Multi-Criteria Decision Making and Applications
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Week 08
Lecture 36

Welcome back my dear friends, a very good morning, good afternoon and good evening to all of you. So, as you know this is the lecture or NPTEL MOOC series where the course title is multi criteria decision making and my good name is Raghunandan Sengupta from the I M E department at IIT, Kanpur. This is the 36th lecture out of this 60 lecture series and 60 lectures means spread over 12 weeks and each week as you know there are 5 lectures each lecture being for half an hour. And as you know after each lecture of 5 that is each week you have assignments. So, there will be 12 assignments, these are already known to you, but I am just repeating. So, there will be 12 assignments and at the end there would be one final semester examination.

The broader set of points, broader set of ideas, broader set of philosophies which we are discussing for a long time for if you are going off on quite a few weeks. The main points are related to multi criteria decision making, multi objective decision making which we are still continuing discussing. Then we will go into multi attribute decision making and multi attribute utility theory and draw conclusions from multi attribute utility theory which I said it will be a little bit theoretical and go into discussing the other so called qualitative methods of ranking. The examples which we were discussing in the last lecture was related to linear programming for MCDM and then as I said we will also go into the non-linear programming.

Hopefully we will be able to start this problem for non-linear programming. Now where we closed the last day was this slide where the top row had different colouring schemes of x_1, x_2, x_3 given in red, violet, blue and the next three column heading were f_1^{**}, f_2, f_3, f_1, f_2^{**}, f_3, and f_1, f_2, f_3^{**} and the colouring scheme of red, violet, blue is clear to you. The values which you are seeing in this table correspond are corresponding to the fact that you get a combination of f_1, f_2, f_3. I am not reading this double star values it should be obvious because this is the slide we are discussing and if I add up the sum what I am trying to optimize based on that we obtained the different type of Pareto points. Now a question would come which I will answer later on. So let me do the other methodology of trying to do the multi objective problem and I will solve the same problem this one which you see considering the idea that how the ideas of if the weights come into the function like if I give weights of 1/3, 1/3, f_1, f_2, f_3 then that is fine because I will be adding them and dividing by 3.

So, in that case for the multi objective case for these three when I add up the functional values and divide by 3 do not change, but I will give different weights like 80%, 10%, 10% for the first function, second function, third function then the values will change obviously because in the case if I put 80, 10, 10 that is 0.8, 0.1, 0.1 so the value if I consider the yellow marked cells only it will be 39 multiplied by this is the multiplication sign is π

of 0.8. Now I will do it the other way, other way means a different so I will use the red color because that is f_1^{**} maximize so it will be easy for us to make the differentiation using the coloring scheme. So, the first part this one so this will be 39×0.8 because we are putting 80% onto the first function + 18×0.1 because we are putting 10% function 2 and 18×0.1 because that is the weight we are putting for the third function. Now if I follow the same scheme 80, 10, 10, 80 %, 10%, 10% then here it is $18 \times 0.8 + 39 \times 0.1 + 18 \times 0.1$. The value on the red shown and value on the violet they would not be same because we are putting different weights. Similarly, if I consider the blue one here so this one would have 18 into 0.1 plus 18 into 0.1. So, this is the blue one and this is the red one. So, this is the blue one and this is the red one. So, this will be $18 \times 0.1 + 18 \times 0.1 + 39 \times 0.1$. Luckily the violet and the blue one are same, but it is not guaranteed depending on the outcome we will have different answer.

So, I want to show you an example later on after we discuss the second one that how we can formulate another simple objective function which will be not dependent. It can be an example which would not be dependent on the weights we are doing, but having said of the weights let us consider the other methodology of trying to solve the multiclass criteria problem. Before we move to the weighted concept, let us consider the 3D plots of f_1, f_2, f_3. So, as this programming were all done in R, so all the graphs and the results which you see for. So, here we have variable 1, variable 2, variable 3 technically variable 1 is on the floor where I am standing, variable 2 is going from my side towards the camera that is towards the students and variable 3 is coming from the ground going vertical up which is 3 dimension one and they are orthogonal to each other.

So, the points which you see I have not been able to plot the plane, but I will try to analyze that later on that you will be able to get the Pareto points as we just discussed that was given the value wise how you can visualize pictorially that will come later on. Having said that let us go in trying to utilize the concept of not transferring the objectives in the constant, but giving weights. So, the weights we consider for function 1, function 2, function 3 are this is function 1, this is so the weights is λ_1, this is function 2, weight is λ_2, this is function 3, weight is λ_3. And if you see the constraints this is the same problem which you have considered, this is the same constraints 5 in number and important point which we all know all the weights should be between 0 and 1 and some of the weights is 1. So, you can basically do a permutation combination trying to find out what are the weights based on that, we will follow that scheme later on also.

Now and this is a linear programming and we can solve the integer part also of this. If you remember one problem we have solved the same problem which was in two dimensional case we took linear programming and integer programming. By the way for this problem which we are solving giving the weights we will solve it using the linear programming. That means, the variables are continuous and we can also solve the problem considering the variables are integer. Now, the question which you all would ask the problem for which we have been discussing in the last class could not that has been solved in the using the variables integer yes they could have been solved.

I did not do it, but I am sure you got the gist on the essence of the discussion. Now I take different combinations and the combinations I have enumerated all of them for simplicity for ease of understanding for getting a better view point whatever you can say. So, I took the values of $\lambda_1, \lambda_2, \lambda_3$. Why I am putting colors because they were corresponding to red violet blue for f_1, f_2, f_3. And if you add up column 1, 2, 3 which are marked for $\lambda_1, \lambda_2, \lambda_3,$ the sum are all 1.

0.1, 0.1, 0.8,
0.1, 0.7, 0.1,
0.3, 0.6, 0.1,
0.4, 0.5, 0.1,
0.5, 0.4, 0.1,
0.6, 0.3, 0.1,
0.7, 0.2, 0.1,
0.8, 0.1, 0.1.

So, all the values here add up to 1 remember that. Now as I solve them I get the following values and the objective function is as it was the weighted values. The weighted values being $\lambda_1 \times f_1 + \lambda_2$. This would be $\lambda_2 \times f_2$ and final being $\lambda_3 \times f_3$. And the values which I get and mark these are the decision variables. So, this is x_1, x_2, x_3 and the objective function which is convex combination of f_1, f_2, f_3 is given.

Now immediately you may jump to the conclusion and that may be an interesting snapshot here. If you consider these two points, two sets of points. So, let me use the if you use this one 32 here, 32 here objective functions are same, but interestingly the decision variables are 0, 0, 20 and 0, 20, 0 which can be considered as points for the same objective function value. Now if you look do we have any other sets of points for the case if I use the green color this is 26, 26 objective function and the pyrite points are 0, 0, 20 and 0, 0, 20, 0. Now if I consider these two, the points are same, but the objective function values are different why because the weights are different that should be the continuing the discussion again $\lambda_1 f_1$ this is $\lambda_2 f_2$ last is $\lambda_3 f_3$ and the values of λ_1 first column, λ_2 second column, λ_3 third column are given here for different values of λs, sum is 1 remember.

So, it basically the first column are all values of lambda bar which is 0.2 and the second and third column are λ_2, λ_3 and based on that when we solve the linear programming problem x_1, x_2, x_3 are given here and the objective function values are given. So, if I consider again for simplicity 20, 20 the objective function values and the points are 0, 0, 20, 10, 10, 10. Similarly, if I consider this 30 here which you already consider earlier. So, it was 0, 20, 0 and the 30 value here was basically 0, 0, 20.

So, again you will have the same decision where the outputs objective functions are different because based on the λ values we have. So, these are $\lambda_1 f_1$ again I will highlight that this is. So, $\lambda_2 f_2$, $\lambda_3 f_3$ these are λ_1, λ_2, λ_3, sum is 1. These are the decision variables and obviously for the case for inter example these are all 20 decision variables are x_1, x_2, x_3 are all 10, but the combinations which we are taking for $\lambda_1, \lambda_2, \lambda_3$ are different here. What they are? They are for the same output which is the decision variables as well as the objective function they are as follows here.

Point 30%, 20%, 50%, 30%, 30%, 40%, 30%, 40%, 30% corresponding to f_1, f_2, f_3. Similarly, λ_1 here, λ_2 here, λ_3 here, sum is 1 remember this is $\lambda_1 f_1$, this is $\lambda_2 f_2$, this is $\lambda_3 f_3$ and interestingly watch this 20, 20, 20 decision variables if you see here these are the whole set of decision variables, but for a value output objective function 20 the values are 20, 0, 0, 10, 10, 10 and 0, 20, 0. Now, if you go back here for 20 we already have 10, 10, 10. So, apart from 10, 10, 10 we have 20, 0, 0 and 0, 20, 0 that means they can be considered as the set of Pareto points corresponding to an output of 20, but one should remember the λ values are also significant here. The λ values were 40, 20, 40, 40, 30, 30, 40, 40, 20 corresponding to the weightages in they are in percentage weightages given to f_1, f_2, f_3.

These are when λ_1 is 0.5 I am enumerating all of them λ_2 second column, λ_3 third column, this is $\lambda_1 \times f_1 + \lambda_2 \times f_2 + \lambda_3 \times f_3$. Sum of all the λ is 1 as usual the decision variables and combinations are given here. And if you see here there are values of the objective function 26, 22, 20 and 24, but obviously we have other such objective function outputs here also 26 is here. So, one Pareto point is 0, 20, 0 another point is for 26 is 20, 0, 0. Here for 24 one Pareto point is 20, 0, 0 here one point is 0, 20, 0. For a value of 20 we have already seen that 10, 10, 10 was there 20, 0, 0 was there 0, 20, 0 is there and finally, the value of 20 can be obtained with values of 0, 0, 20, also depending on the combination. For a values of λ_1 as 0.6 column 1, λ_2 column 2, λ_3 column 3. So, this is first part of the objective function, second part of the objective function which is $\lambda_2 f_2$, third part of the objective function $\lambda_3 f_3$ and the sums of the lambda is 1 as you can see adding up the corresponding cell values for first, second, third column and row wise you are adding. The values of decision variables are given here and 28, 24, 22 is given here you can just double check with 22 is there here if you see is 20, 0, 0 22 here is 0, 20, 0 and if I want to see any other 22 let us see there is a 22 here is 20, 0, 0. So, any combination so here you have 22, 0, 0, 0, 20. So, 22 can be obtained by all combinations of 20, 0, 0, 0, 20, 0, 0, 0, 20, but remember the weights are changing which is λ_1, λ_2, λ_3. Second last table here is for λ_1, 0.7, λ_2 as given as 0.1, 0.2, λ_3 given remember sum at up to 1, this is $\lambda_1 \times f_1 + \lambda_2 \times f_2 + \lambda_3 \times f_3$ and the decision variables are this 20, 0, 0, 20, 0, 0 for objective function 30, 26. Finally, for λ_1, λ_2, λ_3 given in column 1, 2, 3 and obviously the sum is 1, the objective function is $\lambda_1 \times f_1 + \lambda_2 \times f_2 + \lambda_3 \times f_3$ decision variables solved are 20, 0, 0 give a value of 32. So, I have enumerated all the combinations of lambda in jumps of 0.1 and shown the decision variable values and the objective function values.

Now, let us collect all the data. So, first we correct this one this is $\lambda_1 + \lambda_2$, this is $\lambda_2 f_2$, $\lambda_3 f_3$ and for different values of λ_1, λ_2, λ_3 we get the value of 32 are the objective function which is shown in the last column which is here and the combinations of the decision variables are as given here 0, 0, 20, 0, 20, 0, 20, 0, 0. So, if I for the time in only concentrate for the values of decision variables which can be considered as a Pareto optimal point because we get a value of 32 for all the three cases of objective function then the Pareto optimal points are 0, 0, 20 again I am repeating 0, 0, 20, 0, 20, 0, 20, 0, 0. Now, if I consider objective function of 30 the Pareto points are 0, 0, 20, 0, 20, 0, 20, 0, 0. Now, the question would immediately come up that we are getting the same Pareto points that is because we are taking the values of lambdas differently. So, I will draw it again for different values of λ_1, λ_2, λ_3.

So, this is $\lambda_1 \times f_1 + \lambda_2 \times f_2 + \lambda_3 \times f_3$ we get 28 in one case for some concepts of combinations of λ, 26 in another case for other different combinations of λ and 24 for another third case for other third sets of combinations of λ. The values which are given here x 1, x 2, x 3, correspondingly the decision variables are 20, 0, 0, 0, 20, 0, 0, 0, 20 that should be looked on from the point of view as the Pareto points. Let us for the time being ignore in the case of the λ values. Finally, $\lambda_1 \times f_1 + \lambda_2 \times f_2 + \lambda_3 \times f_3$ the values are 22 objective function and 20 objective function and the decision variables continue to be 0, 20, 0, 0, 0, 20, 0, 0, 0, 20. Now, the reason why I am going to draw it for the last one which is this a little bit orange and yellow and little blackish one color here is because it has got 4 points and it will be easy for me to analyze.

So, let me draw this set points and I am ignoring the concept that the lambdas are there, but I am only concentrating on the objective function value and the Pareto points. So, I will basically do that and create a now here comes the. So, I will only consider the last 4 whether objective function is 20 and the points I will repeat are 20, 0, 0 the third one and 0, 20, 0 the second one 0, 0, 20 the first one and last one being 10, 10, 10. If I draw the surface consider this is x_1. So, these are all orthogonal one is on the floor, another is vertical up and another is the wall going towards you.

So, based on that I draw. So, this mark is x_1, this I mark as x_2, this I mark as x_3 what are the points 20, 0, 0. So, let us take 20, 0, 0 and use same color they would not be much of it. So, same color. So, if I mark 5, 10, 15, 20, 5, 10, 15, 20, 5, 10, 15, 20.

So, if I consider this, this is 20, 0, 0. So, this is 20, 0, 0 this point is 0, 0, 20 because x 3 is value and this value is 0, 20, 0. So, if I draw it this is the slanted triangle which I have. So, all the points and interestingly if I consider the point 10, 10, 10. So, I will go 10 on to the right x_1, then I go come forward for x_2 and then I go up. So, it will be some point in the three dimension which is 10, 10, 10.

So, it will come out then it comes out then it goes up. So, this is the 10 would be the point. So, it will be somewhere here even though at actually is in equilateral triangle it should be the midpoint, but I have tried all the set of points would be the pareto surface where the objective function is as I calculated was given as 20. So, with this we will end this class and continue discussing more about pareto points and all were linear programming and the same analysis can be done for the integer programming. We will basically come to the quadratic programming and non-linear programming later on.

Thank you very much for your attention. Have a nice day.