**Multi-Criteria Decision Making and Applications**
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**Week 07**
**Lecture 33**

Welcome back my dear friends participants, this is the multi criteria decision making course under NPTEL MOOC and my good name is Raghu Nandan Sengupta from the IME department at IIT Kanpur in India. So this is the 33rd lecture out of the 60 lecture course which is spread over 12 weeks and each week we have about 5 classes each class being for half an hour. Now if you remember the broader umbrella or this concepts under which we were discussing lot of concepts and ideas is multi criteria decision making and then we are undergoing the understanding of multi objective decision making. Then we will consider multi attribute decision making and multi attribute utility theory and then proceed accordingly as was discussed in the syllabus when we started the class. So first part is quantitative then qualitative like objective, subjective concept, more mathematical modelling, lesser mathematical modelling. Now if you remember in the 32nd class we were discussing the problem where harvesting has to take place and the person has 4 different crops to cultivate, irrigate and there was one point which I mentioned that when you consider the concept of revenues, I will come to the problem.

When you consider the revenues you have one set of objective function but the constraints are same. Then you had the cost function, so consider separately with the same objective functions. Then you can analyze the problem from either a net profit perspective or a net loss perspective. Now the interesting part for this problem was that they were both negative and positive terms considering the net profit and the net loss.

So the problem would basically if considered from the net profit or net loss would have both the positive and negative terms that means producing or not producing would basically either push up the profit or lower down being it more on the lower side for the loss such that a balance has to be made. So considering that it is a linear programming example for whole programming and we will consider more such problems but the problems will be discussed not from the practical point of view that means there is an example it will be considered with a very simple mathematical model. The reason being it is in three dimension, so it will be easier for us to communicate how the overall the feasible space and how the solutions would look like. And having said that if you remember we had discussed in simple concept how multi objective problems can be solved either you weight them with the function of lambdas or you basically consider the maximization or the minimization of only one objective function and bring the other objectives as a part and parcel of the constraints. So we will highlight that.

The same concept where we consider a convex combination of the objective functions and also the considering the case where you can consider the one of the objective function has to be optimized and the rest can be passed on to the constraints we will consider for the nonlinear programming also. By the word nonlinear programming I mean the simple concept of quadratic programming. So to understand the general concept of the problem where there were four crops, so they were rice, wheat, maize and bajra. So the problem is the farmer cultivates these crops. So the problem we have discussed but I am just mentioning selling price per kg are given as respectively as 28, 25, 30 and 24.

The cost, selling price would be a more to for the revenue, the cost per acre for the four crops are respectively 33,200, 30,500, 35,750 and 27,000 while the output per acre for the same for these four crops are 1250 that is 1250, 1150 and 1200 and finally being 1100 and based on that we already have the information that the total acreage is 1000. 2% of that cannot be utilized and another two were obviously the crops have to be produced in positive quantity either 0 or more than 0. And also there were other two constraints depending on the area where the person is growing the crops. So due to water shortages rice cannot exceed as it was mentioned, rice cannot exceed that of the combined acreage of wheat and bajra and wheat being the staple diet hence its production should be more than 50% of the combined production of rice, bajra and maize. So we had the revenue function which I am marking in green, the cost function which I am marking in red. So either you optimize the revenues or optimize the cost. Optimize the revenue means increase it, optimize the cost means decrease it, but the constraints remain the same.

When I consider the constraints those four which were mentioned they are as follows. Constraint one I will use the dark red color, you can use 98% of the total acreage. So $X_1 + X_2 + X_3 + X_4 \leq 980$. So that $X_1, X_2, X_3, X_4$ as we have said and I kept repeating is the acreage for each and every crop rice, wheat and bajra and maize. Similarly due to the production constraints about water requirement for rice and wheat being the staple diet, the second constraint I am marking in blue and the third constraint I am marking in violet and obviously I have considered each of them as less than 400. There in the problem we had considered between 100 and 400. Now comes the formulation, I would not solve this problem for a reason because this being in four dimension it will be easy not very easy for us to visualize. So in order to visualize we will take a three dimensional problem with variables as $X_1, X_2, X_3$. Now if I consider the maximization case only, so I will put the maximization case as in green. So this is the maximization and the constraints which were already discussed I will mark them with the dark red color as 1 for total acreage, 2 being for rice and 3 being for staple diet of wheat and 4 being less than 400.

Now if I consider the minimization problem then it was basic minimization of the cost. Now what we have seen that if you only minimize the cost best would only nothing else to be considered. So obviously it is best to produce nothing, so the minimum cost would be 0. Now what we do is that the moment we start producing the cost would be more than 0 step by step. So I am not considered the sign cost in the sense it is a negative flow of money out of the pocket but I am considering the quantum wise. So if I only concentrate on the cost, the cost function was this. It was obviously greater than has to be greater than 0. In case we produce 1 unit of any of this crop of these 4, the cost will be negative but if I consider without considering the sign it will be positive. So that is why I am increasing the value 0 which is the minimum value with the amount of $\delta$. So $\delta$ =10, 20, 30, 40, I can choose the value accordingly and find out the values of the maximization function based on the fact that the last constraint which is marked in blue is true for different values of $\delta$ and here $\delta > 0$. Now which means that we are as there are only two objective functions, so when we optimize one the other comes into the constraint that means the number of constraint increases by 1.

Now the question would be can we formulate the problem in the similar way where we try to minimize the cost with the constraint corresponding to the revenues coming to the constraint. Answer is yes. So here is the problem. So we can minimize the cost, so it is minimization problem because cost has to be minimized and the corresponding constraints are exactly the same as they were in the last slide. So this was one related to the acreage of 22% not being utilized, this is related to the wheat, what are utilization was for the rice. Third one was because wheat was a staple diet and fourth one was basically $\leq 400$ and if I consider the added constraint coming from the cost, if I solve the revenue function the value for the revenue function maximum value would be this value as given here which is 211 lakhs. Now that was the best profit.

Now the moment I want to basically increase some production that means cost will become negative when I am minimizing it. Obviously the amount of the profit will decrease. So a negative component has to be added. So that is why it is $- \delta$, where $\delta > 0$. So if I look at this constraint, added one which is related to the cost revenues considering we are trying to minimize the cost, it is $\leq$ the max value - the $\delta$ value and for the previous problem, when we were trying to maximize the revenue based on the one of the extra constraint being the cost which has to be $\geq 0$ because 0 was the minimum value greater than, then it will be addition of this positive quantum $\delta$. Now how it would look like in a simple problem which we can understand let us solve it.

Now to make things a little I am not saying to make it difficult but to make it much more comprehensive I have not considered two objectives but I have considered three objectives, this point one. Point number two for simplicity it is not that the problem cannot be solved for simplicity I have taken all the optimization being maximum. So maximum of the first function which I mark as $F_1$, maximization of the second function which I mark as $F_2$ and maximization of the third function which I mark as $F_3$. So the point of maximization could also have been replaced by minimization. So this could have been also minimization of first function. Similarly it could also have been minimization of the second function, it could also have been minimization of the third function. So the idea remains the same

how we solve it only greater than less than sign the $\delta$ value which I mentioned will be adjusted accordingly. Now what are the constraints? So it is a 3D one we have to visualize and I will draw it. The first one is the $X_1 + X_2 + X_3 \leq 30$ and by the way all $X_1, X_2, X_3 > 0$. The second one is $X_1 + X_2 \leq 20$. The third one is $X_1 + X_3 \leq 20$. The fourth one is $X_2 + X_3 \leq 20$ and obviously as I said I am repeating it $X_1 + X_2, X_3 > 0$. Now to start the problem we will forget that it is a multi objective problem. We will only concentrate on the single objective. Single objective which is $F_1$, single objective which is $F_2$, single objective which is $F_3$. Optimize them and then proceed. So we will consider maximization. Again maximization could also have been minimization. So I will again write it if it is a repetition please excuse me but I am trying to highlight it accordingly. So the max could have been min                                                                                                                      also.

So here it is maximization of the first function which is shown in red. By the way why I mentioned red, why I mentioned violet and why I mentioned blue would become very clear as we solve the problem. So the transformation of the objectives into the constraints step by step for these three problems Y3, I will come to that later. Step by step as we do so the coloring coding will make things much much easier for you to all to understand and for me to explain. Now Y3 problem is because if you consider there are three objectives, so technically we would basically be considering the first objective and transforming the other two objectives into the constraint.

Then take the second objective and transform the first and the third into constraint and the last case would be take the third objective and transform the first and the second into the constraints. So the first objective I will denote the optimum value I will denote by $F_1^*$ and it is a red color C. The second objective which is in violet, so when I call the coloring scheme you will understand. So the first objective just for repetition I am saying it is $2X_1 + X_2 - X_3$ shown in red color. The second objective shown in violet color is $-X_1 + 2X_2 + 3X_3$ and it is marked the optimum value when you find out the single objective $F_2^*$ again violet.
Third objective shown in light blue is $X_1 - X_2 + 2X_3$ and when we consider the four sets of constraints, so if you read it when you consider the four sets of constraints what are these? These are they are already there the four sets are technically these 1, 2, 3, 4 and the last one which is related to $X_1, X_2, X_3$ being $> 0$ is 2 because they are all positive. So when you consider these four constraints the objective functions solution for separately for first, second, third are $F_1^*$ in red color shown, $F_2^*$ in violet color shown, $F_3^*$ in blue color shown. Now let us solve it, it is a simple linear programming and we solve it. The problem is maximization of the first objective I will denote as objective 1, number 1. The constraints remain exactly the same as they are, so it is 1, 2, 3, 4 and obviously the fifth one is this one,            so            I            will            write            it            5.

When I solve it the solution point is for $X_1 = 20$, for $X_2 = 0$, $X_3 = 0$. So the objective function when we solve it, it is 40, how does it come? It comes as let me use a darker color, so this is $X_1, X_2, X_3$ solution is $20 \times 2 + 0 - 0 = 40$. This is the value of 40. Now let us solve the second objective separately again separate single objective. So when we solve it the objective function is here what I am marking the five constraints are here.

When we solve it the value of $X_1$ which was 20 now 0, $X_2$ which was 20 now it is 20 and $X_3$ was 0 is 0. So the objective function is $-0 + 2 \times 20 + 0 = 40$, as shown here in the violet color. Second problem also solved a single objective. Let us solve the third problem. Third problem is given here which I have already repeated the coloring scheme.

So it is blue in color which is $-X_1$, so this would be plus. So this is the objective function the constraints are again 1, 2, 3, 4, 5. When I solve it $X_1$ which was first was 20 then 0 is still 0, $X_2$ which was in the first problem objective function 1 was 0, objective function 2 solution was 20 now for objective function 3 is 0 and $X_3$ which was 0 in both the cases when we solve the objective function 1, objective function 2 now it is for objective function 3 is 20. The solution is $+0 - 0 + 2 \times 20 = 40$ as shown here by $F_3^* = 40$. So this coloring scheme now makes sense that we can differentiate.

So the corner points in one sense 20, 0, 0, 0, 20, 0, 0, 0, 20 are the corresponding optimal points for objective function 1 shown in red then for objective function 2 shown in violet and last being for objective function 3 which is shown in blue and these objective functions are $F_1$, $F_2$, $F_3$. Now we will slowly bring the constraints to be in a such a way that the objective functions which are not being considered to be optimized will be considered as a part and parcel of the constraints. So the problem is first case is I am taking the objective function 1, so I will write it OBJ number 1 and the constraint 1 is there perfect, constraint 2 is there, constraint 3 is there, constraint 4 is there and constraint 5 is also there. Now the violet and the blue one are the corresponding objective functions 2 and 3 being brought into the constraints. So now is an objective function maximization problem we have solved them individually and we found out the values individually for each $F_1$, $F_2$, $F_3$ which is $F_1^*$, $F_2^*$, $F_3^*$ was 40 respectively for all the three cases.

So generally if you want to find out we will try to basically change the values of these $F_1$, $F_2$, $F_3$ corresponding to the case when they are in the constraints and the solve the problem accordingly. Now in this case objective function 2 and 3 are in the constraints, so $F_2^*$ violet, $F_3^*$ blue should be decreased from the value of 40 in order to see that how that we can reach the values corresponding to the case that we have fixed the values of $F_2$ objective function 2, fix the values of $F_3$ objective function 3 and trying to optimize the objective function 1. So, what we mean is this let me use the violet color, so this was actually 40 single. So in case if I subtract say for example 20 which means the objective function 2 is being maintained at 20 and if objective function 3 which is blue and use the blue color which originally was 40 single objective and I am taking subtracting 30 which is 10. So based on the fact that the right hand side for the last two constraints this should be $\leq 20$ and this should be $\leq 10$ and based on that we will solve and find out that what is the value of the objective function $F_1$ which means that once we have fixed $F_2$ at 20, $F_3$ at 10 the value of $F_1$ which will get this whole set corresponding to the values of $F_1$, $F_2$, $F_3$ obviously $X_1$, $X_2$, $X_3$ which are the decision variables will come later on will give us the best optimum solution which is required and we will forward it the correspondingly problem accordingly.

Now the issue is that this value of 20 and 10 which I have considered considering that I have subtracted 40 from the constraint second last one which is related to the objective

function 2 and I have subtracted 30 from the constraint last one which is corresponding to objective number 3 can they be changed answer is yes they can be changed. Would we get different results? Yes we will get different results. Can they be plotted? Yes we can plot them and we will consider them accordingly. Now we solve it. If we solve it if you remember there were values of $\delta_2$ and $\delta_3$ which was corresponding to the second last constraint and the last constraint second last being for the objective function 2 and third last constraint being for objective function 3.

So I have also continued using the coloring scheme as it is $\delta_2$ in violet, $\delta_3$ in blue and obviously the $F_1^{**}$. Why I am using a double star because $F_1^*$ was the best value, single objective 40 for each $F_1$ 40, $F_1^*$40, $F_2^*$40, $F_3^*$40. So we will take $F_1^{**}$ to denote different combinations of the constraints corresponding to second objective, third objective how they affect $F_1$. So I will mark take it in the last column. Now I take a value of $\delta_2$ 22, $\delta_3$ 22 which is what the left hand side actually would be basically 40 - 22.
So that means $\leq 18$. Here also it will be so this is = 18. This is 40 - 22 = 18. So $\delta$ 23 would give me 17, this is 17, this is 15, this is 15, this is 15, this is 18, this is 18. If you look at this means the first and second column I am proceeding. This is = 15, this is 15, this = 13, 13 and 15. So based on that I will solve. So the first problem which I will solve which I will mark in this one. This is let me use a different color. This is case 1 for maximization of objective 1 with the objective 2 or objective 3 in constraints and less than equal to respectively as 22, 22. The optimum value comes out to as 19, 1, 0 and the objective function $F_1^*$ is 39. Then I take case 2 which is 23, 23 that means 17, 17 less than equal to, for the second last constraint, third last constraint values come out to be for $X_1$, $X_2$, $X_3$ as 18.5, 1.5, 0 and the value is 38.5 for the objective function. Similarly for the less than equal to constraint for the second last and last constraints are given which I have told I am not going to repeat it are shown in the first column, second column and the corresponding values of X's are found out $X_1$, $X_2$, $X_3$ when X's means the vector and the values of $F_1^{*s}$ are calculated. So having said that it will mean that with the addition constraint we have been able to solve the first objective giving very interesting results. We can repeat that for objective function 2 being the object in the function to be optimized then objective function 3 to be optimized. But having said that the other objectives which are not being optimized would automatically come in the constraints.

We will continue discussing this problem for objective function 2 then objective function 3 draw the diagrams and then go into the concept of trying to understand in the quadratic programming case which is the nonlinear programming. With this I will end this 33rd lecture. Have a nice day and thank you very much. Thank you.