

**Multi-Criteria Decision Making and Applications**  
**Prof. Raghu Nandan Sengupta**  
**Industrial Engineering and Management Department**  
**Indian Institute of Technology, Kanpur**  
**Week 06**  
**Lecture 26**

Welcome back my dear friends, a very good morning, good afternoon and good evening to all of you and this course under NPTEL MOOC is titled multi criteria decision making and my good name is Raghunandan Sengupta from IIT Kanpur, IME department IIT Kanpur in India. And as you know that we will be starting the sixth week for the classes, so this is the twenty-sixth lecture out of the total number of lectures which is sixty for this course and as you know each week we have five classes each being for half an hour. Now, if you remember that in the last class we were discussing about Pareto optimality the concept that how for a problem formulation you can get different setup points of the decision variables like we considered a by objective one because it was easy for us to fathom diagrammatically, understand diagrammatically and also see that how the overall idea of Pareto frontiers considering that we were interested to maximize both the functions separately and then collectively. And we saw that it was a integer programming if you remember at the two corner points and then in between you had different combinations which was basically integer point one I mentioned that the corner points inside the feasible space were integers and obviously in between they can be fractions also. And at each point as the indifference curve or in a way the Pareto frontier moved the overall objective function remain the same it was summation of this two objective functions will further see such examples later on. So, the broader outline we are still continuing in our discussion is multi criteria decision making under that there were two broad areas one was multi objective decision making where you have optimization problems to solve they can be mathematically understood the problems for objective in nature could be illustrated and analyzed and there are different techniques which are there.

The other set of problems or ideas is multi attribute decision making and multi attribute theory which we as I said or I keep saying you will definitely consider in different methodologies of trying to solve the problem where the analysis would be much more subjective not objective. The coverage for this twenty-sixth lecture we will start off would be concepts of Pareto optimality in 2D space and we will continue discussing more problems effective versus inefficient solutions we will also touch upon Karush-Kuhn-Tucker conditions only the statements if possible we will do some problems later on. We will also consider the scales of measurements the nominal scale, ordinal scale, interval scale, ratio scale and we will discuss with a simple problem and why scales of measurements are important will be relevant later on when we come to multi attribute decision making and then we will move on to goal programming the two methodologies of goal programming will be discussed later on and as we proceed. Consider two

optimization problems as stated below and I will highlight them with two different colors such that the illustrations are clear cut and easy for me to explain also and easy for the students and the participants to understand.

So, the first one is they are maximization if you remember just before I start the problem if you remember using a hypothetical Pareto frontier set of points we consider they can be four different combinations considering there are two objective functions. In the first case where both are being maximized second case the first is maximized second is minimized third case the first is minimized second is maximized and the last case being both being minimized and based on that we found out that how the so called Pareto frontier would look like considering these four options and if you remember I did also highlight the fact that the ideas of the concept of Pareto optimal frontier or so called trying to bring us understanding about indifference curve was considered when we consider in the first part of the set of lectures about the person Mr. Murthy buying a house in the city of Vizag or Vishakhapatnam considering different types of criteria he is considering some were quantitative very clear cut in rupees and some were qualitative and in both the cases we analyze how you combine and you can get the idea about the illustration of the frontiers. So, here we will consider a maximization problem for both of them. The first problem is maximization of function  $f_1$  where the objective function consist of two decision variables why two because easy for me to illustrate and obviously the concepts can be carried onward for the more than two decision variables also.

The first decision variable is  $x_1$  second decision variable  $x_2$  the objective function is  $6x_1 + 5x_2$  and what are the constraints if you see there are four constraints. Constraint 1  $3x_1 + 2x_2 \leq 24$   $x_1 - x_2 > 1$  and  $0 < x_1 < 6$ ;  $0 < x_2 < 6$ . Now, we when we come to the second problem again I will use the different color as I said it is still maximization, but of a different function  $f_2$  with same two decision variables  $x_1$  and  $x_2$  the objective function is  $12x_1 + 7x_2$  and the constraint in both these problems are same. So, if you see the left hand side which had marked in red the constraints and the right hand side which had marked in blue the constraints are same objective functions are different. Now before I move on further another important point can the way of analysis methods of solutions are different, but way of analysis can also be extended the question may arise can it also be extended for non-linear programming with constraints being linear or non-linear programming with constraints being non-linear the answer is yes.

And to further give an explanation if you remember that this without in going to the details we did not go into so much details as we are going now it was given as an illustration in the first week of slides or may be in the second week I do not remember distinctly where you are trying to basically optimize a quadratic function based on constraints being linear. So, the idea that they were two quadratic functions of the form of a circle and you want to

find out the maximum the minima for them separately and then combining them. So, the analysis what we did there and what we are going to do now are exactly the same. Now, let us if when we solve the two separate linear programming now remember one important point whichever highlight because I made a different coloring scheme for the slides also so that we all do not miss that point I will mark that in black. So, we are considering continuous variables for this problem.

So, problem 1 which is  $f_1$  problem 2,  $f_2$  with the same constraints when we solve it the decision variables optimal decision variables for problem 1 which are marked by I will use the red color because that was the norm which we decided we will use for the first objective function  $x_1^*$   $x_2^*$  where  $x_1^*$  comes out to be 5.2 and  $x_2^*$  comes out to be 4.2. So, putting those decision variables  $x_1^*$   $x_2^*$  in the objective function would give because the objective function was  $6x_1 + 5x_2$ . So, it is  $6 \times x_1$  which is  $5.2 + 5 \times 4.2$  which is  $x_2$  4.2 is  $x_2$  the value comes out to be 52.2 only objective function 1. Now, coming to the second objective function again the problem solution is linear and continuous variables linear programming because the objective function is linear and also to make our life simple the constraints are also linear.

So, the optimal values for of  $x_1^*$   $x_2^*$  considering the fact that we are trying to maximize  $f_2$  come out as follows  $x_1^*$  value is  $6x_2^*$  is 3. Now, they are discrete integer but we solved the problem using linear programming and the objective function was  $12x_1 + 7x_2$ . So, let us double verify  $12x_1 \times 7x_2$  when we put them in the solution with those  $x_1^*$   $x_2^*$  it will be  $12 \times 6 + 7 \times 3$  the value comes out to be 93. Now, let us solve the same set of problem again separately, but now considering discrete variables which is integer linear programming the methodologies of solutions are different linear programming continuous variable integer linear programming there are methods. By the way we are not in this course as I said we are not discussing the basics of operational research because that is presume the participants and the students who are taking this course are well aware of that and those concepts for from which from the books where you can pick up that was already mentioned in the first lecture there was a detailed set of text books and the syllabus both for probability statistics as well as an operational research.

Anyway, let me continue when I solve this integer linear programming the first objective  $f_1$  the  $x_1^*$  comes out to be 6 and  $3x_1$  comes out to be 6 and  $x_2$  comes out to be 3. So, putting that in the objective function  $6 \times x_1$ ,  $x_1$  is  $6 + 5 \times x_2$ ,  $x_2$  is 3, the value we get is 51. Now, note down this integer programming problem is 51 with values of  $x_1^*$   $x_2^*$  as 6 and 3 and in the first case problem which was linear programming. For the objective function  $f_1$  the values of  $x_1^*$   $x_2^*$  were 5.2 and 4.2 and the objective function value optimal was 52.2 here it is 51. Now, when we again separately solve problem number 2 which is  $f_2$  the  $x_1^*$  value comes out to be  $6x_2^*$  comes out to be 3 and luckily this exactly matches the problem when

we have solved the linear programming because both the methodologies gave the same answer because coincidentally the decision variables are the same 6, 3 in both the methodology of solution. So, considering 6 and 3 for  $x_1$  and  $x_2^*$  respectively the objective function is now  $12 \times x_1^*$  which is  $6 + 7 \times x_2^*$  which is 3, the value comes out to be 93, 93 here 93 for the first case are same and for the linear programming for  $f_1$  the first case when it was continuous the values was for the objective function 52.2 and in the second case when it is integer the value is for the objective function 51.

Now, even though it is a little bit difficult for you to see, but once you solve this problem and have that feasible region and the objective function which I solved. So, we measure  $x_1$  value along the x axis  $x_2$  value along the y axis because it is a Cartesian coordinate system it is easy for us to explain. So, if I consider the case for ease of diagram I will just mark them and keep erasing them the coloring scheme. So, there is no confusion the movement of the objective function separately in order to obtain the optimal point for the maximization case for  $f_1$  maximization case for the  $f_2$  are these points as the move shift because it is the maximization problem. So, obviously we know from basic operation research concept that the maximization can happen as the objective function moves to the right top corner the moment it leaves the feasible region the boundary point one should satisfy if there is unique solution.

There can be multiple solutions also or there can be no solutions also it can be unbounded also depending on how the feasible region is. So, based on that the objective functions are I will mark them here this is as it moves parallel and the other set I will use a different colors say for example, green and then I will use objective function is this. So, moves parallel out till reach the optimum value at the corner point and the objective function the other one moves parallel out till it meets the boundary point and is optimal. Now, if I consider the region, so let us go back to the feasible region how it is found by the constraints. So, the constraints were there, so I will come to the diagram, but just for clarity and clarification this is  $x_1$  this is  $x_2$  the first one is  $3x_1 + 3x_2$  and the second one  $+2x_2 = 24$  is the straight line.

So, when  $x_1$  is 0  $x_2$  is 12, so consider it is 5 10 15 12 will be somewhere here and when  $x_2$  is 0  $x_1$  is 8, so it will be somewhere here so and it is less than, so the constraint is this. Which is the first one I have used black color I should not have used black color let me check if we can use a different color let me use the green one this goes I am erasing it, so there is no confusion. So, the first one was this, so this is less than, so this is the equation first one where I put a arrow green. The second one which I will use the violet color is  $x_1 - x_2 > 1$ , so when  $x_1$  is 0  $x_2$  is - 1, so - 1 would be somewhere here somewhere here down. So, and just one minute this extends somewhere here and when  $x_2$  is 0  $x_1$  is 1 is here, so the line goes like this and the points would be they are greater than 1.

So, this is the set of points are here and the equation is this one for the third constraint let us use the red color  $0 < x_1 < 6$ , so it will come somewhere here. And similarly,  $x_2$  for which we will use the dark red color it will be somewhere here, so the overall region is somewhere is very difficult mark here, but somewhere here which we have depicted here. And as the objective functions move which have already illustrated you get the feasible region and the optimum points. The optimum points had been solved both for the case of the linear programming as well as for the non-linear for the integer programming separately for both of them. Now, we are going to solve the linear programming continuous variable case, now the multi-objective bi-objective.

So, again I am repeating the statement plus please bear with me the concepts is bi-objective here it can be multi-objective also depending on how many different problems you have. Now, the question can be asked from the student side or can be asked by the participants that can we have a multi objective problem for two variables answer is yes. In that case they should have been more than two objective functions which means like depicted by  $f_3$ ,  $f_4$  and correspondingly which I have not taken I did not want to make it too cluttered, but the ID would remain the same for the two dimensional case. Now in a simple way of trying to analyze the bi-objective problem would be that we will give weights give weights to the functional value  $F_1$  and  $F_2$  depending on what is the importance we want to place on the functions like consider this example this is a mathematical problem, but consider problem where we want to buy a car and there are two objectives both are linear in nature one is say for example, mileage and another is cost of maintenance mileage I want to increase which is maximize cost I want to decrease which is minimize. So, in this problem the first objective would have been maximization second objective would have been minimization for the example of the car which I am giving.

So, the analysis which we will do now would remain the same, but only the second objective would be minimized here. Here we are going to consider both as maximization this is just an illustration. Now why did I mention this the idea being that for the example of the car mine age and cost. So, I will put more weightages for the mine age and less weightages for the cost and less weightages for the cost that means cost should be minimized. So, in that sense that a value of weighted function or weight value which we denote by  $\lambda$  this one.

So, we will put weightages for the first function as  $\lambda_1$  second as  $\lambda_2$  and then we have  $\lambda_2$  both should be between 0 and 1 such that the sum of the weightage is 1. So, in case if it is like this so the function would be  $\lambda_1$  weighted to the first function and  $\lambda_2$  being weighted to the second function and this bi-objective case is a sum. So, if  $\lambda_1 = 0$  that means all the weightages for function 2 is  $\lambda_2 = 1$  all the weightages for function 1 and if  $\lambda_1 = 0.5$  and  $\lambda_2 = 0.5$  that means we are giving equal weightages to the function  $f_1$  and  $f_2$  respectively.

In case it is  $f_1$  weightage is 0.25  $f_2$  weightage is obviously 0.75 which means the weight weights ratios are  $\frac{0.75}{0.25} = \frac{3}{1}$  that means I would be giving thrice amount of weightages to the function corresponding to which it is getting 0.75 as  $\lambda$  value. For R is

illustration we did not restrict our self to any particular weightage we took all the different combinations of  $\lambda_1$  and  $\lambda_2$  and illustrate the problem. So, now the problem is maximize  $\lambda_1 \times f_1 + \lambda_2 \times f_2$  and constraints are same. So, they are denoted the constraints are denoted. So, when we bring the functional form of  $f_1$  and  $f_2$  in the equation this is the equation we have which is  $\lambda_1 \times f_1$  this is  $f_1 = 6x_1 + 5x_2$  is  $\lambda_2 \times f_2$  where  $f_2$  is  $12x_1 + 7x_2$  constraints remain the same. So, now we take different values of  $\lambda$ .

Let us start the iteration and we have that in front of us. So, obviously as I said when  $\lambda_1 = 0$   $\lambda_2 = 1$  which means that we are putting all the weightages on function  $2f_2$  and that problem we have already solved we have illustrated when we are discussing or when we started the problem. Now, we keep changing if you see here. So, the  $\lambda_1$  values start change from 0.1 to 0.2 to 0.3 to 0.4 to 0.5 and correspondingly  $\lambda_2$  which is  $1 - \lambda_1$  is also changing from 0.9 to 0.8 then to 0.7 0.6 0.5 and correspondingly we solve the problem and find out the optimum decision variables which is  $x_1^*$   $x_2^*$  and interestingly this is just an example the values of  $x_1^*$   $x_2^*$  initially are integer then for  $\lambda_1, f_2, f_3, f_4, f_5, f_6, f_7, f$  50%  $\lambda_2$  which is 0.5 as it is an linear programming where there is no restrictions on the variables they can be integers non integers also for 50 % weightages the  $x_1^*$   $x_2^*$  value changes from 6 3 to 5.2 4.2 and correspondingly if you see we have written down the object the combine optic objective function which is  $\lambda_1 \times f_1 + \lambda_2 \times f_2$  and it continues.

So, this is when  $\lambda_1$  is 0.1 this is when  $\lambda_1$  is 0.2 this is when  $\lambda_1$  is 0.3 this is for  $\lambda_1 = 0.4$  this is the value for  $\lambda_1 = 0.5$  and the values of this combine objective function by objective case are given 88.8 84.6 80.4 76.2 72 and we continue. Now further on because we have in the last slide was still  $\lambda_1$  was still 0.5. So, considering  $\lambda_1$  as 0.6 then  $\lambda_1$  as 0.7  $\lambda_1$  is 0.8  $\lambda_1$  is 0.9. So,  $\lambda_2$  correspondingly corresponding values are 0.4 0.3 0.2 0.1 the last value which we have not written, but we have already solved is when  $\lambda_1 = 0$  and  $\lambda_2 = 1$  which is when we are putting all the weightages to objective function  $2f_2$ . We are putting all the weights. So, my apologies  $\lambda_1 = 1$  and  $\lambda_2 = 0$  that means we are putting all the weightages to objective function  $1 f_1$ . Now if we check the  $x_1^*$   $x_2^*$  values they are as given 5.2 into 4.2 and the corresponding values objective functions are  $0.6 \times f_1 + 0.4 \times f_2$  then  $\lambda_1$  changes to 0.7. So, it is  $0.7 f_1 + 0.3 f_2$  then  $0.8 f_1 + 0.2 f_2$  then finally,  $0.9 f_1 + 0.1 f_2$  and the values are given as 68.08 64.08 60.12 56.16. So, the problem is not yet finished, but considering that duration for each lecture is half an hour I will keep the limit of the time frame and end this discussion here, but continue this problem in more details later on in the 27th class. Thank you very much and have a nice day. Thank you.