Multi-Criteria Decision Making and Applications

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Week 05
Lecture 22

A very good morning, good afternoon, good evening to all the participants and the students for this NPTEL MOOC course titled multi-criteria decision making and my good name is Raghunandan Sengupta from the IME department at IIT Kanpur. If you remember in this class which would be the 22nd class and as you know that this is a 12 week course with 60 lectures each lecture being for half an hour and each week there are 5 lectures. The broader umbrella for the coverage which will continue for many lectures would be multi-criteria decision making, multi-objective decision making, multi-attribute decision making and multi-attribute utility theory. So, we will try to analyze them simple problem wise. So, the overall coverage of this would be obviously as we consider the coverage concepts and the writing in the slides will change and they would be uploaded and repeating that such that there is no confusion in what I am teaching and what the information the participants and the students have. So, we have already considered MADM and versus MODM, they would be coming up time and again hence the coverage I have written.

We will try to cover Pareto optimality, property of dominance, strong Pareto optimality, weak Pareto optimality, the concepts of Pareto optimality in two dimension, the effective versus ineffective solutions, Karusch-Kuhn-Tucker conditions, scales of measurements, there are different scales which will be important and goal programming and further on. So, if you remember in the last class which is the 21st lecture, we were discussing two simple methods of trying to solve multi-objective programming and one was which I am trying to highlight in the yellow color is basically you optimize each individually, each functions individually, objective functions and bring the rest of them whichever you are not optimizing as a part and parcel of the constraints and solve them and we will see that later on as we proceed. Here if you remember the a_s part which is the constraints which are relative to the objective function which are not being considered in the optimization problem and they are part and parcel of the constraint. This a_s would be decided accordingly and the decisions taken likewise which I have already mentioned in 21st class and if I consider this other method, there are many methods but I am considering these two initially if I am putting a green mark in order to highlight that.

You put weightages w_i are the weightages given to the each ith objective function and such that the sum of the weights add up to 1 that is important. The actual threshold a_i, a_s and the weights w_i can actually give some meaningful solution but as I said deciding on these values whether the threshold or the weightages depending on which methodology you are trying to utilize to solve the multi-objective problem may be difficult, practically difficult theoretically you can get answer but whether it is practically feasible that has also to be analyzed and what is the practicality of the answers. Now let us consider the bi-objective optimization problem which are integer valued and I will show you that if I consider this bi-objective very simple bi-objective linear problem bi-objective Y the first objective f_1 is given by $-2x_1 + 4x_2$ and the second one which I am trying to highlight with the different color is given by $3x_1 + x_2$. So there are two different objectives. Now under each objective or collectively for each objective the constraints are the same which I will put the constraints in red mark.

So the first constraint is $x_1 < 5$, $x_2 < 5$ with the second one and the third, fourth, fifth constraints are given as $(x_1 + x_2) < 8$, $(-x_1 + x_2) < 4$, $(x_1 - x_2) \leq 4$ and x_1 and x_2 as I mentioned are all integers. Now the question would be can we solve individually the objective functions using considering integer problem? Yes we can solve it is a very simple concept and there are algorithm methodologies in operation research based on which you can solve this objective functions considering their single objective function. Single in the sense we can consider the first objective of f_1 solve it with this constraints which are marked in red. Constraints marked in red and the second problem we can solve it which, I am putting a tick mark, which is f_2 again based on the set of constraints which are same which are again marked in red. So this is what I do and their maximization, minimization, maximization, ideas of how you solve it would be the same ideas means how you structure the problem. So the first problem which is this one maximize the first objectives constraints are same and the second problem is basically maximize objective function 2 again constraints are same.

So they are marked in different colors green and blue. Now let us solve them and let us analyze the answers in details. The single objective problem solution for problem 1 which I have marked as f_1 and which I will be using green color for the second one I will use the blue colors to auto differentiate. So if I use objective function 1 on optimize the value comes out to be 18 and the respective decision values are x_1 and x_2. Now for simplicity let us check whether $x_1=1$ and $x_2=5$ satisfy them.

So if I consider x_1 as 5, so if I consider x_1 as 1, x_2 as 5, so the first one $x_1 \leq 5$ perfect, $x_2 \leq 5$ perfect, $x_1 + x_2 \leq 8$ perfect because $5 + 1 = 6$, $-x_1 + x_2 \leq 4$. This is $-1 + 5 = 4$ is fine, $(x_1 - x_2) = (1 - 5) = -4$ is ≤ 4 is perfect and x_1 and x_2 which is 1 and 5 are integers and if I put them in the objective function which is $-2 \times 1 = -2$, 4×5 is 20, $20 - 2 = 18$ which is perfect. If I consider the I am going step by step, so if I consider the second objective function which I marked in blue, the objective function comes to be 18, f_2 is the objective function and the decision variables is x_1 is 5, x_3 is 5. So let us consider 5 and 3, so

$x_1 < 5$ which is fine then $x_2 \leq 5$ which is fine because that is 3, $x_1 + x_2 = 5 + 3 = 8$, $8 \leq 8$, $-x_1 + x_2 = -5 + 3$ which is $-2 \leq 4$ is fine, $x_1 - x_2 \leq 4$ which is $5 - 3$, $+2 \leq 4$ perfect, x_1, x_2 are integers fine and when I put them in this equation 3×5 is basically 15, 15 + 3 is 18 which is also fine. But what is the problem? The problem is when I singly objectively optimize f_1 and f_2 the values are same which is a different question the decision values based on which you get individually x_1 maximize individually f_2 and x_1 is not the same. Why not the same? Because if you see here x_1 is 1, x_2 is 5 for the first objective which is coming from here and if I consider the second objective value $x_1 = 5$, $x_2 = 3$ they are not the same.

So, different points give you different objective function optimization for x_1 and f_1 and f_2 which is not the idea of multi objective criteria. In case if you are interested to find the best solution for this bi-objective model then interestingly the objective function which will optimize them individually, but taking in a collective manner would be in case you have this the objective function and the decision variables where I will concentrate first x_1 and x_2 for both the bi-objective problem solutions are same as it should be. So, you would have x_2 as 2, x_1 as 2, x_2 as 5 and when I put them in the objective function it will be if I go back 2 and 5. So, if I put $x_1 = 2$, $x_2 = 5$ and then if I put $x_1 = 2$, $x_2 = 5$. So, 2 (-2) = -4, 4×5 is 20, $-4 + 20$ comes out to be 16.

If I put 5 in the objective function 2, 5×3 is 15, 15 + 3 is 18. Let me check no sorry my mistake, $2 \times$ in this first case -2 (2) is -4, I have 20 which is 16 when I put 2 and 5, $3 \times 2 = 6$, 6 + 5 is 11. So, this is 16 and 11 as is the answer we get here. So, this would be basically called this superior solution or the maximum solution would be basically the utopian solution which we all want to in the sense collectively we want both to reach the maximum which if you see the first bullet point where they are marked in green and blue is true, but separately they would reach that. But when I consider both of them to basically proceed in the same direction and optimize or maximize at the same set of points of x_1 and x_2 there is a different answer.

Both for the decision variables as well as for the case when you are considering the objective functions. Now just important point, you may be thinking that well that does the collective sum of the objective functions give us best results when I consider both of them together answer is no. Because if I consider them individually the answers worth 16, 16. So, in some remote sense if both this points of x_1, 1 and x_2, 5 and then other point is x_1, 5 and x_2 is 3 they were reached f_1 and f_2 would be 18, 18 individually. But when I am reaching the collective point which is x_1, 2 and x_2, 5 then the individual values are 16 and 11 and if I want to add them without giving any weights add them and find out the objective function then is 16 + 11 weights if the word I mentioned.

So, if I consider 50% of the weight for f_1, 50% of the weight for f_2 then the objective function would be (16 + 11)/2 because 50% is 0.5. Similarly, if I want to find out the objective function based on the weight of 10% on 16 and 90% on 11 which is f_1 10% f_2 90% then the answer would be 0.1 this is the multiplication sign 16 + 0.9 × 11 and you will get the answer this is I just wanted to mention.

Now more detail analysis pictorially will be done. Now this is the real part which should give us a lot of information and pictorially you can appreciate that. If I go back to the original problem the overall space if you remember so I will write them such that the information loss is not there. So, one the objective functions which I will write $x_1 < 5$, $x_2 < 5$. So, $x_1 < 5$ which is this one $x_2 \leq 5$ is this one and obviously they are > 0 because they are and we have considered the problem accordingly. Next is $x_1 + x_2 \leq 8$. $x_1 + x_2$ if I consider ≤ 8 this would be the line because when x_2 is 0 x_1 is 8 that means this line will touch the point (8,0) and (0,8). So, this would be the line I am just putting a tick mark because I do not want to clutter it. The other point will be– $x_1 + x_2 \leq 4$. So, when I put x_1 is 0 x_2 is 4. So, x_2 would be 4 would be this point when x_1 is 0 x_2 is 4 and when x_2 is 0 x_1 is – 4.

So, yes - 4. So, x_2 being - 4 would basically give me the point somewhere here. So, this would be the line and similarly the last constraint which is $x_1 - x_2 \leq 4$ would give me the last point this one. Now, the overall boundary is that light blue part and this is basically starting from here till here A B this point has not been marked. So, I will just put the highlighter there I am not marking any letter there is another point C this is the third the next point after C this is the second last point. So, if I see the overall area which I will try to mark. So, this is the origin one boundary A which is marked here this one C this one this one. So, the overall space is there and the integer points are given. Now, interestingly let us consider them one by one, one by one means the objective functions. So, I will remove all of them such that there is no cluttering when I write. The first objective was – 2 x_1 + 4 x_2.

So, if I consider that what was the decision variables and the objective function. So, let us go back to the decision variable was 1, 5 and 18. So, 1, 5, 18 is this point 1 means x_1 is 1 and x_2 is 5 and if you remember the point is this. Now, if I consider the objective function this one is basically the line red it shown in red in color which is moving in this direction. So, obviously these points which are there red the red one which is basically I am marking in green here because the green color is used here this point will move outside and the moment it crosses the individual area the feasible space is A it will give you the best solution and that is an integer is satisfied.

If I consider the other objective function which I will mark in blue which is 3 x_1 + x_2.

This is the green one and this is moving outside. Moment it touches the point C it gives you the best optimum solution for f_2. Now, what is the coordinate of C if you remember the coordinate of C is as from the diagram it is (5, 3). If you see these are 5 and 3 as solved individually, but now comes the interesting part let us combine them. If I combine them the overall point which is satisfy both these objective function should be 1 not individual separately like A and C no and the objective function reached would be objective function maximize for that collective bi objective problem not individually anymore.

So, let us see where they are same. So, what I do pictorially is that I keep shifting the red one inside the feasible region and I also keep. So, if this red one which is objective function 1 for which I had used green color I will continue using this it should be moved down in order to satisfy both of them together and the blue color which I have been using which is green shown green here should also be moved in this direction such that the moment they meet at the one of these points which is B which I will mark in red that will give me the best solution which will satisfy both objective function 1 and objective function 2 simultaneously at the same point and the decision variables are same. So, let us see what are the decision variables the decision variables if you see from the graph is $x = 2$ and $x_1 = 2$, $x_2 = 5$. So, here you see $x_1 = 2$, $x_2 = 5$ and the values were 20, 60 and 11.

So, now individually the objective functions were A and B. So, this is A which I am circling now A and C sorry and the other one was C, but collectively when I consider it is B and the solutions have changed for the first objective from 18 to 16 for the second objective from 18 to 11 and also the individual decision variables have also changed. Now the question would be that it has to be optimal and we will see that in examples later on more details. It should be noted that an ideal solution in MADM which is multi attribute decision making is a subjective one because that will depend on how the individual is trying to analyze the problem. It should be noted that an ideal solution in MODM is an objective one.

So, MADM which is multi attribute decision making which where attributes characteristics were considered is more qualitative in nature hence they have to be analyzed accordingly and multi objective decision making is more ideal solution in the characteristics because they give us a much more quantitative feel of the answer. For MADM which is multi attribute decision making generally an ideal solution may not exist because what is best which is utopian one we do not know, but the concept of an ideal solution is important as it helps once to find the compromise solution which is near or as near as possible to the ideal solution. So, in ideal solution concept would be utilized in both the concepts of MODM and MADM, but in MODM it will be more objective in nature while MADM it will be more subjective in nature and ideal solution concept would be utilized in such a way that that is the ultimate goal I want to reach or what is the answer I want to reach and we will try to basically be as close as possible to the answer. The broader idea of MCDM which is multi criteria decision making determines the best feasible solution based on different criteria and characteristics. Unfortunately, finding a solution highlighting different criteria characteristics with different effects may not be possible all the time or may not be possible in many situations.

Hence, the concept of Pareto optimality is used to find the best solution whereby the important concept of non dominated non inferior or efficient solution which is a set of admissible alternative choices in statistical decision theory concept will be utilized and we will consider that with simple examples to highlight that. The concept of Pareto optimality to find the best solution such that the idea of non dominated non inferior inefficient solutions concept would be utilized. The term Pareto optimality is named after Vilfredo Pareto and Italian economist he used the concept in his studies of economic efficiency and income distribution. It is true that as per this concept they can be non inferior solutions as introduced by the person Vilfredo Pareto in 1896 though the formalization of this idea which led to the mathematical understanding in more detail can be credited to Francis Edgeworth. The author is best known for the Edgeworth box, but it must be remembered that in the field of multi objective optimization the concept is known as Pareto optimality rather than the Edgeworth optimality and we will be considering the concept of Pareto optimality pictorially in more details.

The origin of Pareto optimality goes back to the following text from Pareto which says we will begin by defining a term which is desirable to use in order to avoid prolixity. We will say that the members of a collective league enjoy maximum optimality the word optimality is basic optimality in a certain position when it is impossible to find a way of moving from that position very slightly in any direction in such a manner that that point of optimality enjoyed by each of the individual collectively basically can be increased. That concept may not be possible because the word optimality is such that everybody has reached his or her best level any slight change would basically make one person at least one person inferior. That is to say in any small displacement in departing from that position necessarily has the effect of increasing the optimality with certain individuals enjoy, but as I said it would decrease the enjoyment level of at least one individual. So that means collectively some may be happy or more happy, but someone's happiness would have decreased which would not be allowed by Pareto optimality because everybody has reached their self goal collectively such that any displacement would basically break down that goal.

There are other source of references for Pareto optimality formally analyzed for production as resource allocation problems where the combination of different activities represent the output of commodities as a function of various factors. If you remember the labour concept or raw material concept of production frontier concept which I considered. Another classical reference of Pareto optimality concept is used in optimization is from the area of decision sciences which is under statistical sciences. Reference about Pareto optimality can also be found in the work of economics and in the domain of multi objective optimization. Having said that we will close this class and continue more discussion about Pareto optimality with examples. Thank you very much.

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