Multi-Criteria Decision Making and Applications

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Week 06

Lecture 29

Welcome back my dear students, dear friends and participants for this course titled multi criteria decision making under NPTEL MOOC series and my good name is Raghunandan Sengupta from the IME department at IIT Kanpur. So as you know that we are in the set of classes for the sixth week and this is a 12 week course for spread over 60 lectures and each week we have 5 classes each class being for half an hour each. So the broader overall context based on which we have been discussing on the slides and the lectures are multi criteria decision making, multi objective decision making, multi criteria decision making is the broader umbrella. Under that you have the multi objective decision making and then the multi actable decision making there is a difference which I keep repeating. Multi objective is more quantitative, continuous, easy to understand and very easy to communicate. Multi attribute is more subjective, qualitative in the discrete space and not that easy to communicate.

Solution techniques would take cue from the concept of multi objective decision making and some concepts or different qualitative methodologies of ranking and further on as we have discussed about univariate utility theory we will consider the concept of multi attribute utility theory also and this is the 29th lecture as I mentioned. Coverage would be Pareto optimality we will again visit that with examples why I will come to that later. The concept of Pareto optimality in more details with example and then go into the concept of goal programming. Now consider the simple example it can have different flavors.

So let us consider this simple example you have I will just mark it you have a 3D problem because we will be able to analyze it much better and the coordinates are discrete for simplicity they are given as x_1, x_2, x_3. x_1 can take only values discrete 0, 1, 2, 3. x_2 can only take discrete values 0, 1, 2, 3 and similarly x_3 can take discrete values 0, 1, 2, 3. The functions to be analyzed are they are not we need to be either maximize or minimize I will come to that later, but first let me state the function. The functions are $x_1^2 + x_2 \times x_3$ so this it is technically quadratic because it is square here and obviously x_2, x_3 is also the concept of square for the variables.

The second function f_2 is $x_2^2 + x_2 \times x_3$ and finally, the function 3 is $x_3^2 + x_1 \times x_2$. What are the constraint or what is the constraint to make our life simple for analysis I am only going to take one constraint can be expanded also. The constraints are $x_1 + x_2 + x_3$ = exactly 3. So obviously if I consider these values few of the values which would not be allowed and that is very obvious. If I take a value I will few examples which are not applicable.

So consider x_1, x_2, x_3. If I take x_1 value of 0 which is applicable x_2 a value of say for example 1 and x_3 a value of 3 is that allowed no because the sum is 4 not 3. So this is not allowed I put a - sign this is a - not allowed or let it better would be if I put a no. Consider next value this is 2 this is 0 and this is 2 x_1, x_2, x_3 are 2, 0, 2 allowed no, because sum is 4. Consider the values of 1, 1, 1 allowed yes because sum is 3.

Sum is exactly 3 value of 1, 0, 1 allowed no, sum is 2. Consider the value of 0, 0, 3 allowed yes. So we have different combinations which are allowed different combinations which are not allowed and if you see I should make this heading of x_1, x_2, x_3 not in red color because red is no and green is yes. So I will use the black color here. This is the situation red is no green is yes and we see it should basically follow the norm. Number 1 and number 2 obviously all these values x_1, x_2, x_3 are discrete and they are in the domain space of 0, 1, 2, 3 discrete as I said. Based on that I mark all the relevant values. So these are the relevant values. So the value of x_1 is 0, x_2 is 0, x_3 is 0 and x_3 is 3 allowed yes because sum is 3. x_1, 0, x_2, 1, x_3, 2 allowed 2 + 1 is 3. Third I am not writing anything because I have to plot it which I will come later.

So I do not want to clutter it. x_1, 0, x_2, 2, x_3, 2, 1 allowed 2, 1 + 1, 3. So I will read the values of x_1, x_2, x_3 corresponding this 0, 3, 0 allowed, 1, 0, 2 allowed, 1, 1, 1 allowed, allowed means yes, 1, 2, 0, yes, 2, 0, 1, yes, 2, 1, 0, yes, 3, 0, 0, yes. All the sums are exactly equal to 3 and the values of x_1, x_2, x_3 are either 0 or 1 or 2 or 3 and you can see it from column 1. So there is no values which is not in that set of 0, 1, 2, 3.0 is there, 1 is there, 2 is there, 3 is there. For x_2 similarly 0, 1, 2, 3 is there and similarly for x_3 0, 1, 2, 3 is there. Now what are the functions? If you go back the functional value of f_1 is given by this. I will use a color, see for example violet. So this is $x_1^2 + x_2 \times x_3$.

So let us go back. x_1^2 0, $x_2 \times x_3$ is 0, so 0 + 0 is 0. Then $0^2 \times 1 \times 2$, so sum them up 2 which is written there in the fourth column. $0^2 + 2 \times 1$ is 2, so $0^2 + 2 \times 1$ is 2, in there. $0^2 + 3 \times 0$ is 0. $1^2 + 0 \times 2$ is 1. $1^2 + 1 \times 1$ is 2. $1^2 + 2 \times 0$ is 1. $2^2 + 2 \times 1$ is 4. $2^2 + 1 \times 0$ is 4. $3^2 + 0 \times 0$ is 9. Fourth column which is F_1 is taken care of. I go to the second value which I will highlight using the light blue color which is $x_2^2 + x_1 \times x_3$. So, x_2^2 x_2 is the middle value here which is the second column. So, if I consider $0^2 + 3 \times 0$ is 0, $1^2 + 2 \times 0$ is 1, $2^2 + 0 \times 1$ is 4, $3^2 + 0 \times 0$ is 9, $0^2 + 2 \times 1$ is 2, $1^2 + 1 \times 1$ is 2, $2^2 + 1 \times 0$ is 4, $0^2 + 2 \times 1$ is 2, $1^2 + 2 \times 0$ is 1, $0^2 + 3 \times 0$ is 0 which is F_2 done.

Now, come to F_3 I will use this a little bit darkish orange not orange darkish yellow which is x_2 $x_3^2 + x_1 \times x_2$. So, we will concentrate on F_3 value it is not colored yet, but I will come to the reason why I have not colored it I will come to that. So, $3^2 + 0 \times 0$ is 9, $2^2 + 1 \times 0$ is 4, $1^2 + 2 \times 0$ is 1, $0^2 + 3 \times 0$ is 0, $2^2 + 0 \times 1$ is 4, $1^2 + 1 \times 1$ is 2, $0^2 + 2 \times 1$ is 2, $1^2 + 2 \times$

0 is 1, $0^2 + 2 \times 1$ is 2, $0^2 + 3 \times 0$ is 0. And finally, the last column which was not mentioned in the earlier slide I have taken the sums why I will come to that later. So, if I take the sums so it is 0 + 0 + 9 is 9 then 2, 1, 4, 7, 2, 4, 1 I am adding them up 7, 0, 9, 0, 9, 1, 2, 4, 7, 2, 2, 2, 6, 1, 4, 2, 7, 4, 2, 1, 7, 4, 1, 2, 7, 9, 0, 0, 9.

Now, let us concentrate on the columns marked F_1 and F_2 which I have marked it here and what I mean by Pareto optimality using this coloring scheme if you see the coloring schemes are yellow, dark yellow and blue and finally green. So, I will create and Cartesian this is two dimension create a Cartesian coordinate in order to explain that. So, I will add a blank file and then discuss that. So, this is the blank slide I have created. So, let us consider only the yellow part 2, 1, 1, 2 and there is this you are measuring F_1 and F_2.

So, if I consider I will measure F_2 here, F_1 here and this is so this is 0.12 this is 12. So, mark it 1212. Now, let us see the yellow mark which I said it is 2112. So, 21 is here and 12 is here. So, what does this point set of points mean? If I join a line these two points are on the same level considering the net worth if I consider the sum of F_1 and F_2 they are fixed, but the individual values of F_1 and F_2 have been changed. So, F_1 if I consider for point A and for point B, F_1 has changed from 2 to 1 and F_2 has changed from 1 to 2 sum remains 3 for the individual points have changed. So, if I consider this dotted line it is akin to an optimum parity of frontier or parity of line. So, as the value of the sum of $F_1 + F_2$ remains same individually they can change. Now let us consider the next set of points yellow we have already done.

So, let us go to the set of points which are given by green 1441. If I join so the set of points is 4114 again the sum of this functions F_1 and F_2 is 4 + 1 or 1 + 4 is 5 individually they are at different values individually, but the collective value is same. Again this violet line is a Pareto so called line where all the set of points have different coordinates, but the sum of F_1 and F_2 remains same. Next let us come to the set of points blue which is 2442. So, again if I join this point is 42 this point is 24 again sum is 6 individually they are different.

Finally, I have the point as 9009 which is in the dark yellow 9009 and use the color 9 would be somewhere here consider and 09 here. So, again if I draw it is out of the screen, but I will draw again it is a straight line. What is and again this is Pareto 9 sum is 9 individually they are different. What is important to note is I will use the blue color if I consider all these lines are parallel to each other they do not intersect the sum of the value for each of these lines red violet green and dark yellow is increasing as I go up onto the right top corner. That means the net value if you are on along any of the line is fixed the net value of the sum of F_1 and F_2 increases.

So, they are just like the indifference curve which are basically parallel to each other never intersect and for the case when you want to optimize maximize both of them. Obviously, if you remember those constraints were brought into the picture in order to give a much more practical feeling. So, whether the line was $x_1 + x_2 + x_3$ was 3 = 3. So, these 4 sets of the points are the Pareto frontier such that a decision maker based on the fact he can choose any one of the points x_1 x_2 x_3 such that each are discrete 0 1 2 3 would obviously lie on any one of these 4 lines which are shown in red violet green and dark yellow anything out is not possible all of them are parallel in the overall net value along each individual line is fixed and differences are there as you move up onto the right top corner the values of the sum increases. Now I consider so we have considered F_1 and F_2. Now let us consider F_2 and F_3 again the same colouring scheme exactly the same.

So, here if you see the yellow part I do the same analysis draw the same curve the yellow part is as shown here for values of F_2 and F_3 respectively as 2 1 1 2 sum is 2 + 1 or 1 + 2 is 3 fixed individually they are different. So, if I draw them the Cartesian coordinate they would be a straight line and each individual point along the straight line gives the same sum but the individual values of F_1 and F_2 are different. Similarly when I move on to the next set of points which is the green one which is 1 4 4 1 1 4 4 1 are basically I am talking about F_2 and F_3 respectively sum is 5 1 + 4 is 5 4 + 1 is 5 and individually they are different coordinates but the sum remains same and also if I compare with the yellow part with the sum is 3 this is 5 that means I have moved from a value from 3 to 5 corresponding to the concept that the constraint is on the already applicable. When I come to the blue set of cells which is highlighted here in the slide the F_1 and F_2 and F_3 values are 2 4 4 6 individually they are different but the sum is 6. So, when I join these points 2 4 4 2 as I have done in the previous figure the sum is 6 they are all the set of points along these straight line they give the same sum of $F_2 + F_3$ individually are different and also yellow with sum was 3 green sum was 5 blue the sum is 6 which means I have moved parallely one step from 3 to 5 then 5 to 6.

Six means 3 5 6 are the sum of $F_2 + F_3$ finally when I go to the dark yellow part which is basically 0 9 and 9 0 which are the respective values of F_2 and F3 the sum is 9 for 0 + 9 is 9 9 + 0 is 9 individual values are different. So, if I join this coordinate 0 9 9 0 is a straight line all the set of points along the straight line gives you the same sum individually are different and also the sum of 9 is more than the sum of 6 that is 2 4 2 + 4 is 6 and 4 + 2 is 6 that means and there is an have been a parallel shape. So, the way of trying to draw F_2 and F_3 would be exactly equal to this only change would be. So, this would be F_2 and this would be F_3 then let us come to I am going a repetition but I am going slow when I consider F_1 and F3 because why I am taking 2D because easy for us to understand again the same coloring scheme yellow green blue orange yellow being for the values of 2 1 1 2 2 1 being for F_1 F_3 1 2 being for F_1 F3. So, I will first say the F_1 value then the F_3 value the sum is 1

2 + 1 1 + 2 is same 3 individually are different when you draw this join this two points is a straight line straight line like this the red part here 2 1 1 2 but in this case let me change it beforehand this would be F_3 fine F_1 when I come to the green highlights itself the respective values of F_1 F_3 is 1 4 4 1 sum is 5 individually they are different when I join this points they give the same satisfaction level of sum of F_1 + F3 individually as I said different.

So, the sets of points are the violet one as you see 4 1 1 4. When I come to the blue cells 2 4 4 2 for F1 and F3 the sum is 6 individually they are different when I join these two points again a straight line all the set of points along the straight line gives me the same value of F1 plus F3 and when I draw it this is the green line 4 2 2 4. Finally, when I take the orange this dark yellow green line and draw yellow cells values of F_1 F_3 0 9 9 0 sum is 9 that is 0 + 9 is 9 9 + 0 is 9 individually the values of F_1 F_3 are different because one is one coordinate is 0 9 another is 9 0 the sum of points joining on line joining them have give me the same utility or same value of $F_1 + F_3$ and if I draw it this is the dark yellow and again if I draw it all the lines for example 1 example 2 example 3 are parallel the values along any line is fixed and the values increase as I go on to the right top corner this is the increasing case. Now I consider the example as given here and I will highlight it. Now when I consider the three cases collectively F_1 F_2 F_3 and the right column is basically the sum of F_1 F_2 F_3 and if you see the values are given 0 0 9 is sum is 9 2 1 4 7 2 4 1 7 0 9 0 9 1 2 4 7 2 2 2 6 1 4 2 7 4 2 1 7 4 1 2 7 9 0 0 9.

So, this is the sum of the three cases and now consider these movements and consider is a minimization I want to do individually for F_1 F_2 F_3 and check whether the if they are individually minimized. So, technically I would think that individual sum should also be minimized whether it is possible or not let us consider. Now concentrate on this dark yellow set of cells consider F_3 is to be minimized so I should go from 2 I should jump to a lower value 1 so let us do that which is shown in the dotted blue part. So, it is decreased I am very happy but see here two things I have decreased for F**3** fine individually but the sum which was 6 initially has now increased to 7 which is not done because collectively I am basically compromising why this compromise is there because if I move from 2 to 1 for F_3 when I move for only consider F_1 it remains as 2. So, this dashed circle part this one for F_1 remains in the orange cell and the green cell same 2 which is fine I am saying and the same level for F_1 F_3 has decreased from 2 to 1 which I am happy but interestingly the value for F_2 which was initially 2 has now increased to 4.

So, which is increasing if I want to optimize so this should not have been the case. So, in this case the value of 2 goes to 4 for F_2 hence the total sum rather than 6 has increased to 7. So, which is not a way of trying to handle the concept of the Pareto optimality and subsequent problem because this is a compromise. If I consider the idea of say for example, trying to maximize so if I consider the concept of maximization in I initially consider the minimization a maximization consider on this concept of F_1 F_1 is increased 2 to 4 fantastic I am happy. So, I am considering this dark yellow and the bottom green part F_2 remain same fine no problem but interestingly it has been maximized 6 has gone to 7 it is a maximization case but the idea of trying to increase F_3 is not happening because it has decreased from 2 to 1 collectively fine I have increased but individually I have compromised in the second example.

In the first example individually 1 is benefiting collectively there is a dis-benefit. So, these two examples were collectively dis-benefit for movement collectively benefit for the movement but individually they are being compromised it will be considered further on in further examples. This example has not finished I will consider more such different combinations of the example later on and with this I will end this class have a nice day and thank you very much. Thank you.