Multi-Criteria Decision Making and Applications

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Week 08
Lecture 38

Very good morning, good afternoon and good evening to all the participants and the students who are taking this NPTEL course by the title multi criteria decision making. And as you know this is a course for 12 weeks spread over 60 lectures and each week we have 5 lectures each lecture being for half an hour and after each week we have assignments. So in totality 12 assignments and they would be one end term or end semester exam and my good name is Raghunandan Sengupta from the IME department at IIT Kanpur in India. So this would be the last lecture not last lecture in total series last lecture on the ideas of basically of multi objective optimization. I will just say few things and then going to the concepts of multi attribute utility theory and the concepts of different type of other decision methods, comparative methods, ranking methods like Topsis method, Elektra method, then Macbeth method, Vikor methods are there which we will consider. All the whole gamut we will not be able to but definitely few of them in quite details.

Now why I said that would be the last lecture in the areas of multi objective optimization and I told that they would not be any slides but I will discuss. So if you remember we had discussed different multi object methods and I told you and I showed you different problems where quadratic programming were solved in the multi criteria methodologies and the discussion was there for Pareto optimality, frontier Pareto optimality points, then concept of linear programming and then combining linear programming to solve multi objective problem, integer programming and all these things were there. part was there about different utility concept because utility concept what we discussed may be you thought that was disjoint from the concept of multi criteria decision making and obviously that thought is quite genuine from your side considering multi objective decision making did not consider as such utility functions. Now the actual utilization of utility functions would be utilized.

So, the broader umbrella is multi criteria decision making, multi objective decision making we have already considered. Now we will go into multi attribute decision making, multi attribute utility theory and proceed likewise. This is the 38th lecture. The main coverage for this lecture with 38th one would be multi attribute utility theory. I will just discuss it conceptually and then go into TOPSIS.

So, when once I go into TOPSIS you will understand the utilization of multi attribute utility theory. So, TOPSIS is one of the methods and as we proceed you will see that as I mentioned few minutes back TOPSIS, ELECTRA, VIKOR, AHP will be discussed as we proceed. So, generally when we consider multi attribute utility theory concepts, so the concept of the multi attribute should immediately ring a bell that we are considering more than one attribute or characteristics to understand the concept of utility theory. So, in utility theory if you consider the ideas we have considered the quadratic utility function, the general power utility function, exponential utility function, the algorithmic utility function they are all in the cases W was the wealth and we considered utility based on the wealth. Now we will consider utility based on more than one such parameters.

Consider very theoretically w_1, w_2, w_3 are three variables based on the we are trying to understand the utility function. So, u w where w is a vector consist of w_1, w_2, w_3. So, continuing the discussion here single attribute utility functions are monotonic where the best outcome is set as one or the best one depending on how the scalarization has been done and the worst is set at level of 0 and then the comparison is made. Single attribute functions are developed to describe the decision makers compromise between the best and the worst alternative. So, he or she is trying to basically make a decision and given the benefit it accrues to him or her the person makes the decision and assigns the score or gives the utility function accordingly.

This best and worst analysis if you remember we have discussed the lotteries, the simple lottery, the complex lotteries. So, based on how the design has been how you can convert and we also consider if you remember the concept of equivalence such that given a particular lottery we can find its expected value and given the expected value we can find out that how a sure event can be balanced with respect to the lottery such that the expected value in both the cases are same. So, under a multi attribute utility theory it tends to deduce the complex problem of assessing a multi attribute utility function into one of assessing a series of unidimensional on utility function. So, rather than have a multivariate case we will basically break down into different combinations of univariate case depending on how the combinations can be done. Either it can be weighted, it can be multiplied, it can be added.

So, depending on the assumptions it could be done. So, what basically MAUT multi attribute utility theory does its stitch is basically a functional form which stitches or joins this unidimensional utility function in such a way that it presents in front of you a multi attribute function such that given all the attributes you can immediately understand that how they can be analyzed from the point of view of the univariate case. It is basically a way of trying to replicate or present the multi attribute concept using unidimensional case because it will be easy for us to basically function. As I said I will only discuss the formulas later on we will consider them as we proceed with the problems. The main problem under multi attribute utility theory as I said is basically to find the utility function u(x) and here x or w whatever we have been utilizing x is basically the variable and it is not a scalar it is a vector.

So, x vector basically has x_1, x_2, x_3, x_4 to x_n. So, we want to basically find it and that is the multi attribute utility concept which as I mentioned few minutes back can be broken up or stitched from a univariate utility function case. A simple method could be to derive a function such that I know or we are aware that utility function is given by some functional value f. So, depending on the parameters here now it is more of u_1, u_2, u_3, u4 where $u_{1, 2, 3, 4}$ are the corresponding utilities based on the univariate case. So, even though that may not be correct but let me see consider that see for example, the univariate cases are quadratic and exponential.

So, given a multivariate case we can break it down into different types of univariate quadratic and univariate exponential utility functions and solve the problem accordingly. This is more for the problem solution point of view. One of the simplest way of decomposition is the additive function. So, you have different utility functions you add them up like if you remember the case where we have considered the objective function or the multivariate case the multi objective optimization case. So, they were two options if you given.

So, given f_1 and f_2. So, which are the objective functions I am considering only two. So, you can optimize f_1, or f_2 bringing the other objective function as a part and parcel of the constraint or else case two method is you combine f_1, and f_2 in a convex combination way that means give weightages of λ_1, λ_2 to f_1, and f_2 respectively such that the weights add up to 1 weights means $\lambda_1 + \lambda_2$. So, in the similar way, rather than have the multivariate at utility function. We basically have the univariate case, add them up and give weights accordingly such that the corresponding weights are from $\lambda_1,\ldots., \lambda_n$ and the weights are given in such a way that combining the addition of $u_1 \times \lambda_1 + u_2 \times \lambda_2$ till the last utility function which has been considered depending on how many such univariate cases are there you can basically present the multivariate case that is a very simple way.

So, another parameter which is also used is basically depending on the scaling factor. So, what is the scales you want to give what is the overall effect. So, you want to scale it up or down. So, here $\lambda_1,\ldots., \lambda_n$ which are considered are normalized depending on the weights you want to give. So, say for example, I am giving a weights of $\lambda_1,\ldots., \lambda_n$ as between 0 and 1 that means the addition of $\lambda_1,\ldots., \lambda_n$ is 1.

So, all of these weights which is which are basically $\lambda_1,\ldots., \lambda_n$ has been normalized in a sense or you have considered the normalized weights such that they add up to 1. And if you want to scale them up you basically multiply or divide depending on whether the scaling is positive or negative such that we are able to get the scaling factors also included as part and parcel of the univariate utility function and add them to find out the multivariate or multi attribute utility function. The scaling factor assures that the compound utility function u assumes values in the interval between 0 and 1. So, if you consider the properties of the utility function in general should hold true for the univariate case and it should also hold true for the multivariate case. In the same way like say for example, if you have a univariate distribution and a multivariate distribution all the distribution properties fundamental things from statistics should hold in the same way the univariate and the multivariate utility function cases the properties would definitely hold.

So, without going to the problem which as I said I am again repeating they would be coming up later on. So, I will just state the models and state the important facts or what are the important points to remember. So, you have the additive utility function. So, in that case and consider the utility functions are any number the first one is that where you add up the utility functions. So, this would basically become $u(x_1,\ldots,x_n)$.

The next one is the weighted additive which means that you first multiply with the weights we just discussed and then add them up. So, if I consider technically the weights here by the way if you see the first additive case there immediately the idea would come out of the weights corresponding to each utility functions $\lambda_1,\ldots,\lambda_n$ are all 1. So, now if I basically give the weights $\lambda_1,\ldots,\lambda_n$ where all of them are not 1 you have the weighted additive utility function given where n is the number of utility functions as general and there are other n number of scaling constraints also which basically add up each individual utility function value to be combined and in the end give you the multi attribute utility functions. The third and the fourth one models are considered as the multiplicative because you multiply the factors or log additive and the fourth one is quasi-additive and the I will come to the utility functions and also in the third column the number of calculations or the number important points to remember of that. So if you consider the utility function the multiplicative or log additive so there are many terms.

So, consider this is the first term which I have underlined with dark a little dark yellow then there is the second one which I am underlined by red then is the dark blue and so on and so forth. Now if you pay attention to the first one and ignored the higher terms so this is just the last example we got which was the or discussed which is the weighted additive. Now the multiplicative term if you see they are coming from the fact that like consider the expansion of a Taylor series in the multivariate case or just a expansion in the multivariate case of any series. So the terms which come up consider there are n so if I consider n number of them so they can be combined in ${}^{n}C_2$ number of ways and if I consider combination with itself is also allowed then you will have basically the whole set of combinations which are possible. So if I consider this concepts and here $i= 1,\ldots,n$ where n is the number of such univariate case j is also $1,\ldots, n$ and in this case if I assume or if I try to visualize the variance covariance matrix.

So the principle diagonal if the variance covariance matrix is what is basically the variance or the covariance with itself and the of the diagonal elements are the covariance of i^{th} to the j^{th} one and it is multiplied by 2 because they are symmetric. In the same way if you consider depending on $i = j$ and $i \neq j$, you can basically replicate and this is of the second order that is why the word multiplicative you basically get all the terms accordingly. So obviously it become complicated but depending on the simplicity we will only stick to the simple additive model and if you look at but the multiplicative one obviously the higher orders can also be considered. So in the higher orders means when you are taking combination of 3 out of ${}^{n}C_3$. So if I consider ${}^{n}C_3$ the corresponding parts which is very simple to understand you are taking $\lambda_i \times u_i$ as the first.

So u_i is basically the first utility λ_i is the corresponding factor to be multiplied then

$u_i \times \lambda_i, \ldots, u_k \times \lambda_k$. So these I am just putting a tick mark λ_j then u_j then I am using the green color $\lambda_j \times u_j$ then I use the violet color $u_k \times \lambda_k$. And similarly if I go I can have 4 terms which means and by the way all another thing which I missed as I was going. So if I consider there is k^2 so depending on the factor which is being multiplied and this factor of k if you remember that was basically in a way given as the scaling factor which you can consider. So if you see the scaling factor here when we started.

So scaling factor being k means you are giving the scales for each of them in the single case as only of one order then k^2 means you are basically multiplying the scaling factor of both. So it may be possible if both the scales are to be increased so obviously both of them will be multiplied and increase the value or else it can decrease the value depending on the values of k. In the case of quasi additive you consider the weights as $\lambda_{i,j}$ depending on the ideas you are going to consider for the multi attribute. So coming back to the multiplicative or lot additive you have n number of utility functions and n number of scaling constraints. For the quasi additive there are n number of utility functions and 2n -1 number of scaling constants because the scaling constants are not lambda i's now themselves they are $\lambda_{i,j}$ depending on how the scaling the factors have been taken into consideration.

And also if you only concentrate on the first term it remains and continues to remain as the simple additive model. When I consider the bilateral one the corresponding functional values are they are actually dependent on the functional value of f. So bilateral one hybrid one if you consider the factors can be increased so depending on number of terms you are considering. So the second term where I put a tick mark it is a square order to 2 terms where I put a double tick mark in red it is cube and if you can understand the cube means it is being summed up for i = 1,…,n; j = 1,….,n; k = 1,….,n but obviously that double counting so j should be > i and k should be > j. And as usual the first term which I circle with violet color is the simple additive one.

In the hybrid one also the f_n is the number of scaling constants where f is some function which depends on n and these scaling factors or hybrid is basically combine them accordingly to achieve the neutral function. So these I am just mentioning but detailed discussions are outside the ambit of the course. So you have the quasi pyramid and semi cube so if you consider so the pyramid structure like this where the base is a triangle or else if you consider the base is a square the pyramid structure would be there and the layers which you can take which will give you the different factors which are there in the extension can be understood as you take the slices of the pyramid accordingly. Similarly for the semi cube one so I am not going to the details so also again another point the first term in the quasi pyramid and the first term in the semi cube are the simple weighted concept of utility function univariate weights are taken multiplied by the factors and scaled up to find out the overall multi attribute case. So for the quasi pyramid number of calculations is $n \times (n-1)/2$ and with non-separable interactions is there and f_n is the number of scaling factors similarly for semi cube f n is the number of scaling constants where the function f is some function which depends on n only in the same case as like the other one.

The interdependent one would be the case where rather than having the univariate case your main focus is now the utility functions which are dependent. So rather than having u_i u_j multiplied separately it is a bivariate case bivariate case I am using the statistical term is basically a utility function where both the parameters or the variables are x_i and x_j i and j are basically the counting between 1,…,n. Similarly if I consider the third factor it is u_{ij} x_i x_j x_k so it is basically not dependent independent but they are interdependent depending on the order we are taking. Similarly for the fourth term which is not written here it will be u_{ijkl} so i j k l which are the four terms and the fact and the parameters of the variables which will decide the new utility functions would be x_i x_j x_k and x_l. So generally the fourth term which as I said is not written here I am not writing the lambda part so i j k l so there are four terms this is x_i x_j x_k and x_l and f is the is some function which depends on n depending on how the utility functions have been developed.

In the multininear one also before I go to the multininear one so this would have become much clearer now the first term is basically the simple additive utility function. For the multininear case you have again the first term is the additive one and the factors which are there are build up so if you consider the second term generally for the earlier case it was basically λ_i λ_j λ_k and then it was u_i u_j u_k here u_i u_j u_k remain the same but the factors λ_i λ_j λ_k are expressed as λ_{ijk} all together which are dependent and there you will get $2(n-2)^{n-1}$ number of scaling constants to be included. Now the reader or the students or the participants to should know two important points about MAU T without going to much theoretical concepts that there is a preferential independence concept that is the pair of attribute x and y is preferential independent of z if the value tradeoff between x and y is not affected by a given level of z. So, consider x and y are there and if I basically try to incorporate the ideas of a independent attribute function of z in both of them the corresponding ranking concept would not change. So, which we will see later on and it has already been discussed in one of the properties on the axioms if you remember when we are considering the earlier part of the course.

And the second point is utility independence where attribute x is utility independent of y when conditional preference of lotteries on x given y does not depend on the particular level of y. So, depending on the on given a condition that y is changing. So, obviously the preference of the condition preference of lotteries based on x would not depend on what is the state of y independent. One can also use goal programming to address the issue of MAU T. So, if you remember in the goal programming we had the objectives and the constraints were less than type, greater than type, equality.

So, here let us consider the criteria which are k in number and they are denoted by $f_1,\ldots,f_k$ and I divide the overall k, k is vesicular number it can be 13, 14, 40 whatever I divide into three sets. Three sets in the sense that they are distinct interaction between these three sets which are elements of k is a null set. So, I consider the first set from $f_1,\ldots,f_k$, k suffix 1, the second one I will use a different colour is from $f_{k1+1},\ldots, f_{k2}$, the third one is basically from $f_{k2+}1,\ldots$, to f_{k3} such that the sets are I am marking this is the third one, this is the second one and this is the first one. So, this is what I mean. So, union of them is the k and k1, k2, k3 are all distinct such that the interaction between k1, k2, k3 is a null set.

Now the question is that when we consider the criteria functions would it be true that all of them are independent answer is no, but we will consider it to be one of the assumption based on which we will try to proceed. The question would be consider the dependence structure like I want to now if you remember the examples which would be discussing are more qualitative in nature, subjective in nature, discrete variables. So, consider you are buying a car and mileage and maintenance are dependent or cost and maintenance are dependent or cost and mileage are dependent, higher the cost better the car we assume and better the mileage. So, mileage is positive increasing better for me, cost increasing obviously is not better for me because I have to pay more. So, obviously there is a dependence between these two criteria and when I mention mileage, when I mention about cost these are the criteria which are used to decide that what are the important factors based on which you are going to buy the car.

Another factor consider for the housing or for say for example selecting the college where you will study or well a person will study for higher studies be it commerce, engineering, MBA, sciences whatever it is. So, consider the buying the house if the locality is good which you want positive feedback, safety, nearer to bus stop, nearer to metro obviously the cost is higher, maintenance is higher. So, the point which is mentioned that they are independent would not be true many of the examples, but we will still consider that to be true and solve the problems accordingly. So, with this I will end this lecture and start with the discussing of more details about one of the methods one by one as I mentioned top-size Electra, Macbeth and so on and so forth. Have a nice day and thank you very much.