Multi-Criteria Decision Making and Applications

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Week 05
Lecture 23

A very good morning, good afternoon, good evening to all the participants and the students for this NPTEL course titled multi criteria decision making. My good name is Raghunandan Sengupta from the IME department at IIT Kanpur. So as you know this is the fifth week set of classes which is going on and this total course duration is for 12 weeks spread over 60 lectures and each week we have 5 lectures of half an hour each and as you know there are assignments after each week. So obviously once you finish the fifth week all the students and participants will take the fifth set of assignments. So the broad umbrella under which we are discussing many of the topics is basically multi criteria decision making and under that you have multi objective decision making, multi attribute decision making and there are sub parts like of MCDM and I have explained that that in multi objective decision making is more mathematical where continuous variables are considered more straight forward and it is more objective while multi attribute decision making and the concept related to multi attribute utility theory is more subjective, more discrete search and there is much more the answers which you may get under MADM and the concept used in MAUT would be much more subjective based on the decision maker. This is the 23rd lecture which is we are in the third lecture for the fifth week.

The coverage which we will be doing and obviously when I mention the coverage I mention the point also that overall under the broad umbrella we will cover merit optimality. We have covered few of the concepts of MCDM definitions, the comparison of MODM then multi attribute decision making and we will consider further on the merit optimality concept, property of dominance, what does dominance mean, strong merit optimality, weak merit optimality, how the concepts of merit optimality can be understood in 2D space. In 3D space it will be we can imagine or we can also analyze but in higher dimension it will be difficult for us to visualize. We will consider effective versus inefficient solutions, the concept of Karusch-Kuhn-Tucker condition this what I am saying is not the total coverage for today's lecture we will cover it and whatever today's lecture is that will be updated in the slides and it will be shared accordingly.

The Karusch-Kuhn-Tucker condition, the scales of measurements which are nominal scale, ordinal scale, interval scale, ratio scale and why scales are important you will understand when we come to the concept of multiple attribute concept of decision making where decisions are more subjective than being objective and then we will go into depth about goal programming and go step by step. So when we consider a multi objective problem which we did discuss and I did mention few important points which will be highlighted with examples. So the points which we discussed and which was the focus I will use the red color so that the differentiation of the colors scheme which I always insist would make things much clear. So if you remember in multi objective decision making where it is more objective oriented, multi objective decision making multi where it is more as I said continuous, mathematical, but more clear, clear cut answers. So you have functions f_1 to say for example f_j, J is a suffix I am using even though in the concept where we used in the last lecture was M, but J is being done such that all the subscripts are absolutely clear cut there is no confusion.

And subject to conditions being $g_1 \leq b_1$ and these continue to say for example $g_K \leq b_K$. And if you remember I did also mention x is not a scalar decision variables is basically in the search space from x_1 to x_L. So it is a L dimension one, but obviously remember when we solve the problems always later on in multi attribute decision making we will always consider there are M number of alternatives which is a and there are N number of criteria and the relationship of M and N which I said that M = N, M can be > N, M can be < N will be clarified as we proceed with different type of problems. So let us consider the multi objective problem of the form where you have you need to optimize, optimize means it can be maximization or minimization there is no problem. You need to optimize f_1 to f_m which is basically f_1 to f_j we are saying x which is the decision variable by the way x when we are considering it can be depending on the problem can be continuous, can be discrete, can be binary, can be mixed integer whatever it is.

So when I consider this problems I am mentioning the examples they may be repetition plus please bear with me continuous case can be say for example you are producing paints in litres, litres can be 2.52 litres, it can be 10.69 litres, it can be 125.6255 litres. So they are continuous variables and obviously the amount of paint being produced cannot be negative.

So they are all continuous variables > 0. Now when I come to the concept of integers it is when you are considering the production of say for example number of chairs, number of tables, number of trucks being utilized when you are transporting goods, number of bins you are packing, packing that means putting in and transporting those are integers that means you cannot have 2.25 number of trucks being utilized for transportation or you cannot basically make 1.259 number of chairs. So they are integers.

In the case of many problems where it is binary it will be whether you build the factory or not build the factory, whether you transport or do not transport so they are binary and in many of the other examples they can be decision variables which we will see later on which can be integers, some of them integers, some of the discrete, some of the binary and accordingly. So when I mention x is basically of L dimension so I am not going to the details but this is the basic framework. Here f_1 to f_m which is basically f_1 to f_j are individual objective functions such that I want to find out the best value of this functions which we considered in the example the last lecture. So if you remember there were two objectives and these two objectives when they are optimized they give you different set of decision variables x_1 and x_2 and interestingly the maximization problem for both f_1 and f_2 gave the same answer for f_1 and f_2 but for different solution or for the decision space. But when we try to basically find out the common point based on the multi objective case then we saw that it was a integer problem if you remember then we found that the objective functions for both this f_1 and f_2 are different but what was important is the decision variable which was x_1 and x_2 was same and I showed you in a very simple way in a two dimensional case how you can visualize how the solution space that is point number 1 and also the optimum point or optimum solution separately for f_1 separately for f_2 and then combining them we saw that there was a difference.

So here we are saying that f_i which is basically each objective function basically is on the real line once you put the value of the decision variable it gives you some functional value which you want to optimize individually it can be done but we want to basically collectively optimize that is what the idea of multi criteria or multi objective decision making is. There may be instances when the objective functions are at least partially in conflict that is we do not have any decision vector x which optimize all of the objective problems or all objective functions simultaneously. So again coming back to these two examples number 1 the example which we consider in the 22nd lecture where we want to optimize a function f_1 and f_2 separately first and then combining them so we found out that they were different results. Now when we come to the concept of multi attribute concept where there are attributes characteristics and where it is much more subjective consider these examples that my means an individual decision maker and consider I am the decision maker I want to buy a car or I want to get an admission for some college or I want to buy a house or I want to recruit somebody. So in that case if in when I am considering buying a car the conflicting criteria here the conflicts are happening for the criteria or it can happen for the alternatives when we basically combine the criteria and try to basically compare the alternatives.

Alternatives are basically buying the car can be based on price can be based on mileage can be based on maintenance cost can be based on safety and so on and so forth. For the case of mileage case of maintenance case of cost it is all very clear cut there is a price component and it can be communicated it is very objective in nature and you can analyze. But what if we consider the concept of safety features what we consider the concept of say for example, style what is the idea that how would we consider the concept of color. So in that case the concepts of multi attribute decision making along with multi objective decision making will be considered for such decisions when the decision is there to buy a car. The other two examples which I mentioned few minutes back is buying the flat I come back to the same example time and again because these are the example based on which we are proceeding.

So it should definitely give you a better idea that how these problems can be utilized or the concept can be utilized to solve the problem. When you are solving the problem of buying a flat on an apartment we know the objective facts are price objective facts are maintenance cost. But the subjective facts can be how safe it is or how near it is to the metro or how near it is to the transportation hub like bus, taxi, auto or how close are the schools, how close are the set of offices where people work where you have to basically go and work. So these are the sets of criteria we want to basically consider when trying when you want to buy an apartment. Similarly when you are going for a higher studies as I mentioned I will keep mentioning these points again it can be based on price of cost of tuition, it can be based on what is the quality of education, what are the different type of courses which are offered, how are the faculties, what is the scenario and the chance of getting good internship, what are the scenario and chance of getting a good final placement, what is the contact and how good is the contact of the alumnis.

So these all these criteria will be considered where some of the criteria's would be objective some of them would be subjective. But coming back to the last point which is there, there may be instances where there conflicts for the car example price is too high which is detrimental to my decision process because I want to basically buy at a lower price but the safety features are excellent or consider there the safety features are very low which is I do not want but the mileage is excellent which is positive to me. So how you basically balance the positive and negative points that has to be analyzed. For buying the for getting an admission in a college it is very costly but the set of alumnis or set of courses or the set of faculty members are excellent or it may be the cost of getting a loan is very cheap because people are willing to banks are willing to give you very good rate of interest rate for getting good loans. But maybe the educational level or the type of education which you will get in that institute maybe of not that quality when you basically try to analyze with other institute.

So those compromises have to be analyzed and what is best that would we will consider considering the concepts of utility theory which have already considered and further on. To make things more formal we state few results which are basically the properties of Pareto optimal solutions which may be considered as relevant and important in this area. Now when I am considering the concept of Pareto optimal solutions if you remember generally consider this case and I will draw the diagram using a different color let me draw the axis here. So there are two products and I mention this product this P is not the price is product just what one product one and product two. Now consider your budget is fixed and you want to spend money for your consumption and a savings.

So obviously if your budget your total income your total salary is fixed and obviously if your expenditures increase then your savings decrease and if your and consider if your expenditure decreases obviously you will try to increase your savings. Consider another example you want to consume some amount of hot beverage so this is a very common example say for example tea or coffee if the prices of tea increases obviously you switch to coffee and vice versa depending on that. So if you consider the concept of Pareto optimal solutions in the two dimensional space and these concepts were definitely highlighted in few of the examples which we did in the first set of lectures in the first week. So generally depending on your indifference curve or your level of budget we will basically have I_1 which is indifference 1 then I_2 then I_3 and so on and so forth where as you move from I_1 to I_2, I_2 to I_3 obviously it means your level of satisfaction is increasing but the balance which you want to do between product one product two tea coffee or expenditure and consumptions expenditures and savings should be balanced in such a way that you maintain that parity. Now these curves can be extended for the higher dimensional also say for example in the three dimension it will look like a circus tent which is just an envelope and any point there which you consider would be the optimum point based on what is your level of satisfaction which you are trying to meet based on all the three products because in that case they would be product one product two product three.

In this example there are only two products it is in a two dimensional Cartesian coordinate. Now continuing these examples so the three properties which we will mention here theoretically and later on it will be highlighted as we consider the examples by the way when I mention the concept of highlighting it will come up naturally as we solve the problems. The first property is property of dominance so we will say in mathematical terms a vector x or the decision which you are going to take will is said to dominate a vector y so x and y are two sets such that the functional values of f(x) and f(y) in maximum the cases maximum when I am mentioning I will come to that later f_i or $f_i(x)$ or $f_i(y)$ so f_i is basically any objective function which you have basically j number of objective functions we are considering each of them individually. So it will give you the fact that for any i from i = 1 to j or of any j = 1 to J, $f_j(x)$ will dominate or will be greater than in value then $f_j(y)$ I am calling it j because we are considering the number of objective functions as j = 1 to J. Here it is basically let me mark it with a different color this is j this is 1 to J so it will be ≥ sign and it will be > for at least one of this capital J's which means that in case if you have the dominance the functional value based on x when you compare with y can be equal for all the cases except one in that case we will say the set solution of x will dominate y.

Now if I consider the concept of strong so property of dominance basically if I go into further on in this concept of strong parity optimality we will consider a vector x again a solution x star x star is basically a vector which is the element of x, x is the set of solutions which you have is defined to be as strong of parity optimal the word strong is important if there exists some x star such that it dominates all the other x's in X and this concept of strong optimality point would basically prove that there are decision variables x star which would be denoted as subsets of x. So this concept of subsets would be used in such a way that they can be proper subsets or just common subsets which are there. Now the concept of weak parity optimality this ideas which I am explaining would be made clear with the examples which I said a vector x star in x is defined as weak parity optimality let me use a different color see for example blue if there exists no other vector of x such that $f_i(x) < x^*$ and this objective vector would be called a weak parity optimality if it corresponds to x^* and is weak parity optimality based on the fact it is less in the other case it was basically based on the fact that that < sign can be replaced by ≤ sign also. Now when we consider the concept of parity optimality we will always consider very simply the set which would dominate or a set of solutions which will dominate the other solutions in at least one of these values of x based on the concept of different type of objective function which I have. So say for example if I have very simply set 1 which I am denoting in black color and the objective values are given here so I am not considering the corresponding x for the time being.

So f_1 is 2 based on that particular x_1^* so this is x_1^* I am just utilizing in order to denote the second value based on $x_1^*$and remember x_1^* which I am writing is basically a vector. So the second objective function is say for example 3 and the third objective function is basically 4. When I want to compare with a different set of solution and I will use a different color consider it is $x_2^*$and x_1 x_2 are vectors again there are subset of the overall set of solutions and the solutions of f_1, f_2, f_3 are now consider 2, 3 and 3 which would mean x_1^*does not dominate x_2^* based on functional value 1 because both are 2 they are equal to x_1^*does not dominate x_2^* based on functional value f_2 because both the values are 3. But interestingly which I will mark here with a different color the value of f_3 based on x_1^* and when I compare the value of f_3 based on x_2^*4 > 3 so in that case we can we would say that x_1^* would dominate the value of x_2^*. But where the problem would lie so here there is no confusion in the sense that 2 = 2, 3 is basically = 3 I am talking about the values of f_1, f_2, f_3 and 4 > 3 so x_1^* dominates x_2^*.

Now consider another example so I will use a different color and let me mark it in green and marking in on the top consider x_3^* the values of f_1, f_2, f_3 are when I am going to compare with x_1^*only. So, this is the example of x_1^* and x_2^*. So, here it is interesting if I compare the values of x_1^* , $x_3^*$for f_1, f_2, f_3 if you see x_1^* when I want to find out $f_1$is 2, but when I compare with the value of f_1 based on x_3^* it is 3. So, obviously x_3^* which is noted in green color is not important. Next when I come to f_2 functional value both the values for x_1^* and x_3^* are 3 so they are equal.

When I come to the concept of the functional value of f_3 based on $x_1^*$and x_3^*you will find $x_3^*$is 4 while the value of x_3^* is 3. So, the dichotomy or the decision based on I will use a different color based on the fact that 3 > 2 here which is fine, but here 4 > 3 which basically tilts the balance again in favor of x_1^*. But the question is that the net worth of 3 > 2 or 4 > 3 has to analyze where the concept of Pareto optimality and dominance has to be considered. So, I may say I as an individual would say that the functional value which I have for f_1, 3 is better other person may say no 3 is not better may be 4 is better than 3. When I am talking about 4 and 3 if you look at this set of diagrams which I have drawn the confusion should be not there.

Now, we will consider a very simple 2 dimensional figure to analyze the ideas of Pareto optimality. So, it will illustrate the Pareto optimality plots for 4 different combinations and we will denote the objective functions. Now, we are going to consider the objective functions by using objective 1 underscore objective 2 underscore where objective 1 and objective 2 are the 2 different sets of values which I want. So, let us consider objective function 1 and objective function 2 in a very simple sense this is f_1 and this is f_2. So, in the other example in the other slides we consider f_1, f_2, f_3 and in general they would be J number of such objective functions.

I am not mentioning anything for this example about the decision variables they can be l in number l can be that is L, L can be 3 can be 4 can be 5 can be 10 accordingly. Now, the bounded area which we will see now would basically depict the hypothetical feasible set and the boundary would be considered to be optimum set of feasible points depending on whether it is a maximization or a minimization of that objective function which you are considering. Now, here it is important if you remember for the other examples when we considering this x_1^*, x_2^*, x_3^* using the different colors and the values of the objective function capital f_1, f_2, f_3 I was always mentioning the point that you are trying to maximize hence 2 being less than 3 would mean that the objective function value is better to have 3 than 2. So, we are always trying to maximize, but it can be the other way round also that means we want to minimize. So, the minimization concept and the ideas what I mentioned in the last slides would continue to hold true, but you have to look at the point that you are trying to basically reduce that value of the functional form that is f_1, f_2, f_3.

Now, in the coming set of the diagrams there are different type of color combinations which I use in order to make it clear. The red colored would basically the Pareto optimal points in A, B is for the case when we consider the minimization of objective function 1 along with the minimization of objective function 2. So, if you notice the second bullet point I did mention there is a minimization maximization concept also. So, in combination they would be 4 different ways of trying to analyze which is maximization of f_1 which is objective function 1 along with maximization objective function 2, then minimization of f_1 and minimization f_2, then maximization of f_1, minimization of f_2 and the last one would be minimization of f_1 and maximization of f_2. So, with this I will close this 23rd lecture and continue this slide would also be shown in the next class or in the next lecture because this will continue with some discussions and have a nice day and thank you very much for your attention. Thank you.