Multi-Criteria Decision Making and Applications

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Week 05
Lecture 25

Good afternoon, good evening to all the students who are taking this multi criteria decision making course under NPTEL MOOC series and my good name is Raghunandan Sengupta from the IIT Kanpur and as you know this is a 12 weeks course spread over 60 lectures each lecture being for half an hour and in each week we have 5 lectures. So in the last class which was the 24th lecture we were considering the detailed analysis in last and last to last class 23rd and 24th we are discussing about Pareto optimality considered that theoretically in using diagrams and the coloring scheme was used green, orange, blue and red and then we are going to consider which we started a simple problem where both were maximization and I also did mention that if we have minimization, minimization problem or maximization, minimization or minimization, maximization when I use this word maximization, minimization I always mean there are two objective function. It can be more than two also there is no problem, but only trying to depict it for us to visualize may be difficult. So continuing this lecture if you read the slide this is the 25th lecture under the 60 lecture series and the broader umbrella is multi criteria decision making under which the main emphasis still now has been multi objective decision making which is more objective in nature easy to understand mathematical models are there and the answers which you get is easily and can be understood communicated. Then also I mentioned the concept of very simply the concept of multi attribute decision making and we will consider further on the multi attribute utility theory. We have already considered the simple utility functions they would be extended.

The discussions coverage would be even though there are many points written I think we will try to cover the concept of Pareto optimality solution in 2D space as I keep mentioning 3D space is fine, but 2D are very easy for us to communicate that is why. We will consider the concept of effective versus inefficient solutions the concept of Karusch-Kuhn-Tucker conditions later on scales of measurements nominal ordinal interval ratio and the goal programming. So, whatever we are able to cover in this last lecture for this 5th week the coverage would be changed accordingly and then all the students and participants can see the slides. So considering the example it was maximization and maximization and the concept we are using the coloring schemes were accordingly even though when you come to the graph we have later on just within few seconds which I will show they gave you the feasible region based on these constraints which you have and the two objective functions the feasible space is given in the light blue color.

And I did also mention in the last class which I will also again mention the green and the black dotted lines are basically the objective function and the corner points which you see which I have did mention corner points I am just marking here in black again and this points would be individual maximum value for the objective functions considering the feasible region is same for both. Now I did also mention the different coloring schemes which are used for this parallel lines like if you see I will move my pointer not mark it because not make it cluttered there is a brown color optimum value 15, blue color optimum value 20, pink with optimum value 23, red optimum value 25, green with optimum value 28 and finally, orange with optimum value 33. By the way this green with optimum color 28 has nothing to do with the dotted part or the red one optimum value 25 has nothing to do with the red dotted part. So, once I concentrate or we concentrate on this parallel movement of these lines the situation is like this and which will cover one by one. When the optimum solution when you are trying to combine $f_1 + f_2$, if you remember we did discuss that part that the optimum value that the objective function when there are multi objective you can either take a combination of f_1 or f_2 with λ_1 and λ_2 where $\lambda_1 + \lambda_2 =1$.

Here just for simplicity I am considering that you are just simply adding them even if you add λ_1 and λ_2 multiply to f_1 and then multiply to f_2 respectively λ_1 and λ_2 the overall concept of trying to analyze the problem would not change only they will basically weight the function. So, if I consider the concept of $f_1 + f_2$ just add them when the value is 15 and mark it which is this point 15 means this one the brown part. Then if I consider all the sets of points along this brown line which is this one brown I will use this dark red this one consider this one I should use I will let me use this one all the set of points along this brown line. So, I will hash it because that is what you are going to consider. So, what are the corner points if I consider the corner points here (5 , 0) another corner point is (0 , 3).

So, this is the one (5, 0) and the other extreme is (0, 3). Now in between there are infinite set of points joining these two lines the brown one the two points sorry anodic combinations and these points like 4.5 for x_1 0.3 for x_2 4 for x_1 0.6 for x_2 3.5 for x_1 0.9 for x_2 3 for x_1 1.2 for x_1 .2 for x_2 and all these sets can be verified they lie on this brown straight line. Interestingly if I take the objective function values f_1 only first and f_2 only next individually the values of them are given respectively in row number 3 where I am putting a double tick mark and row number 4 where I am putting a triple tick mark.

So, put the objective function value find it out for x_1 x_2 for (5 , 0) you will find the value as 20. So, let us double check that. So, there is no confusion between what I am saying and what is being understood by the participants and the students. So, let me write down the first function $4 x_1 + 3 x_2$. So, I will use the same color $4 x_1 + 3 x_2$ is for the first function which I will denote as f_1 and the f_2 function is $- x_1 + 2x_2$ $- x_1 + 2x_2$.

So, let us double check there is no confusion $4x_1 + 3x_2$ for the first function f_1, $-x_1 + 2x_2$ is the for the second function f_2. So, let us check when I put (5 , 0). So, 4×5 is 20, $20 + 3 \times 0$ is 20 again. So, here you see 20. If I put the value of (5 , 0) for f_2 which means the fourth row. So, that value would be $-5 + 2 \times 0$ which is -5 which is true. Let us consider the other extreme points because other points can be verified I will consider the other extreme which is (0 , 3). So, if I put (0 , 3) for f_1 which means (0 , 3) for x_1 and x_2 and then try to find out f_1 it will be $4 \times 0 + 3 \times 3$ it is 9 which is 9. If I put (0 , 3) for x_1 and x_2 and try to find out f_2 function it will be $-0 + 2 \times 3$ which is 6 which is verified. And similarly, all the values which are there in the third row and the fourth row can be verified.

Now interestingly if I add up these values of f_1 and f_2 for each corresponding values of x_1 and x_2 that coordinate system which is all the points along the brown straight line. The values are $20 - 5$ 15, $18.9 - 3.9$ 15, $17.8 - 2.8$ 15, $16.7 - 1.7$ 15 and if you continue till the last point $9 + 6$ is 15 which means the combinations for all this set of points which are along the brown line. The coordinates may be different decision variables may be different, but the combined objective function value is same which means in such in a way you can consider that I am getting the same value of 15 gaining the same value as 15, but for different combinations of x_1 and x_2 which means that my level of satisfaction of getting the value of 15 is fixed, but how I can get it would basically depend on the combinations of x_1 and x_2 which I have which means the set of points on the brown line are giving me the same level of satisfaction. So, I am indifferent between any points so long the level of satisfaction is being met.

Let us now come to the second line which if you have noticed I did not mention that if you have noticed all lines are parallel. So, let us consider the blue one. So, I will use the blue color. So, I will use the blue color which I am hashing and the points are as here and here. So, this points which have been marked are the corner points are marked as 5 1 0 4. 5 1 0 4 let us see. If I consider x_1 and x_2, I will have x_1 and x_2. So, I will have the blue one point. So, this would be just one wait one minute the diagram, but generally when you solve it they would be the case 5 1 0 this has been not done accurately, but I will try to highlight this. So, if I consider the points of 5 1 and 0, I will have x_1 and x_2 0 4 for the function form of $4x_1 + 3x_2$. So, f 1 this is not coming up let me write it down $4x_1 + 3x_2$ and f_2 function is $-$ x_1 x_2.

So, for these values let us proceed similarly, all these points which you see which is there in this first row where I put as one tick mark, second row where I put as two tick mark they are always on the same blue line. They can be infinite points I am just taking a set and when I put these values of x_1 x_2 collectively one at a time in the function f_1 and f_2. So, it will be $4 \times x_1$ which is $4 \times 5 + 3 \times 1$ will give me 23. If I put it in the function of value f_2

it will be $-5 + 2 \times 1$ which will be -3. Going on to the other extreme if I put 0 and 4 for x_1 and x_2 for f_1 to find it out and then for f_2 it will be like this respectively.

It will be $0 \times 4 + 3 \times 4$ which is 12 and then for f_2 it will be $-0 + 2 \times 4$ which is 8 which is correct. Similarly, all the values which are there on the third row which I put a triple tick mark and the fourth row where I put a four tick marks would confirm the fact that I can get the values of f_1 as 21.9 and -1.9 for a value of x_1 as 4.5 x 2 as 1.3. Similarly, for a value of 4 and 1.6 for x_1 and x_2 respectively I will get values of f_1 as 20.8 and -0.8. Similarly, if I go to the extreme for a value of x_1 and x_2 which is respectively 0.5 and 3.7 I will get values of 13.1 for f 1 and 6.9 for f_2. Now, if I combine them interestingly it is the same thing comes $23 - 3$ is 20, $21.9 - 1.9$ is 20, $20.8 - 0.8$ is 20 go to the other extreme if it is 13.1 + 9 6.9 is 20. So, all the values objective function values for those straight lines are same that is the value or the net worth is same, but the combination in which I am able to get those values would differ that means my level of satisfaction is fixed along this blue line.

Now, if I go back to this diagram which I mentioned, but just for double clarification this lines the brown, blue, pink, yellow, green, yellow, green, yellow, green, yellow red, green, orange are all parallel to each other that means they would never intersect. And also another important thing is that as you move up go on to the top right corner if you remember the pareto optimality 1 the orange part diagram it was always given that we should move in this direction when it is maximization maximization for objective function 1 and objective function 2 and that is what is happening is I move parallel on to the tight hand side the values keep increasing from 15 then to 20, 20 to 23, 23 to 25, 25 to 33 and so on and so forth. When I come to the optimum value of 23 I will just without going to the diagram if at all necessary I will go back, but if I consider that coloring scheme which is better for me, so the value is 20, 20 was done for the value of blue then I come to the value of 23, 23 was given in pink. So, let us use try to use pink color if it is there, it is not there, it is not there let me use another color say for example, light green. So, the value is 23 and we will see that the extreme points which are (6,1) and (1,4), so if I plug in this values (6,1) for f_1 and then plug in (6,1) for f_2, so the values respectively come out to be $27 - 4$. If I go to the other extreme 1 and 4 plug 1 and 4 for f_1 and f_2 and the respective value for f_1 and f_2 come out to be 16 and 7. And similarly, all the sets of points which are there and I will just read out few of them as x_1 is 5.5, x_2 is 1.3 for which the respective values of f_1 and f_2 is 24, 25.9 and -2.9. When the coordinate or the decision variables are 5 and 1.6 the respective value of f_1 and f_2 are $24.8 - 1.8$. Similarly, for 4.5 and 1.9 respective values of x_1 and x_2 the corresponding values of f_1 and f_2 respectively are 23.7 minus 0.7. And skipping few of these values if I go to the third last and the second last one values where the highlighted is for a value of x_1 and x_2 is 2.5, 3.1 the respective value of for f_1 and f_2 is 19.3, 3.7 and for x_1 and x_2 is 2 and 3.4 respectively the respective values of

f_1 and f_2 are 18.2, 4.8. Again interestingly if I add up these functional values f_1 and f_2 separately is 27 – 4 is 23 which is what is written here. Then if I add up 25.9 – 2.9 then in the next step do 24.8 – 1.8 then next I do 23.7 – 0.7. So, all the values add up to 23 that means again the net value is same, but the overall decision variables can be different, but the net satisfaction level remains the same. So, considering the next optimum solution which is f_1 and f_2 is 25.

So, let us go back to the diagram which is red in color. So, I will use the red color. So, this is 25 sorry sorry this is 25 and the values first I will consider is 5, 2 and 0, 5 for the extremes which is 5, 2. I think the diagram has not been drawn very accurately, but this would be basically (5, 2) and (0, 5). So, if I find out and put the values individually for f_1 and f_2 in this equation I am not writing the equation because that has been done for the three sets. 5 and 2 as x_1 and x_2 values would give you a value of 25. So, 26 for f_1 and – 1 for f_2. Other extreme for a value of x_1 and x_2 as 0 and 5 the respective values of f 1 and f 2 is 15 and 10. And all the corresponding values which you see in the first row, second row, third row, fourth row are respectively like this. The first row corresponds to all the x_1 values, the second row corresponds to all x_2 values, the third row and the fourth row respectively correspond to individually f_1 values and then f_2 values.

If I add them final row value which you see is f_1 and f_2 is 26 – 1 is 25 which is what is adding up 24.9 + 0.1 is 25, 23.8 1.2 is 25 to the other extreme if I go 17.2 + 0.2 is 25. 15 + 7.8 is 25, 16.1 + 8.9 is 25, 15 + 10 is 25. So, again it is a set of points which give you the same level of satisfaction, but they can be achieved by different combinations of decision variables. Finally, without going to the details this is basically the optimum solution when I want to find out based on the points of where the sum is 28. So, this is what line let us make it 28 is basically the point the green one shown and the last one is basically orange. So, if I use the color green again, so I have f_1, f_2 functions values and the two points extreme points is 6, 2, 1, 5 and all the values which you see in the first row, second row correspond to x_1, x_2 respectively. And the third row, fourth row which I am putting a single tick mark and a double tick mark correspond to the individual values of f_1 and individual value for f_2 corresponding to the different combinations of x_1 and x_2 I take.

So, when I put 6, 2, 6 for x_1 and 2 for x_2 the value of f_1 is 30, the corresponding value for f_2 is – 2. On the other extreme when I put 1, 5 for x_1 and 5 for x_1 and x_2 respectively the values for f_1 is 19 for f_2 is 9. And individually all the values which you see in the third row and the fourth row corresponding to correspond to the individual values of f_1 and f_2 when you put x 1 and x 2 accordingly. Again interestingly add up f 1, f 2 30 minus 2 is 28 which is exactly what is matched then 28.9 – 0.9 again 28 to the other extreme 20.1, 7.9 is 28 and 19 plus 9 is 28. Again same level of satisfaction different sets of values for decision variables. This is the optimum solution for 33. So, it was orange in color I will just mark it. So, the extreme point just taking is basically 6, 3, 1, 6 if you put it in f 1, f 2 respectively the values are f 1 for 6 and 3 for x 1 and x 2 is 33 and value of f 2 for a value of x 1 and x 2 as 6 and 3 is 0.

And on the other extreme if I consider the other point extreme means the points which I am considering along the straight line would be for 1 and 6 x 1 and x 2 the corresponding values are f 1 and f 2 are 22 and 11. And all the respective values of x 1, x 2, f 1, f 2 are given correspondingly in row 1, 1 single tick mark, row 2 double tick mark, row 3 triple tick mark, row 4 tick marks. And finally, when I add them add these values f 1 for f 2 33 plus 0 is 33 matches 31.

9 plus 1.1, 33 matches on the other extreme 23.1, 9.9 is equal to 33 again matches 22 plus 11 matches to 33. Now, let us consider the problem as stated below which is a different type of maximization again maximization there is no change. But here I would not be going to so much details of trying to tabulate as I did. Now, the question would be before I start the question would be you may be thinking that the so called set of points which gave us the same level of satisfaction to come back to this diagram. So, these parallel lines brown, blue, pink, red, green, orange should they be always straight line answer is no they need not be they can be curves also.

Because if you remember in the diagram which I drew to basically using this four coloring scheme red, green, blue, orange they were curves and this case was basically for the maximization one. So, when we are using the concept of maximization obviously we will move to the right top hand corner. And this lines which you consider need not be straight line and also the constraints point one the objective functions need not be linear they can be non-linear also quadratic also anything. But the idea which will be conveyed is that they would be set of points where the value of the combination of the multiple objective function which you have would be same. But how you can achieve it would basically depend on different combinations of the decision variables which you have.

So say for example, just I am drawing a theoretical one consider I have x_1 and x_2 and the feasible region I am drawing in the blue one is say for example, this. Now consider the objective function can be like the non-linear. So, what we will try to achieve is consider this objective function and obviously the internal points would also be considered. It may be possible that if I want to maximize or minimize consider that I want to maximize it may be possible that the combination of the objective function at point A and at point B are same level of satisfaction. But the combination based on which I am trying to achieve x 1 and x_2 would be different.

So, this is a non-linear part we will consider those non-linear parts later on, but what I want you all of us to understand and which I am trying and I am maybe I am going a little bit slow is in the sense that the objective functions here are linear the constraints are linear. The idea which is to be conveyed in trying to tackle the problem on the Pareto optimal solutions the concepts and trying to visualize would be same. So, even though I switched to this the non-linear part later on the problem which is stated here is basically again a maximization problem with the same sets of condition and we will try to basically analyze the problem accordingly. Considering the time shortage which we have for this end of this lecture for the last lecture and the fifth week we would basically take this example in more details in the sixth week. Thank you very much and have a nice day. Thank you.