 which shows only one level of comparison this can be replicated for different levels of comparison for the hierarchy and also considering the alternatives with the criteria. Now here what we have considered in the last class I will just repeat. So, consider A is the comparison matrix at each stage, each stage means considering the criteria, sub criteria or considering the alternatives with the criteria and if we continue different combinations. So, A is the comparison matrix of size $n \times n$. Now here n in the last diagram is you saw was 3×3 because there were 3 criteria along the row

row which you are considering along the column.

So, obviously, when we come to the portion which is marked in blue dark blue not the violet one. So, obviously, it will be size 5 5 5. So, n basically denotes the size of the matrix and it will always be a square matrix. Similarly, when we consider the alternatives and proceed accordingly. So, A is the comparison matrix of size $n \times n$ of n criteria also called the priority matrix.

Here I am considering the criteria obviously, they can be sub criteria also. X is the Eigenvalues of size $n \times 1$ for that $n \times n$ matrix and λ_{\max} is the Eigenvalue. So, Eigenvalues and Eigenvectors are found out in order to find out the level of importance and how the criteria or the sub criteria can be broken up in order to understand their level of significance. Because for person 1 which is the decision maker 1 may be fuel cost is very important for me. For decision maker 2 on trying to buy the same set of alternatives or the cars which are there in front of him or her and the same level of criteria which are there for him or her may be purchase price is important.

So, based on that depending on the scores which have been given at each level criteria sub criteria the Eigenvalues and Eigenvectors will give you the level of importance how is being differentiated. So, here X is I am just putting the highlighter is the Eigenvector and λ_{\max} is the Eigenvalue. Now, to find the ranking of the priorities namely based the concept of ranking which I may is depending on the score which was given we will use the Eigenvector X and we need to basically normalize the entries by dividing each by the sum of the columns. So, if you remember the concept of trying to normalize along the column the word column was mentioned time and again we had considered in electro method we had considered in epsilon electro-method we had considered in topsis method we had considered in vikor method also. Even though I did discuss in details about the concept, but few of the ideas were repetitive in all this four methods which I just mentioned.

So, in many of these methods I may have just mentioned it considering that those ideas were already considered in the previous set of lectures. Normalize the column entries by dividing each set by the sum of column and you take the overall row average. So, what we do is that if you see the $n \times n$ which is 3×3 for the first level of criteria which is the first hierarchy the values along the principle diagonal were always 1 as they should be and the of the diagonal elements were 2 : 1/2 then 1/3 : 3, 4 : 1/4. And this basically we analyze and normalize along the columns. So, once we do that the normalized values are this which are if I am reading along the column first column 0.3 0.6 0.1. So, that adapts to 1 the next value is 0.28 0.57 0.15 and the last value is 0.37 0.51 0.12. So, if I consider 7 + 1 8, 8 9 10 3 1 4 4 + 5 is 9, 9 + 1 is 10. And similarly if I do 8 + 7 is 15 15 + 5 is 20 then 2 + 2, 4 4 1 + 5 5 + 5 1. So, I am double checking that they are 1. Now, what does that mean that

actually means the level of importance when I am considering for the set of criteria is when I only concentrate on criteria 1 which is basically along the first column. Then the level of importance which the person will be placing is 30% 60% 10%.

That means, all his concentration or her concentration is only on criteria 1. Similarly, when I consider the level of importance for criteria 2, it is 28 I am not reading the decimals is 28 57 15. And for the third criteria if I read the third column is 37 51 12 again I am not reading the decimal. Now, what are these comparison this comparisons is concentrating only on criteria 1 or only on criteria 2 and only on criteria 3. Criterion 3 which you have you are trying to find out that if your concentration was only on C 1 this is criterion 1 then how it would basically be analyzed considering C 1 C 2 C 3 which are along the rows.

Similarly, when I consider column 2 it is paying attention to C 2 the second criteria and then how the overall analysis would be done based on C 1 C 2 C 3 again along the rows and finally, for the third column accordingly. So, what we need to find out is the priority vector which I just find on the average. So, the average one along the rows would be add up the values and then divide by the number of such values which you have. So, it is 3. Now once you have that the criteria weights are given and the criteria weights are as mark and I will mark it with red color.

So, style cost and fuel economy what it actually means is that trying to find out the weights accordingly we would give or the decision maker would give about 30% weightage to the style 56% which is to the cost and 12 percent weightage to the fuel economy. Now, if you remember in this matrix which I have just circled in red they were individually for each criteria and considering the comparison was being done for C 1 C 2 C 3 which was along the rows. Now we have found out the averages that means, on the long run what would be the overall importance the decision maker would like to place on C 1 C 2 C 3 this is the set of values these are the set of values which we have obtained. Again I am repeating 32 56 12 I read them without the decimal. So, if I basically add up is $6 + 2 + 8 + 2$ is 10 then $3 + 1 + 4 + 4 + 5$ is 9 9 + 1 is 10.

So, the add addition basically adds up to 1 as it should be. Now what it means if I look at the that level of hierarchy then selecting a new curve which should add up to 1 would basically be in the level of importance the decision maker would place if the total score is 100 the total level of importance is 100 or the total amount of weightage is 1 the person will be putting a weightage of 32% for style which I just mentioned 56% on cost and fuel economy being 12%. Now this is decision maker 1 it can change depending on the decision maker it is point 1. Point number 2 is this type of hierarchy can be made for the second level of hierarchy and we can continue and basically draw the overall diagram accordingly. Now what is the diagram which I have drawn and I still highlight which we have done in

the last class we have already done in this class also which I will again highlight here and go back to the last slide.

So, the hierarchy which we want to find out and I will basically highlight it with the light green is this is this is the hierarchy. So, the points which you see here in the along the black so called lines which are or the arrows which is joining hierarchy 1 to hierarchy 2 hierarchy 2 which is the last one to the alternatives are the points which you are trying to find out which we found out for the level of style cost economy which was accordingly found out. What about those values I will again repeat it the values were 32, 56, 12. Now if I pay attention to the one of the diagram which we have done or the diagram which we considered the Ram and Sham making a decision to choose which I am to join A B C and they had level of importance on placement potential and the academic rigor. So, the diagram was like this first level which is the decision you want to take consider next we have hierarchy 1 and then we have say for example hierarchy 2 and consider there are only 2 alternatives in order to make our life simple.

So, if I want to basically draw it here again I will mark the alternatives as I say there is only 2 A1, A2 and these are the criteria which are again marked for completeness of the diagram. So, this is C1, C2, C3 and the next level which was shown in blue was C11, C12, C21, C31 and the points which are being mentioned are along the black arrows. So, they would basically be multiplied accordingly. So, if I have I will mark them in black color. So, see for example, if this is A from going from A capital A1 to C11 this is b going from C11 to C1 and this is c going from capital C1 to the decision.

So, decision will be basically I will mark as decision which you are going to make. So, if I consider A, B, C and then I have see for example D, E then we have F, G, H, I, J, K. So, if I am basically trying to find out the overall score for A1 to reach the decision what are the points? Points would be from capital A1 to capital C11 it will be A and the corresponding points which will be there would be A into B into C the first root. Then the next root would be plus DEC which is from capital A1 to capital C12 and capital C1 till the decision. The next root would be plus F, G, H, F, G, H which is capital A1 to capital C21 to C2 and the final decision and finally, the last one would be $i \times j \times k$ which is from capital A1 to C31 to C3 to the decision.

So, likewise we will basically find out all the roots sum sum them up considering alternative 1 till the decision alternative 2 till the decision and if there are say for example, m number of decisions we will basically find out such values multiply along the root add them up for all the roots and basically rank them from the maximum to the minimum. Now, one thing you should remember which would be very intuitive because if you remember when we are finding out average. So, let me go back to the average here when

we considering the averages we were considering technically if you remember for epsilon electre method, topsis method, vikor method we always consider there was a weights based on the criteria. Here we are considering the weights are being equal that is why you are basically putting one third one third one third weight. So, that weights can be changed according to the level of importance somebody wants to put on the criteria here we are considering one third equal weightage.

Now, once we have done for each and every set of criteria sub criteria we need to find out what is the consistency ratio. The consistency ratio would be for non parametric methods consistency ratios are important and I will mention why which I think I have discussed, but I will just mention it once again considering that in the multi objective case if you have the real line and you want to basically mark the level of the numbers. So, say for example, if I say the property of 3 being if I consider the numbers on the right hand side they did increasing 3 is better than 2 or > 2 and 2 is basically $>$ or better than 1 then obviously, 3 is better than 1 as per the numbering scale, but here if I have the alternatives it may happen this one A1 is better than A2, A2 is better than A3, but there may be cases where rather than having A1 better than A3 you may get a different answer which I am marking in red is that A3 comes out to be better than A1. So, in order to basically check for consistency inconsistency which would arise like this we need to basically check with the consistency ratio and that consistency ratio for the AHP would be given. So, the next stage is to calculate the consistency ratio to measure how consistent the judgments are relative to the large number samples of purely random judgments.

Now, why this problem occurs because if you remember in all the methods I will keep repeating the matters please bear with me for epsilon electre method, topsis, vikor and this AHP method also when we are comparing the alternatives among themselves based on the criteria or when we are considering the criteria also in the example which we saw the decision to compare is only 1 is to 1. That means, taking 2 criteria or only 2 alternatives one at a time at each go and such a way that we ignored the influence of the rest which practically is not true theoretically is true here in this problem. Now, what problem does it create that means, if the number of criteria of the number of alternatives is used then trying to basically measure perceptible logical judgment may not be right always because in that case you are only considering 1 : 1 and in that process ignoring the rest of the levels of influences. So, in order to overcome that that is why the consistency ratio and calculating it and comparing is important. AHP evaluations are based on the assumption that the decision maker is rational as I said if $A > B$ and $B > C$ then $A > C$ which may not be true in many of the decisions.

If the CR is greater than which is consistency ratio is greater than 0.1 the judgments are untrustworthy because they are too close for comfort to randomness and the exercise is

value less or must be repeated and the questions may be asked by the decision maker or the decision maker should analyze repeatedly. So, the next stage is to calculate the lambda max so as to lead to the consistency ratio and the lambda max is basically given by the formula. So, we have already found out A, A was the level of scoring which you are given for any comparison the criterias or the alternatives and x is basically the eigenvectors which you have found out and based on that I find out the λ_{max} . So, we know the formula which I will just highlight $AX = \lambda X$ this is the known very important known equation in basic mathematics where again I am repeating A, is the matrix of the scores given and the scores given if you remember the principal diagonal is one of the diagonal and are not symmetric λ_{max} is basically the eigenvectors and eigenvalues and X is the eigenvectors.

Based on that we find out the value of the λ_{max} . So, what we do is that $AX \times \lambda_{max} = X$. So, these values of A into x are already found out based on the average values of of x we have found out along the rows and based on that lambda max is found out as the average. So, the averages are given so you have the averages values which are already there 0.32, 0.56 and 0.12 based on that I find out the λ_{max} which comes out to be 3.04 and based on the formula given by Sate we find out the criticality index Ci and here n remember is basically the size of the matrix based on which we are working the value is basically comes out to be 0.02. And these values which are calculated lambda max are given here. So, the scoring system for the criticality index or ratios is given the random index values for n n 0 1 2 3 4 n is basically the size of the matrix the values of random index are given critical index has already been found out which is 0.02 n is 3 size of the matrix was 3, based on the table we have so 3 and 0.52. So, we are considering for 0.50 the value of the ratio would be basically Ci by Ri and if you if you remember the criticality index was 0.02 calculated and the value of the Ri is basically is 0.52 which comes out to be 0.04. So, as it is less than 0.01 which includes sufficient consistency is there for the decision. So, you have to basically find out for each and every step. So, I will try to finish this AHP part in this 55th lecture.

So, I may take a length determinant extra. So, once you have that I have now I will repeat the same set of calculations for all the comparisons. So, if you remember I mentioned that and here it is what were the alternatives or the brands of the car was Civic, i20, Escort and Alto. So, I compare the alternatives based on the first set of criterion which is style. So, criteria is style I compare them how do I compare again the same methodology the principle diagonal which I will mark in light green is 1 and the comparison which is happening for one example let us consider Alto to Civic. So, I placed a score of 6 which I mark in blue for Alto with Civic and obviously, the corresponding level of points which I will place on Civic with respect to Alto is 1 6.

Now, this 6 and 1 6 is placed by the decision maker. So, it may differ from decision maker

to decision maker. Similarly, when the person the same decision maker is comparing Alto with Escorts he or she puts a score of say for example, 5 : 1 5th or the person is comparing Alto say for example, i20 the person is putting a weight of I will use an another color violet 1 4 : 1 4. So, look all the values are reciprocal with each other with respect to the principle diagonal. So, he or she does for the style and the priority vectors which have been found out in the same way like finding out the average first normalizing them along the row columns finding out the sum is 1 then finding out the average finding out λ_{max} repeating the same set of steps of criticality index criticality ratios and so on and so forth. For that the priority vector for style when being compared for all the 4 alternatives is given here.

Similarly, when the same decision maker tries to find out for the cost that criteria for all the 4 alternatives again the principle diagonal is 1 of the diagonal are asymmetric the priority vector comes out to be this. So, this priority vector if you are find if you want to compare this value we have done it for the case of selecting the curve the priority vectors or the weights were coming out to be 32 56 12 again saying them values without the decimal. Now, here if I go back it means the level of priority or the level of importance based on style only for the 4 cars is 13 percent for civic I 20 24 percent escort 0.07 and alt of 56. When I go to the cost factor one of this the important factors based on which the choice for the 4 cars is made then the priority vectors have changed they are I am again repeating without going to the decimals 38 29 7 26.

So, obviously, we will see the priority vectors based on the set of important points which is style and cost would differ because the percent decision maker has different level of importance for this style and cost. Now, coming to an additional one which point which may be quantitative here consider the kilometers per liter and the priority vectors are found out kilometers per liter basically means arbitrarily I consider 34 27 24 and 28 is basically kilometers per liters for all these 4 alternatives and the priority vectors based on them are found out priority means the same level of importance which I put comes out to be again I am reading with the decimal now 0.30 0.24 0.21 0.25 add up to 1. Since, fuel economy is a quantitative measure fuel consumption ratios can be used to determine the relative ranking of the alternative. So, obviously, in this case higher the ranking priority vector means more the level of importance you will try to place. So, we will consider both the quantitative one as well as the qualitative one and then combine them have the final decision tree such that at multiplying for along that any path and then adding up all the different paths leading from one alternative to the decision maker will basically prioritize them accordingly. So, here selecting the cars the levels of importance are for style we have this for cost we have this fuel economy we have this fuel economy was quantitative and style cost fuel economy on an individual scale if I compare was 0.32 0.56 0.12 and in between for all the 4 alternatives for style again I repeat 13 24.07 56 similarly for cost and similarly for fuel economy. Once I have that ranking I basically find out and now the

matrix which I have is the alternatives along the rows and the criteria along the columns. Now, this is what is the final decision we are going to make. So, this is the priority matrix the criteria weights are already been found out for the criteria which was for the 3 levels 32 56 12 and based on that we basically have the overall score as given as 28 25 0.07 and 34. So, adding cost as a new criteria is very difficult in AHP a new column has to be added new row has to be added in the evaluation matrix and each and every stage depending on the addition. However, the whole evaluation should be repeated since addition of a new criteria might affect the relative importance of other criteria as well. So, instead one may think of normalizing the cost directly and calculate the cost benefit analysis. So, consider the cost effect note from the fuel one the cost effects are given and the normalized cost are given.

So, consider the benefits are on a scale of 100. So, what the person wants to find out is basically the cost and benefit obviously, this is in the reverse order cost being in the numerator benefits being in the denominator. So, obviously, it will go in the reverse order and found out accordingly. So, the highest one would be the lowest rank and based on that you calculate. So, these are values which are taken from the data and they have been normalized accordingly. So, in the overall scheme the Escort would be the winner with the highest benefit of cost to ratio.

So, that means, in the reverse order second being that of i20, third being alto all the fourth position will go to civic. So, what are the advantages of AHP is multi criteria is a multi criteria decision making where you can consider both quantitative and qualitative evaluation it is applicable to for group decision making environment also. The hidden consumptions are about consistency. So, in many of the cases repeating the evaluation may be cumbersome same thing is being done and when the number of criteria alternatives is more than 7 or some number it may be difficult to compare because in that case inconsistency may crop up is a as I told you. It is difficult to take out an existing criteria alternatives and basically because the effect of the criterion alternative would already be subsumed in the overall set of calculations which are being done.

So, for consistency just on a on a very simple scale for a matrix to be consistent the first criteria is when I consider along the rows and the columns which is a $i j$ into $j a j k$ should always result in a $i k$ these are just points nothing to go into the derivation a 2 by 2 matrix is always consistent and consistency would imply that all rows and columns are linearly independent. So, this is just a simple case of an of a matrix is consistent if I have n which is n cross n matrix the whether way not the weights the values are given by w_1 and then I basically find out the matrix A and what is A is here. So, basically w_i 's are calculated based on the criteria and they can be considered as consistent. So, hence A multiplied by the matrix w would be consistent if and only if the sum of the average weights which is w

bar is basically equal to all the average values corresponding to the the values along the rows which I have. So, I will went a little bit fast in the last later in the later portion of the AHP because rather than continuing in the next week it is best that if you spend a little bit more extra time and cover the AHP it will be better.

Thank you very much for your patience and in the next week last week we will start with the remaining portion of the course. Thank you very much for your attention and thank you for your patience. .