Multi-Criteria Decision Making and Applications
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Week 01
Lecture 04

Hello, a very good morning, good afternoon, good evening to all the participants and the students. This is the course titled multi criteria decision making and application under the MOOC platform. As you know this is a 12 week course and each week we have 5 lectures and each lectures being from for half an hour. And this is the fourth lecture or slide whatever you say in the area as discussed about multi criteria decision making and my good name is Raghunandan Sengupta from the IME department at IIT Kanpur in India. So, if you remember in the last lecture we were discussing about the optimization problem and I said that considering the optimization problem you have set of constraints which are linear in nature while the objective functions are in the initial stages they would be two separate objective functions which are basically non-linear in nature which is basically a very simple quadratic equation based on the fact that you want to find out the optimum point for the finding of the maximum on the minimum for a circle and there are two circles. And we will see that how we try to find out pictorially, obviously solution, is given in the background pictorially we try to find out how we can find out the optimum point both for the minimum maximum for objective function 1 then for objective function 2 and then we try to find out the combined objective function which will be $f(x_1)$ which is basically the first one or f_1 as we say and f_2 which is for the second objective function and we try to find out the combined bi-objective function which will be (f_1+f_2).

So the coverage would be continuation of the example 6 which we are doing and then as it proceeds we will try to basically start with the last example under the introduction which is the seventh one. Now considering the example which we were discussing. There were four constraints to find out that the overall area first will be a rectangle and then we bring the last two constraints which is the fifth and the sixth it will result in an feasible area which had noted down in yellow color I will again note it down in this lecture accordingly. So the constraints are, I will put a tick mark the first constraint is x_1 >15 second one is x_1 < 90 the third one is x_2 > 15 the fourth one is x_2 < 90, so that forms the rectangle and then we add the fifth and the sixth one which is accordingly the fifth one is $(x_1 + x_2) > 60$, $(x_1 + x_2) < 90$ and once you have that the overall area I am just marking in yellow color; this is the feasible region. I am just marking the outline. So if I color the whole area, this is the overall space where you want to search for the optimal points.

Now considering the objective functions if you remember I did mention in the third lecture that there was a color combination for f_1 which is the objective function 1 and f_2 which is object function 2 and that is why I marked those circles accordingly. So if you concentrate on f_1 so this would basically have $(x_1 - 15)^2 + (x_2 - 30)^2$ and as I keep increasing the circle at some point of time it touches I am not considering f_2 only now so if I keep increasing f_1 at some point of time it touches the feasible region at some point and then that would basically give me the minimum point and then I keep increasing that at some point of time it will leave, just touch the outermost boundary which is

$(x_1 + x_2) < 90$ and then keep increasing. So we need to find out number 1 the point at which the objective function f_1 will be minimum and also maximum inside at the boundaries of the feasible region or in certain internal points and similarly if I draw f_2 which is the second objective function which is shown in blue color as I mentioned, similar analysis will be done that it keeps increasing its concentric circles it touches the feasible region space at some point I am not going to derive it immediately and then it goes inside the feasible region and then slowly touches the line $(x_1 + x_2) < 90$ and then increases accordingly. In the process do remember at some point of time depending on how the rate of increase of the radius of this f_1 objective function and f_2 objective function is they will intersect and at that intersection point some would be in the infeasible region which is outside the yellow and some would definitely be inside the feasible region which is shown in yellow and it will continue accordingly based on the fact we want to find out the minimum as well as the maximum. So this is the idea and obviously if you can visualize we will try to solve it accordingly.

Now first let us consider trying to minimize them separately. So as it says when they are minimized separately that is $f_1(x)$ and $f_2(x)$ are minimized, when you solve them the solutions are accordingly. We will go into the details of trying to find out the optimum solution when it is a bi-objective or a multi objective problem later on, but as I mentioned in the first lecture solving this simple linear programming single objective linear programming single objective quadratic programming would be easy and based on that fact we have that solution that is what I am going to now highlight. So if I separately minimize at the point x which is 22.5 which is for x_1 and 37.5 which will be for x_2 it will interestingly satisfy the line $(x_1 + x_2) = 60$ which was basically greater than 60. So just as it touches that line which is the corner line or corner point of that feasible region. So that will be the point where f_1 will be minimized and f_1 when I put the value coordinates, which I am now circling the coordinates. are which have been found out are (22.5 and 37.5). Putting that we get the objective function f_1 is minimized and the value is 112.5. So this is single.

Similarly when I concentrate on the objective function f_2 which is shown in the blue color and with the different coordinates of the center with respect to f_1, the optimum point where it will give me the minimum solution would be 37.5 for x_1 and 22.5 for x_2. Based on that, when I solve and also very interestingly I should mention this is also the point which is on the boundary of the feasible region and based on the fact that the line is

$(x_1 + x_2) = 60$. Obviously, $(x_1 + x_2) \geq 60$, are the set or the boundary conditions which was set. Based on that, when we find out the objective function f_2, the value also comes out to be very interestingly as 112.2. The objective functions for f_1 and f_2 separately are 112.5 for case 1, which is f_1 and for case 2, which is f_2, are same. But when we try to find out what is the coordinates based on which we get that objective function, they are different. For f_1 the optimum point as I mentioned again I am mentioning is 22.5 and 37.5, while for the case, when I want to find out the optimum point for f2 it is 37.5 and 22.5. The corner points and the functional values as mentioned are illustrated in the diagram and we will find out and they highlight that the minimum values as well as the coordinates based on which we want to find out the optimum points for x_1 and x_2 such that we separately get the minimum point for f_1 and the minimum point for f_2. Now let us consider some other sets of points also and why, I am going to say that. If you remember that I will draw it here if you remember the figure looks like this. So this is the (x_2,x_1) coordinate. So this, I am trying to draw as the feasible region. There are two sets of points.
So this whole area is the feasible region. So you have to find out the points accordingly. Now for example for the first objective function I should use a different color as it was there, red. So the concentric circles increases like this and it goes on. So the point which you see here where it touches is the case where we get the optimum point for f_1 only and if I concentrate on the second circle, second circle increases like this I am trying to draw it as nicely as possible. Maybe the other diagram gives you that idea. So for the case when I am trying to optimize only f_2 , the point which is given is here. So this will denote by a_2 'a' as the optimum point for the minimum case and the suffix 2 is for the objective function 2. If I denote the first point this as a_1 again optimum point is for the minimum case is denoted by 'a' with a suffix 1 is for the objective function f_1. Now obviously if you see both of them start intersecting as they expand and if I consider the corner points accordingly and for our example these corner points which are going to consider are, I highlight again (15, 45) and (45, 15). If I do that very interestingly, just for an example, that the objective function value only for f_1 at (15, 45) comes out to be 225. So (15, 45) would be some point as I keep x_1 as 15 and x_2 as 45 so I go up. So this circle, center remains same but it expands. So it is 225 and if I put the same coordinates for f_2, because f_2 is larger in the concentric radius considering that point of 15 and 45 the corresponding value for f_2 is 1125. So I am doing it separately. So if this is the case when we are considering the minimum point just for example if I go back to the earlier slide where it was mentioned that the values of f_1 and f_2 separately along with the coordinates if you see f_1 was minimized and the minimum value was 112.2 at 22.5 and 37.5 while f_2 was minimized the minimum value was 112.2 again and the coordinates were 37.5 and 22.5. So if I want to compare, 112.5 is much, much less than these values

of 225 and 1125. So obviously the points (15, 45) and (45, 15) would never be the minimum points based on the fact that we want to find out the minimum value for f_1 separately and f_2 separately. Now if I consider the coordinates just for example, (45,15) and I want to find out the values of f_1 separately and f_2 separately very interestingly

f_1 = 1125, which you see as I draw, is exactly equal to the value of f_2 which was at the point (15, 45). So obviously a change in the coordinate and the corresponding change in the values of f_1 and f_2 are exactly the same, they are the same value.

So to further corroborate, check the value of f_2 at (45, 15), and the value comes out to be 225 now for (45, 15). (45, 15), I am writing here. (45, 15) are the coordinates for (x_1, x_2), and if I find out the corresponding value of f_1, so this would be f_1, is also 225. But very interestingly the coordinates are just switched. So if you concentrate and try to visualize the values of 1125 or 225, the values are definitely not minimum, based on the fact which you have found out, but the rate of increase of these two concentric circles, if they are at the same rate, increase. then the loci of the minimum points would slowly be a straight line moving along this direction. This is not really needed for this course, but I thought if you can visualize it will give you an idea that how both the objective function separately and collectively can be optimized. I am not going into the maximization yet. If we now consider the minimization of the both objective function together simultaneously, so this is what is important for us to understand. Under the same set of constraints, constraints as you remember was 6 in number then the common point which satisfies considering the fact we want to minimize would be very interestingly a different point which is now x_1 is 30, x_2 is 30 and it will satisfy the line $(x_1 + x_2) = 60$. because the one of the boundaries of the feasible region was $(x_1 + x_2) \geq 60$. So, if I draw it for point 1, so this is x_2, this is x_1. At a point which is (30, 30), I will try to draw. The point would be (30, 30) inside, such that, at the point the surface with their satisfies so some point here which will be satisfied by both of them and the optimum value which we will denote by A, sorry the color should be red. Again 'A' is for the minimum and I denote it as $A_{1,2}$ corresponding the fact that I am trying to optimize both 1 and 2 combined and when I put the values of the coordinates (x_1, x_2), (30, 30), and I try to find out the overall combined objective function, which is (f_1 + f_2), very interestingly it comes out to be 225.

Now remember the values 112.5 was the single objective function minimum value for f_1 then again 112.5 was the single objective function for f_2 with coordinates which were different but interestingly when I combine them, the combined (f_1 + f_2) coordinate system is just the sum which is you see which is 225 as shown in the second bullet point and the coordinates are also different. So for minimum for f_1, f_1 minimum which were denoted by A_1 was some x_1^*, the values have repeated many times so I am going to repeat it again. When I am trying to separately minimize f_2 which was A_2, you will have another x_1^{**} say for example which was already given x_2^{**}, and very interestingly which you know these two values were equal and these two values were equal separately but combined very interesting again the sum is basically (f_1 + f_2) minimum which is 225 and the coordinate system is no longer (x_1^*, x_2^*) or (x_1^{**}, $x2^{**}$), the point comes out to be (30, 30).

This is intuitive based on the fact that the common coordinate in x which is (30, 30) minimizes both the functions simultaneously. So one thing should be remembered leave aside the minimum point for x_1 or x_2 f_1 or f_2 or combined. They are different, obviously they would be and the coordinate system based on which we are trying to find out the minimum point of f_1 or the minimum for f_2 or the combined would also be different. So this is the last part which will be concentrate for the throughout the course is basically to find out the point which we found out as (30, 30) which will give the best solution for the combined objective function of f_1 and f_2 and that common point which is (30, 30) here need not give us the optimum point for f_1 separately or f_2 separately and this would basically be the main idea of the study throughout the course in different examples. For illustration just to extend that example one can check and I am stating few values here for the coordinate system and the coordinates and the corresponding values of f_1 and f_2 also. So there are 5 bullet points star, given the star mark, and the values of the coordinate system (x_1, x_2), I am repeating the first one is (22.5, 37.5) and the values when you put it in that equation the combined objective function comes out to be 675. Similarly, when the values are 37.5 and 22.5 the combined objective function value also again comes out to be 675 and this is a very interesting fact if you see repeatedly as I go the combined objective function value remains same which is 675. For a third case if it is the coordinates are (45, 15), the value which I found out for the combined objective function (f_1 + f_2) now comes out to be 1350 again by interchanging the coordinates of wait one just one minute, interchanging the coordinates of (x_1, x_2), as they become 1545 very interesting the objective function remains the same and for the case when they are (30, 30), the combined objective function, remember the combined objective function is 450. Very interestingly at (30, 30), they give us the minimum point which was 225 minimum point for that point (30, 30), not the global minimum which is the best solution for f_1 which was 112.5 for f_1 separately 112.5 for f_2 also. So combined if I find it out and then solve the problem based on the fact that the solutions have to be found then the combination comes out to be 450 accordingly. Now if I plot the coordinates in a two dimensional diagram and what are the coordinate system on the x axis I have x_1 and on the y axis I have the combined objective function it is no longer a single objective, (f_1+f_2) and if I draw the graph, so it becomes a very nice quadratic function obviously this is a quadratic equation if you remember, f_1 was a quadratic equation f_2 was a quadratic equation combining them would definitely give me a quadratic equation. What is interesting is when I find out the values for (f_1 + f_2) for different coordinate system of x_1 so obviously x_1 it can be done for x_2 also but I am just highlighting for the case when it is x_1 similarly you can have a diagram for x_2 also. So the minimum point, where the objective function would be if you remember the values were given as (30,30) coordinate system and the objective function comes out to be 450.

So as I keep increasing x_1 obviously x_2 will also change or I keep decreasing x_1 , x_2 will also change the minimum point is always found out to be an optimum point which is now (30,30), it is not the best point if I consider separately for x_1 separately for x_2. So based on that you will understand that trying to find out the best optimum solution for a combined objective function or a bi-objective or a multi objective would not give you the same result based on the decision variables which is (x_1, x_2) here as well as the objective function accordingly. So this was the discussion for us the example 6 based on the fact that we are considering trying to consider the minimization. Later on in the next class which will be the fifth lecture we will consider the ideas of trying to find out the maximization for the same problem and once you understand that considering that the objective function as I mentioned is nonlinear which is quadratic and the constraints are linear then slowly we will try to expand that example for other cases and then discuss other topics under the ideas of MCDM. Thank you very much have a nice day. Thank you.