Multi-Criteria Decision Making and Applications

Prof. Raghu Nandan Sengupta
Industrial Engineering and Management Department
Indian Institute of Technology, Kanpur

Week 05
Lecture 24

A very good morning, good afternoon, good evening to all my participants and students for this course of multi criteria decision making under NPTEL series and my good name is Raghunandan Sengupta from the IME department at IIT Kanpur. So as you know this set of lectures or this course is for 60 lectures which is spread over 12 weeks and we are in this fifth week and each week we have 5 lectures each lecture being for half an hour and just for repetition I do repeat this things such that things are clear that there are each lecture is for half an hour and after each week there are assignments. So after I finish this fifth week you will have the fifth assignment. Now if you remember in the last lecture and I will come to the lecture number very soon what we are going to do today. We did consider the concept of Pareto optimality, the concept of dominance of strong dominance, weak dominance, strong optimality and I did consider three sets of solutions x_1^*, x_2^*, x_3^*in the last lecture if you check the slides and the lectures and with the different colors and that compared that how when you are trying to maximize how dichotomy or confusion can occur if values of f_1, f_2, f_3 which are the three functional values we which I am considering have different values in sets such that the decision where the decision variables of either x_1^* or x_2^*or x_3^* is better on words is difficult to fathom or difficult to basically come to the conclusion. Now in the last lecture if you remember by the way this is the 24th lecture.

In the last lecture 23rd lecture we just started we trying to analyze that we will consider in the two dimensional space and why two dimensional it is easier for us to understand two different objective function based on whatever the sets of decision variables are they can be three decision variables, four decision variables but at the end of the day there are two objective functions f_1, f_2, which is objective function 1 and objective function 2. Now we will consider as I mentioned four combinations maximization of f_1, maximization of f_2, minimization of f_1, minimization of f_2, maximization of f_1, minimization on f_2, and minimization of f_1, maximization of f_2 . So all this concept which wec will consider and these points are under the concept of multi criteria decision making where there are two broad outlines one is multi objective decision making which would come out later on in more details. The concept of multi attribute decision making which is more subjective and utilizing this concept of multi attribute decision making we will also go into further discussion of multi attribute utility theory.

So we did discuss about utility theory but those were all unidimensional scalar. The coverage which we will do and we will continue the concept of Pareto optimality in two dimensional space the effective versus inefficient solutions what is the idea of Karusch-Kuhn-Tucker conditions and this will be dealt in more details later on. What are the scales of measurements? You are measuring the concept of length, breadth, height, temperature so how they can analyze. So the scales of measurements would be nominal, ordinal, interval, ratio and then we will go into the goal programming. Whatever is covered as I am I keep mentioning in any lecture would be complete set of slides for that lecture only and that will be uploaded and available for the students to refer to.

Now I will repeat this slide because the continuation will tacke some time for discussion. In this example as I mentioned there are two objective functions f_1, f_2, which are mentioned as objective 1, objective 2 and whatever the decision variables that is not important for us for the time being. Now we will consider different coloring scheme for us like both the participants and the students who are there listening to the lecture and for me I am explaining it is easier for us to communicate. The first one and I will use the coloring scheme also when I use the marker accordingly. So the first one is basically given in red color which is basically minimization of f_1, minimization of f_2 so the first combination.

The second one we will see the diagram soon, second one is basically used in blue color which is minimization of f_1 along with maximization of f_2. So I am not really calling the boundaries what the boundaries or the arcs what are the names are rather than mentioning the boundary points I will mention them in the coloring scheme here and when you see the diagram that would be much better for me to explain it will be much more clearer to you. We will use the concept of orange for the case when we are considering maximization of f_1 along with maximization of f_2 and finally we will use the green coloring scheme to use the concept of maximization of f_1 which is objective functional 1 along with minimization of f_2. These four distinct Pareto optimal plots may not be exact in shape I have just drawn curves but they may not be exact curves as smooth as I have drawn both in their shape as well in size but it should give a good idea to the participants or the reader when he or she goes through the slides about the different concepts of sets of Pareto optimal point curves such that depending on the four different combinations maximization, maximization, minimization, minimization, minimization, maximization, maximization, minimization how these two objective functions would be depicted and this is the diagram which I wanted to show. Now if I consider the objective function 1 along the x axis which is f_1, object function 2 along the y axis which is f_2 and you see four different colors blue, orange or light orange, green and red.

So let us go one by one. If you remember the concept of trying to basically find out the objective function of red color and going back to the slides it is minimization, minimization. So if I consider the minimization, minimization consider only here and let me use the coloring scheme as red here. I will draw it and erase it one by one. So when I consider the objective function 1 it means that the point in this two dimensional space I will always try to say for example, there are two points.

So consider this blue, orange, green are not there in this case only the red one is here. So there is point number consider let me consider point number x_1 and there is point number x_2 here. If I consider the objective function 1 and objective function 2 both are to be minimized on the scaling of objective function 2 which is along the y axis both are at the same level. So one is not better than that we cannot say, but when I consider objective function 1 the value x_1 and x_2 by the way do not confuse x_1 and x_2 as the decision variables x_1 and x_2 are just points marked here. If required in the subsequent discussion which I do with the blue, orange and green line I will change this nomenclature of x_1 and x_2, but let us continue with that.

But if I consider the minimization of objective function 1 then the value for objective function 1 based on x_2 point is less which means that the person would always try to go in the minimization minimization point in this direction reach minimum for both. So trying to reach the minimum for both may not be possible all the time. So you have to basically make a compromise. So if I consider this red point it is like this if I consider the bundle of x_1 x_2 or bundle of objective function objective function 2 it may be possible that the point which is here I will use x_2 here again x_1 and x_2 are not the decision variable just points and x_1 here. So if I consider x_1 x_2 points here the objective function values for x_2 and x_1 based on only f_1, f_1 is less for x_2 and more for x_1 if you concentrate on the diagram.

But if I consider the objective function 2 the objective function 2 value for $x_1 < x_2$. So compromise has to be made depending on the minimization because I want to reduce both of them. Now let us consider in this diagram so this red part is over we will consider the next combination and I will go the coloring scheme wise. So let us go back to the just previous slide the blue one is minimization of f_1 maximization of f_2. So blue one is f_1 is basically being I will use the color 2.

So it is minimization of f_1 f_1 is objective function 1 and it is maximization of f_2. So let us clarify it what that is what you are doing minimization of f_1 maximization f_2. If I consider 2 points here and let us consider these 2 points as consider them as what nomenclature should I use as there is no consider them as Y y_1 and y_2. So by the way this y_1 and y_2 are the other depiction which I did as x_1 x_2 when we are considering the red counters minimization. If you see here just concentrate here.

So you want to minimize this minimize f_1 that means you want to come there are 2 points. So consider this is f_1 y_1 and y_2. So obviously you will try to minimize f1 go in this direction, but for f2 oh sorry sorry sorry my mistake my mistake. So for f_1 you want to minimize so you want to go in this direction and for f2 you want to maximize you want to go in this direction. So the best optimal solution would be actually you would like to move in this way.

So if you consider $f(y_1)$ and y_2 the actual value of y_2 for f_1 so let me write it down the value for f1 for y2 f1 we want to basically always minimize and if I consider this one is less than for the value of f_1 for y_1 which is good this is what we want go to the left. If I consider the objective function of f_2 which is I will put a tick mark which is good if I consider the f_2 function of y_2 and if I consider the f_2 function of y_1. So the functional value of f_2 should be increased so y_1 is better which is this which would be also good. So y_2 value on the functional part for f_1 is low is on the lower side and which is ok so this is good and for f the functional value for f_2 for y_1 is more which is also good. Now if you see this they move in different direction one pulls up that means you want to go on to the top another basically would go to the left.

So actually I would like to move in this left corner as given by this plot. When I consider so remove this diagram and I use the different coloring scheme as we have been using. So this objective function f_1 f_2 remains when I come to the orange one it is maximization of f_1 maximization of f_2. So it is max write it here f_1 and also max of f_2 both are to be maximized. So if I consider the orange one there are two points let me mark it as y_1 y_2 so these are the points.

So if I want to consider the maximization I will always try to move up for f_2 and also try to move to the right for f_1. So technically I want to move on to the top right corner. So if I consider the functional value for consider f_1 for y_1 and I consider f_1 for y_2 as marked. So obviously the as I want to move on to the right so obviously f_1 of y_2 is better which is good if I consider f_2 for y_1 f_2 for y_2 I want to move on to the top so f_1 f_2 y_1 is better so this is good. So obviously the movement would be placed on the direction of movement.

Finally if I consider last part being its green in color so let me change the color of the highlighter which I am going to use. So if I consider the final one which is maximization of f_1 minimization f_2. So it is maximization of f_1 minimization of f_2 so consider two points y_1 y_2 so maximization on f_1 I will move on to the right direction if you are seeing me on the right and minimization f_2 would be in the bottom direction so technically I would move to the bottom right corner. So let us consider the points so this is $f(y_1)$ y_2 so if you consider the functional values f_1 for y_1 and f_1 for y_2 so what do you want to do for f_1 f_1 you want to reduce so y_2 is better this sign which I am putting is better ways it is not greater it is better. So I should have basically use I should have used the word some other symbol but whenever I am using this greater than it means I am getting a better benefit.

If I consider f_2 of y_1 and continue with f_2 of y_2 so in that case for y_2 I would move on to the right so f_1 is better so I will it is this so this is one. So considering all the combinations if you have seen the movements actually are in different directions so I will remove this black color for f_1 f_2 the movements are for the green one actually I should draw it here it is to move in this way reach the best for the orange one the best for the red one when I mean the green orange red and the blue one this what is the coloring scheme I have already mentioned. And if you remember when we are discussing this Pareto optimality curves I did mention the curves nomenclature like what is AB what is CD what is EF what is GH. So AB is basically for the red one where you want to have the case where the overall ID is trying to minimize minimize both of them in the case when it is blue one which is CD you want to minimize f_1 and maximize f_2 when it is basically the orange one it is basically maximization of f_1 maximization f_2 and finally, when you are trying to consider and that portion by the way that portion was EF and when I consider GH curve which is shown in green is basically maximization of f_1 and minimization in f_2 of the functions and the curve portion is GH which I mentioned. Now let us consider the concepts with a more detailed example and in this example I have taken the objective functions as given and they are both maximization it can be minimization maximization or minimization minimization or minimization maximization all those four combinations.

So, the overall analysis of the problem how you depict those Pareto optimality concepts would remain the same only the way of trying to analyze would depend on the way how this blue this orange this red and this green have been depicted. So, this is maximization so, I will mark the objective functions with different colors. So, consider maximum by the way also to keep things in the similar way of analysis if I consider the maximization problem of the case it was basically given in orange, orange was this part this one. So, if I consider the orange part so, this functional f_1 functional f_2 and the functional forms is given f_4 (x_1) + 3 x_2 and f_2 = - x_1 + x_2 and the constraints are same to make things very simple, but even if you consider different constraints that overall concept of the problem will change, but the overall analysis would remain the same why the concept will remain same I will come to that later with examples as we keep discussing. So, you have the functions the decision variables x_1 and x_2 two dimension y_2 dimension easy for us to analyze the values are $x_1 < 6$ $x_1 < x_2 < 7$ the third fourth fifth sixth are accordingly
$x_1 + x_2 < 10$ - $x_1 + x_2 < 5x_1 - x_2 \leq 5$. So, they are $\leq$ 5or else less than would basically can mean any point equality would mean only the straight line and finally, $x_1 + x_2 \geq 2$ and
x_1 and x_2 are always > 0. So, now comes the interesting part as it is two dimension I first denote variable they are by the way we are not going to depict the objective functions along the x and y axis in the other problem where we consider the Pareto optimal fronts with different four different colors red orange green and blue that objective function concept is being removed we are going to consider only the variables, but removed when I use the word remove of the objective function does that does not mean I am not going to

depict that I will depict that in a different way. So, you have the variable x_1 along the x axis variable x_2 along the y axis which means the decision variable which is a vector consists of x_1 and x_2 and based on this boundary conditions which will give you the feasibility feasible space. So, these are the points if you consider. So, $x_1 < 6$, so x_1 being < 6 is this one I will just mark it in black and then remove it because I do not make it to be too cluttered.

So, this is on the left hand side $x_1 < 6$ the second point is $x_2 < 7$. So, this is 7. So, it will come. So, here it is on to the left this is on to the bottom part you have $x_1 + x_2 < 10$. So, if I consider x_1 and x_2 actually this is the curve this one because it will touch 10 here and it will go on touch 10 here this 10 for x_2 is not drawn this just depicted.

Then the value is $- x_1 + x_2 < 5$. So, if I have $- x_1 + x_2 \leq 5$ it is ≤ 5 let us verify yes. So, when x_1 is 0, x_2 is 5 which is here and when I consider $x_2 < 5$. So, x_2 is 0 x_1 is – 5, - 5 would be here what which is not drawn. So, this would be the other curve which is this one then you have $x_1 - x_2 \leq 5$ $x_1 - x_2 \leq 5$.

So, if you consider x_1 is 0 x_2 is - 5. So, this is so it will be this point this line. So, this would be this one on the curve last one is $x_1 + x_2 \geq 2$. So, ≥ 2 would be this one this is $x_1 + x_2 \geq 2$. So, with this we have been able to first analyze the feasible space.

So, I will first then remove because I said I do not want to make it cluttered. So, x_1 x_2 remains I will remove the black portions here. Now the feasible region is the light blue part and the concept is if you see the green dotted part line and the red dotted line these are the optimum solutions if you consider individually separately objective function 1 and objective function 2. And the corner points which you see, so the corner points which I will mark further time being in black this one and this one the decision variables x_1 and x_2 would be I will write it down as the coordinate system would be (2 , 7) and this point would be basically (6 , 4). So, if you want to find out the objective function values you can find them accordingly and solve the problem.

Now what we need to do is basically try to find out the optimum solutions for the combined objective functions. And before I come to the table one by one I would like to draw all the participants and the students attention to the coloring scheme which has been used. Here it is, if you see that these lines the light brown one where optimum value 15 is written the dark blue line where optimum value 20 is written then the light a little bit darkish pink one where the value 23 is written then red one where value of 25 written then the green one where the optimum value 28 written then the value of a little bit orange one where the value of 33 is written they are moving parallel to each other. And if you see the parallel movement is in such a way that the combinations of x_1 and x_2 you will see would give you the best optimum solution for those set of points which are along those colored lines and there are infinite set of points for any color. If I consider the green one which is here which I am basically marking with green color this one.

The set of points which you will have for x_1 and x_2 would always give you the value of the optimum value of 28 which I just mentioned. Similarly for the other lines how it can be done and why it is intuitive I will consider in the next class. I will start with this slide and also with the previous slide where the objective function is given along with the constraints. So, that means any references to the diagram and the objective function f_1 and f_2 along with the constraints can be easily referred to.

Thank you very much and have a nice day. Thank you.