Multi-Criteria Decision Making and Applications

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Week 06
Lecture 27

Welcome back my dear students and friends and participants for this course titled multi criteria decision making under NPTEL MOOC series and my good name is Raghunandan Sengupta from the IME department at IIT Kanpur in India. So, if you remember this is the lecture 27th which is the second lecture in this week, the week which has started for the lecture series and this total set of lectures for this course is 60 number and each week there are 5 lectures each week for half an hour. If you remember we were discussing in details of bi-objective problem and the initial analysis for the linear programming part for the bi-objective was over, the detailed discussion has not yet started. So, we will continue with the problem also continue the detailed problem solution for the integer programming part for the same type of problem, then do the discussions. So, few of the slides which you will see now may be a repetition, but because I did not want to break the continuity. This set of lectures are under the broader umbrella title which I keep repeating just to make things be in a sequential order about the concepts of multi criteria decision making, bi-objective decision making, multi attribute decision making and multi attribute utility theory.

The coverage would be for the upcoming lecture and set of lectures would be concept of parity optimality effective versus inefficient solution, Karusch-Kuhn-Tucker conditions, scales of measurements and goal programming. The problem was this that is why I said few slides would be repeated. Maximization of f_1, maximization for f_2 constraints are same, I am not highlighting time and again, but if the students pay attention the objective is both are maximization. The first step which we were discussing in details in the last class, when we solve the two separate linear programming continuous variables, the decision variables were as given. x_1^* 5.2, x_2^*decision variable was 4.2, objective function value f_1 optimal was 52.2. For the second objective function x_1^* is 6, $x_2^*$is 3, objective function is 93.

When we solve the problem this why I am repeating is because they would be utilized. When we solve the problem using integer decision variables, integer linear programming the decision variables x_1^* is 6, $x_2^*$is 3, when we put and find out the objective function value terms are to be 51. For the second function f_2, the integer values at the optimum solution point $x_1^*$is 6, $x_2^*$is 3 and the objective value is 93. So, this was the feasible region if you remember we have discussed. So, feasible region discussion has been done in few of the problems in details here also.

So, that we will skip for the time being for this lecture, if required later for other problems we will discuss and come directly to the linear programming analysis which you have already done. So, we take different values of λ as mentioned, λ changing from 0 to 1 or considering. So, obviously the other value would be $1 - \lambda$, but here we are taking λ_1, λ_2 such that the $\lambda_1 + \lambda_2 = 1$. λ_1 is the weight for function 1, λ_2 is the weight for function 2 and the bi objective case. Bi objective in the sense they are two objective function, but we look at the problem very intently it is basically combinations resulting in a single objective, but the weightage functions can be considered as a formulation of the bi objective problem.

We took different values of λ starting from 0.1 to 0.5 in this slide. So, obviously λ_2 changes from 0.9 to 0.5 and the corresponding important things which I will highlight because they will be analyzed. The first one is the are the set of values of x_1^*, x_2^* as λ and λ_2 change. Remember $\lambda_1 + \lambda_2 = 1$ and the corresponding. So, interestingly for $\lambda_1 = 0.1$ to $\lambda_1 = 0.4$ the first four values the x_1^*, x_2^* are all discrete for λ_1 0.5 λ_2 0.5 it is not discrete. The corresponding objective values are given here 88.8 when $\lambda_1 = 0.1$ then as λ_1 changes the objective function value comes out to be 84.6, 80.4, 86.2, 72. Finally, continuing λ_1 changing from 0.6 to 0.9, λ_2 changing from 0.4 to 0.1 and the corresponding decision values are here 5.2 to 4.2 all the values are same and all the values same means x_1^*, x_2^* values are same. The objective function values as lambda changes from 0.6 to 0.9 are 68.4, 0.4, 64.0, 8, 60.1, 256.16. Now we plot and plot two important things plot in first case based on the value of λ_1 on the x axis we plot the objective function value for the bi objective case which is $\lambda_1 \times f_1 + \lambda_2 \times f\ 2$ and the line is linearly sloping. Now if I only highlight the first point here blue.

So, what is this is the value when $\lambda_1 = 0.1$ that 0.0 0.0 means we are putting all the weightages in the second function and this value if you see clearly would be equal to the case when we have 93 here continuous variable 93 value. Another case f_1 continuous case it is 52.2 93 52.2 let us come back. So, this is 93 for λ_1 is 0 and this value is 52.2 for the case of $\lambda_1 = 1$ and these corresponding values are for different λ values the objective function. I am just drawing them in order to illustrate.

So, for 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.91 all values are λ changing in steps you mark all the values of the bi objective case. The question would be you could have also drawn for the λ_2 case where rather than have λ_1 in the x axis. So, this is removed what if we have λ_2 you could have also done it in that case the extreme left point which is 93 for $\lambda_1 = 0$ in that case it would be $\lambda_2 = 1$. So, we are moving in the reverse direction λ_2 would be on the left most $\lambda_2 = 1$ on the left most point and $\lambda_2 = 0$ being on the right most point. So, you would have basically the curve accordingly.

Now more interesting is this case if you see the graph we are plotting the values of x_1 and x_2 with respect to λ_1 change. So, we plot λ_1 along the x axis and along the y axis we plot both x_1 and x_2 separately. So, these are the decision variables. Now for the case when $\lambda_1 = 0.1$ the optimum values for the decision variables are $x_1 = 6$ which is here $x_2 = 3$ which is here.

Now as λ_1 changes from 0 then to 0.1 then 0.2 to 0.3 till 0.4 you see for the for the linear programming part again I am repeating for λ_1 is point 0.0 to 0.4 they are 6 and 3 what are 6 and 3 the values of x_1^*, x_2^*. So, let us double verifying. So, see here the values for λ_1 to $\lambda_1 = 0.1$ to $\lambda_1 = 0.4$ the values are which I am circling are all discrete 6 3 as a solution is nothing to do with whether we are solving the integer programming it is just a answers coincidentally are discrete.

And obviously there is a value which I have left out is $\lambda_1 = 0.0$ $\lambda_2 = 1.0$ there also the decision variables are 6 and 3 and the objective function was 93 if you remember why 93 because the function value is only for f_2 because f_1 is not there because λ_1 is 0.

Now after 0.4 λ_1 0.4 starting from $\lambda = 0.5$ till $\lambda_1 = 1$ we see the values are now I will use the is 5.2 and 4.2 throughout. So, does it match our analysis? So see here for the case when $\lambda_1 = 0.5$ the decision variables are 5.2 and 4.2 and if I consider λ_1 0.6 λ_1 0.7 λ_1 0.8 λ_1 0.9 and there is $\lambda_1 = 1$ the values are all same in the sense 5.2 4.2 and that has been illustrated. And the objective function value for $\lambda_1 = 0.1$ we already know is this 52.2. So, if we plotted it so this is basically the x_1 and x_2 values with respect to λ_1 the answer the question can be that can we plot it with respect to λ_2 yes we can plot it. So, λ_2 is basically 1 - λ_1 and we can plot it accordingly.

Now the analysis is for the case when we have the bi-objective case integer programming again the same setup same analysis no change. So, we will have the bi-objective case as given it is $\lambda_1 \times f_1$ $\lambda_2 \times f_2$ and the problem is as given constraint being same only integer part to be solved. So, let us when you solve it and before that for each step we keep changing the values of λ_1 again same thing steps of 0.1 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 and 1. So, in this slide you see the values of λ_1 is changing from 0.1 to 0.5 and obviously λ_2 is what $\lambda_2 = 1 - \lambda_1$. So, the values are given starting from 0.9 to 0.5 and the decision variable values interestingly are 6 and 3 always for the continuous case linear programming it did change for $\lambda_1 = 0.5$ it changed from the discrete values of 6.3 to 5.2 4.2 correspondingly for x_1^*, x_2^*. The following values on the right right most part of the slide which I will mark now are the so called bi-objective problem value which we have which is $\lambda_1 \times f_1 + \lambda_2 \times f_2$ starting from 88.8 84.6 80.4 76.2 72 further analysis with λ_1 changing from 0.6 then to 0.7 to 0.8 0.9 1 is not there λ_1 is 0.1 because that is already solved corresponding values of x_2 change from 0.4 to 0.1 and the corresponding values of x_1^*, x_2^* are 6 and 3 because they are discrete and the and the corresponding objective function by objective case they are 67.8 63.6 59.4 55.2 and for λ_1 here if you see in this slide λ_1 is 1 which is that means we are putting all the weightages for the first function for the integer problem the answer was 51. So, for λ_1 for and the weightages for the other one being 0 λ_2 is 0 the objective function value optimum is 51. So, again what we do we plot in the integer case we plot λ_1 along the x axis and the functional value along the y axis and again it is a straight line as it should be which for λ_1 is 0 the value is 93 which I am circling and each value for different λ can be found out given the graph and the extreme way a part on the x axis is λ_1 is 1 which is basically the would give us the objective function value for λ_2 into f_2 because λ_1 is 0 and λ_1 is 1. So, in that case it will be the first value. So, there is one error in in my saying when we solve the integer programming part this 51 would be here 5193.

So, sorry for this slip this value would be this is 93 and this is 51 and all these values are integers. Why integers? Because the decision variables are integers and the factors which multiply each decision variables are also discrete numbers. When I plot the x_1 and x_2

value with respect to $\lambda_{1,}$ λ_1 being on the x axis interestingly as there is there are no solutions which are non, integers the values always given 6 and 3 6 for x_1^* 3 for $x_2^*$and they are plotted and if you see they are a straight line for the other case they did dip which was 5.2 and 4.2 here. So, interestingly the these graphs based on which we are trying to plot the functional values bi-objective case for the linear programming and the integer programming look different. Now, this idea why did I illustrate because and also for the functional form of the bi-objective case is that in the multi-dimensional case when there are many variables and quite complicated objective function you can have very interesting results corresponding to the fact that linear programming is true or else non-linear this integer programming is true. Now, we will discuss some ideas for effective versus inefficient solutions. The definition of efficient which is dominant solutions can be nicely illustrated from with the following example it is as follows if x_1 and x_2 are two decision variables in X we say that x_1 dominates x_2 if and only if that for any of this objective function the functional form when we put $x_1 \geq$ the functional value of the same functional value when you put x_2 greater than equal to not less. So, functional value of when we put x_1 is always either = or > that for the functional value when x_1 is replaced by x_2 and for at least one the greater than sign will hold.

So, not for everything it is not greater than equal to it can be in the case that all of the values of x_1 and x_2 are equal except one in the sense when the value of that function based on the fact when you have put x_1 would be greater than this is the case when I am talking about the dominance part. The fact of the dominance part is let us not confuse that concept of dominance with respect to maximization when you are minimizing a problem the lower the point value of the objective function better. So, it does not mean dominance from the perspective of the highest value. So, this is I am again highlighting for at least one it would be true. Now, considering the concept of Lagrangian multipliers are known to all because which I said would be a prerequisite concepts in this course and I would request the participants to please brush up the concepts of Lagrangian.

The Lagrangian multipliers help one to solve the equality constraint problems in case the constraints are inequality type then one can use the concept of KKT conditions or Karusch-Kuhn-Tucker conditions in this case also. So, here I will basically discuss about the theoretical fact of KKT conditions and as we proceed we later on definitely discuss about the concept of with the simple problem. So, let us consider the problem formulation where you want to optimize now the word optimization has been used depending on whether it is a maximization case or the minimization case. Let us consider the problem of the formulation of a multi objective decision problem where there are m number of objective functions to be optimized some can be maximization some can be minimization that is a different question, but you want to optimize find the best solution. Ask that there are k number of constraints of the less than type and n number of constraints of the equality type.

The question may be asked that why have not we included greater than sign, greater than sign can also be converted into less than sign depending on reversal putting on negative value. So, if we have this m number of objectives to be optimized k number of less than type and n number of greater than type. Another important thing is that here in case this is $\leq$ say for example, some b_k then obviously we can convert bring b_k to the left hand side and can be $<$ 0 and formulate the problem accordingly. Now, let x be a feasible point and suppose θ, λ, μ are the multipliers corresponding to the objective function the first set of less than type constraints and second set of equality constraints. So, θ would be basically from θ_1 to θ_m, m being the number of objective functions.

So, if there are three objective functions m is 3 which will mean we will have θ_1, θ_2, θ_3 and consider the less than type constraints are 4 in number. So, λ which are the multipliers Lagrangian multipliers for the less than type would be λ_1, λ_2, λ_3, λ_4. And finally, if equality type are 5 in number we will have Lagrangian multipliers given by μ_1, μ_2, μ_3, μ_4, μ_5. So, corresponding to this case of this Lagrangian multiplier we would have under the KKT conditions is the factor based on which we are trying to find out that these constraints are applicable.

So, if you remember the constraints were given by G and H. So, if they are with corresponding to G and H and the multipliers were given with respect to as you see F, H and G. So, my notational concept as I mentioned and what is written here is just we have H first and then G. So, λ part would be for F, θ would be for F, λ would be for H and μ would be for G. So, corresponding to G and μ as if you remember K in number. So, μ_K into G K would be 0 for all this K number of constraints and the first differential or the first derivative of f_i's i = 1 to m then H for all this n's and G for all this k's when they are multiplied with the corresponding Lagrangian multiplier and the value should be 0.

So, we will illustrate that later on. So, then X is called a V KKT condition if the above condition holds with θ = 0, θ means the vector θ, θ_1, θ_2, θ_3 till m in number and it would be a strong one if the other condition holds for θ_i's is > 0. So, we will consider these examples later on for KKT conditions and illustrate that how they will be utilized for the multivariate cases. Thank you very much and have a nice day. Thank you.