Multi-Criteria Decision Making and Applications

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Week 08
Lecture 37

Good morning, good afternoon and good evening to all the participants and students for this course titled multi criteria decision making which is run on the NPTEL MOOC series and my good name is Raghunandan Sengupta from the IME department at IIT Kanpur. And as you know this course is spread over 12 weeks which is 60 lectures each lecture being for half an hour and as the information is all there with you each week we have 5 lectures of half an hour each. So, the broader outlines for this 37th lecture is as follows which we have been doing it for a long set of weeks is about multi criteria decision making as the main topic under which we have been discussing multi objective decision making with different examples, multi attribute decision making and multi attribute utility theory. About the concept of theory we said that we will discuss that when we start the non-parametric methods. So, the general ideas or problems we have considered is the set of problems in initial set if you remember in introduction we had discussed both pictorially as well as solving the problems, quadratic programming and took binary two different objective functions by objective case and the constraints were all linear. Then we went into discussion of the concept of using linear programming with linear constraints then integer programming with linear constraints and we did discuss the two different important concepts of multi objective programming.

We took only two problems multi objective programming where the idea was to discuss that how we can either give weights to the objective functions. So, consider the convex combination of the objective functions or else consider the objective functions in such a way that when objective function one is considered in the objective two and three would be considered and a part and parcel of the constraints and we did consider a problem where these combinations were taken both where weights were given as well as when objectives were one at a time considered as part and parcel of the constraints also. So, now we will expand our discussion and consider a whole set of ideas but the example will be given in one and it you will understand all the things in details which is the bi-objective quadratically constrained quadratic programming. So, as the name says is here we have a bi-objective case because there are two objectives and if you look at the problem it will make sense to you.

The problem which I will keep highlighting the basic idea is this. So, there are two objectives one is the quadratic part and one is the linear part. Now the concept which you see is 1/2 it is just a constant it would not matter the reason mean that when you want to basically optimization mean what you are trying to optimize or trying to find on the maximum minima and then you differentiate. So, as this function has a quadratic component of x^2. So, when you differentiate it becomes 2x.

So, that $2\frac{1}{2}$ cancel out that is the only simple way of trying to explain that. So, you have one again to repeat one component which is quadratic and the other component which I will highlight using the green color is the linear part. Remember one thing in this case the X^T p and then x then Q^T X all are vectors or matrices accordingly like X^T and X are vectors Q^T obviously is a vector and p, a set of values p is basically a matrix. So, obviously depending on the size matrix multiplication and the rules should apply in the sense that if X is basically a vector which is of size $p \times 1$. So, in that case X^T would be of size $1\times p$ which means p means small p and in and next p capital P not obviously you have to pre multiply p not with X^T.

So p not has to be small $p \times p$. So, this is a multiplication sign and hence. So, on the minimization side which you are trying to minimize it is basically $(1 \times p) \times (p \times p)$ which is $(1 \times p)$ then $(1 \times p) \times$ again x which is $p \times 1$ gives you a scalar and similarly when I try to find out Q^T X. So, in that case as X is $(p \times 1)$ Q obviously has to be $(1 \times p)$ to give the corresponding result also for the linear part of the scalar. So, it will be basically a sum of two scalar quantities which you want to basically minimize.

Now, when I was discussing I did mention the word minimization here. So, here rather than using the word minimization maybe it would have been apt for from my part to use the word optimization, but minimization can also be considered as a maximization problem with the science conversion happening which you all know who are taking this course and who have the requisite background in operational research. So, maximization minimization are just opposite like a mirror they are opposite sides on the mirror. Now, if you consider the constraints. So, it is a bi-objective.

So, it has been explained two parts quadratic linear. Now, it is a quadratically constrained quadratic programming and why constraints are quadratic is as for the following reason. You have again a quadratic component in such this condition and this p_i that is also a matrix because if you consider $X^T \times p$ which is a matrix multiplied by x obviously, the same concept of matrix rule multiplication based on the rows and columns should hold. And this I would basically depend on the number of such constraints are there. So, as you see $i = 1,\ldots,m$. So, m is 3. So, i = 1, 2, 3 that means there will be three such constraints. The second part would be which I am highlighting using the violet color. So, this is the constraint which consists the linear component part corresponding to x and that also matrix multiplication vector, vector multiplication concept should hold. The last term on to the left hand side of the $\leq$ sign is r_i which is basically they would be r_1, r_2, r_3 depending on m is 3 is a constant such that is less than time.

So, obviously the question would be immediately which you would have understood by the way of explanation I have been doing. If it is less than type then obviously you can convert into greater than type or the greater than type is there can be converted into less than type. Equality can be added with a less than type greater than type two equations. Why I am mentioning that because many of these optimization tools which you will use presupposes that how you give the constraints. So, the problems which I am solving have been solved in R.

So, if you check R, I am not sharing the codes I am only sharing the results. So, if you check with the R codes there is specifically mentioned whether it is linear programming package you are using non-linear programming, quadratic programming, integer programming. So, obviously the constraints have to be written very specifically depending on how the code has been written. And another part of the equality concept is which I will use again a different color $Ax = b$. So, obviously if it is equality to I can be converted into less than type greater than type.

So, overall framework is quadratic component in the objective function, linear component in the objective function in the constraints also quadratic and linear components there. So, in mathematical optimization are quadratically constraint quadratic programming which is QCQP is an optimization problem formulation in which both the objective functions and the constraints are quadratic in functions. In its general form it is written like this which I explains first I did the explain and then I read what is written in the slide. Now one by one I will give you all the flavors and then solve a problem in details. Now if you see the slide obviously the slides are being shared with all the participants and the students who are taking this course.

If you see it is mentioned very specifically with different color coding scheme it is like this. So, the linear it is basically a linear programming with both linear and quadratic constraints if the component which is $\frac{1}{2} Q^T p$ not x is not there as I am underline here it mentions if it is not there then you have a linear programming with quadratic constraints. So, obviously they can be different type of flavors of the problems to give an example what example I will solve is a very simple one which will give you a lot of idea, but example can be say for example, in the area of finance you want to rather than minimize as I said it can be converted into maximization problem also you maximize the return of the portfolio subject to the case that the risk should be $\leq$ some stipulated value. So, this type of problems what I mentioned qualitatively can be structured as given in the slide where the quadratic component for the objective function is not there. So, it is a linear programming with both linear and quadratic constraints.

Next case is where I have a quadratic programming with both linear and quadratic constraints which means the linear component $Q^T x$ is not there again the same type of in the area of the example which I am considering you are considering the minimization of the risk because risk which is variance is a quadratic term while return is a linear term. So, if $Q^T x$ is not there which I just circled in red it is not there then you minimize the risk and subject to constraints are quadratic as well as linear, linear would be where you want the returns to be more than equal to some stipulated value as well as the risk to be less than equal to some stipulated value. The idea which I just mentioned about the constraints would also hold for the problem which we just discussed where it was trying to maximize the return subject to the constraints which is both for the risk as well as the return when I would use the what the risk is basically related to the variance. The third flavor of the same problem is where you have linear and quadratic programming in the objective function but the constraints are linear. So, if you remember the first problem which we have solved in the introduction part it was basically trying to maximize and the equations were related to f_1 and f_2 where f_1 and f_2 were that of a circle.

So, if you consider here also you want to minimize or maximize again that can be

changed depending on the problem the objectives are quadratic and the constraints are linear. Now, again going back to the problem which we have discussed in the introduction part the constraints were all linear in nature on a Cartesian coordinate. So, this flavor of the problem can be where you do not have the quadratic component of the constraints. So,
$\frac{1}{2}$ X^T p_i x is not there. So, it is basically a problem of that sort and we will solve a problem of this type within few minutes.

The fourth flavor of the problem is where the corresponding concepts related to in the constraints which is quadratic and the constant term whether r_i is there or not they would not matter because it can be ≤ 0 or $\leq - r_i$ depending on r_i is removed to the right hand side that would depend on what type of qualitative information is there about the problem. So, here this $Q^TX + r_i$ is not there, but the objective function remains as it is. It is a part of quadratic part of linear while the constraints are all corresponding to the case that they are all quadratic. Quadratic means when I am using the word quadratic linear to the constraints I am not talking about Ax = b that is there that can be there that need not be there. So, depending on if it is there not there the concept of equality constraints I would be discussing.

So, in this case if constraints all the concepts of the linearity part is removed only quadratic part is there then obviously Ax = b is also not there. But as I am highlighting the quadratic part the linear part in the less than equal to constraint as well as going back to the objective function that is why my focus of discussion is the objective function and the set of constraints which is i = 1,…..,m. So, the similar logic if I talk about Ax = b can also be included or not included and if it is not included along with Q^TX being not there. So, obviously the constraint is absolutely totally quadratic in nature there is no linear component in the constraint. Now we will solve a problem and as I said these problem solutions are based on basic concept of optimization and of level 1 that means any masters level courses or in bachelors if they do and specialized courses in operational research and the discussions about what books can be used which are good books which are old, but they are very classic books I had discussed that in the first class.

So, considering that all the participants and the students are generally aware of this concepts and as usual I am not going to the actual problem solution like using linear programming these are using inter poly algorithm or these gradient descent methods for the quadratic programming. I did mention this world, but these are different type of algorithms which are there to solve this non linear linear programming I will directly come into the solution as I have been doing for other different lectures also. So here you have the consider the first being the linear as well as the quadratic programming problem now where the problem is interesting is this. The component related to the objective function it consists of both linear and quadratic component how quadratic? So, I will mark them with one color at a time. So, I am putting a tick mark for the blue where the quadratic component is first ½ x_1^2+ x_2^2 and - x_1 x_2.

So the sign of +/- minus is just I said, but it would not matter what is important is to look that there are x's which are quadratic three terms x_1^2 x_2^2 x_1x_2 linear part which I will mark in light blue is x_1 and x_2 here. So, it is basically combination of quadratic and linear part. Now let us come to the constraints, constraints are the I am only considering the linear components the constraints are like this which is $x_1 + x_2 \leq 2\ x_1 + x_2$. So, the $-x_1 + 2x_2 \leq 2$ and $2x_1 + x_2 \leq 3$ and $x_1\ x_2$ are on the real line that means for simplicity they are positive. So that means they would be can be considered as continuous can be considered as integers also.

Now considering this problem I solve them and consider continuous variables. So, when I solve them the value of these two decision variables come as I will use the so they come as 0.67 and 1.33 and when I put the values of 0.67 which is I can consider as one third and this I can consider as one and one third.

So, if I put them in the equation objective function f 1 the value comes out to be minus 8.36 we are minimizing it remember not maximizing. So how would the slope like this graph I have drawn again using R and if you consider is a 3D one I have tried my level best to draw it and I will mark the axis for better understanding and better communication between me and the students who are taking this course. So, along this axis which I am drawing that means here I take variable x_1. So, on the plane where I am standing is basically the x_1 is being measured x_2 variable is going from my side towards you towards the camera.

So, and they are orthogonal. So, it is basically like this x_2 is going towards you x_1 is on the right and the functional value f_1 I am measuring vertically up. So, if I consider the corner of the room going towards the roof is f_1 from the corner of the room coming towards me or going towards you depending I have considered x_2 and from the corner of the room going to the right considering the point is on my left is x_1. Now if I consider that the constraints which were given here let me mark them one by $x_1 + x_2 \leq 2$. So, if it is $x_1 + x_2 \leq 2$. So, if I consider these values x_1 and x_2, the corresponding value which I would have and $-x_1 + 2x_2$ $-x_1 + 2x_2 \leq 3$ and finally, $2x_1 + x_2$, $2x_1 + x_2 \leq 3$.

If I plot them accordingly here in the diagram which as you can see there is one light yellow plane this one which I put a single tick mark and there is a light pink plane which I put a double tick mark and there is a light green plane where I put a triple tick mark. Now if you imagine obviously they are opaque so it is difficult to visualize, but if they are transparent or translucent then you can think of these planes as a case where you have all the values of corresponding to these three planes which give you the boundary of the surface of the feasible region in considering the case where I am plotting this graph. And technically you can say that it could have been done in the two dimension one I agree, but in this case I am plotting the function f along the third dimension. So, if I consider minimization problem again this whole white grid which you see is the function and it is quadratic in nature. So, depending on that it is like a hammock or like a net structure and if you see that the minimum point would be such that it is negative that means the values of x_1 and x_2 are the f value is <0, but the value of x_1 and x_2 are two-third and one-third.

So, depending on that the value would give you somewhere so x_1 is here that side and x_2

is also here so it will be some point here which is negative value being 8.36. So, based on that I can find the values of the objective function. Now, I consider a second objective function and consider the second is where again you have the quadratic part which is marked I am marking in this dark red x_1^2, 4 x 2 square are quadratic component and the other blue one which I am marking is I am not reading the constants it is basically x_1 and x_2 constant if I read it will be - x_1 8 x_1 - 16 x 2 and that would be the linear part and if I consider the constraints they are exactly the same. So, $x_1 + x_2 \leq 2$ - $x_1 + 2 x_2 \leq 2$ and $2x_1 + x_2 \leq 3$ and obviously x_1 and $x_2 >0$.

If I solve this problem again quadratic objective function with and also linear component of the objective function with linear constraints the value comes out to be which is 0.8 for x_1 and 1.24 x_2 and the objective function is - 19.2. So, both the x_1 and x_2 for the first objective which gives the minimum value and x_1 and x_2 which gives you the minimum for the second objective being minimum are different 0.1 and obviously f_1 and f_2 if I consider they are also different as they are different objective functions form formulation will be different. Now if I plot it again. So, the first constraint which was there $x_1 + x_2 \leq 2$ then you have - $x_1 + 2x_2 \leq 2$ and third one is this one will be $2 x_1 + x_2 \leq 3$. So, if I see the constraints again same thing one light yellow part the green and the pink yellow light pink yellow constraint the last one being the light green one again if they are if you can visualize them they are opaque but if you visualize them to be transparent transliteration you can understand all the three constraints have been denoted accordingly. Now the objective function is not totally different.

So, the mesh which you see on the net type of graph which you see here is basically the objective function 2 and as usual I am measuring x_1 going from my left to the right x_2 going from my side towards you or towards the camera and f_1 f_2 is being measured vertically from the bottom to the top. Again I am trying to draw the objective function along with the constraints. Now comes the interesting part. So, in the interesting part is I will consider the bi objective problem and the bi objective problems are marked them with so I will put weights λ_1, λ_2. So, this is λ_1 wait for the first function and I use a different color λ_2.

The second function which is f_2 so this is f_2. So, f_2 was basically this one and if I consider f_1 the constraints are all the same. So, constraints are this is $x_1 + x_2 < 2$ - $x_1 + 2x_2 < 2$ $x_1 + x_2 < 3$ and only other information which will change accordingly is basically $0 < \lambda_1, \lambda_2 < 1$ both inclusive and sum is 1. Now when $\lambda_1 = 0$ then obviously all the weights is on function 2 when $\lambda_1 = 1$ all the weight is in function 1. So, obviously when $\lambda_1 = 1$ you solve the first problem which has discussed and $\lambda_1 = 0$ where which means the $\lambda_2 = 1$ we solve the second problem where f_2 is being solved as a single objective.

First case f_1 is being solved as a single objective. Now I solve it and I will discuss the answers later first but first let me understand how the values of x_1 and x_2 would look like. Now read the what the graph says the graph gives you the value of λ_1 along the x-axis the values of x_1 and x_2 which is the decision variables along the y axis and you see the graph which I will use one as a red color. So, this is the graph which gives me the value of x_1 as I solve and the green one which I will mark gives the value of x_2. So, depending on λ values I would solve find out the minimum value.

So, if λ_1 is point see for example 2. So, consider point 2 then in that case I will basically weight function 1 with point 2 and I will give the weightages for function 2 as point 8. So, I solve it and when I solve it I will get the values of x_1 and x_2. So, those x_1 and x_2 are being marked here I have not written the values but we will discuss that accordingly in the corresponding class. With this I will end this class and discuss further about the this bi-objective quadratically constraint quadratic programming with the example which we are discussing. Thank you very much and have a nice day. Thank you.