Multi-Criteria Decision Making and Applications

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Week 04
Lecture 20

A warm welcome to all of you, a very good morning, good afternoon, good evening to all the participants and the students who are doing this multi-criteria decision making under the NPTEL MOOC series. As you know this is a 60 set of lectures and we are at the last lecture for the fourth week which is the 20th lecture and each week we have 5 lectures each being for half an hour and as you know that I am mentioning it for the first time there would be assignments for after each week. So after this lecture you will be taking the fourth assignments and obviously the coverage would be only related to the week coverage. Now the broader umbrella which we should end today, we have been discussing the broader concepts for a long set of lectures is the definition of MCDM, concepts of utility theory, I am mentioning the concepts of expected value, the four different properties of non-satiation, risk averse, risk lover and risk neutral person, then the concept of absolute risk aversion, relative risk aversion, the concept of certainty. Then we considered the four different examples of utility function and found out their corresponding values of A'(W) properties, values of R'(W) and their properties for all the four different utility functions which are normal, exponential, logarithmic and power. And then we considered the concepts of how certainty value can be utilized, few of the simplest actions of utility function.

Then we went to geometric mean method, considered few of the concepts, then went into the rate of return R, r, went into safety first principle, three proponents of safety first principle of trying to minimize the probability, maximize R_L, Max [$\bar{R}_P$]. And in the 19th lecture we considered the concept of stochastic dominance and only briefly considered, briefly I would not use the word briefly, we went into details of trying to understand the concept of first order stochastic dominance with an example. So the coverage for this lecture which is the 20th one would be about stochastic dominance and hyperbolic absolute risk aversion concepts. So this slide is a repetition of the same thing in the 19th lecture and here our concentration is on the first order dominance.

We have already considered, we will consider the second order dominance and consider that with an example. Example in the sense that like there is a decision in the first order dominance there was a decision A, B and there were states of nature as 1, 2, 3 and the values were given and the corresponding probabilities were given and I did mention important thing the probabilities for each decision is 1 and we will see that that concepts still remains true, it has to be. The idea is like this and I have taken the states of nature which was amounts which are there it was given in the row wise, I am considering as the column wise just a pictorial representation nothing different. In the first order concept of dominance I considered decision A and B, here also I am considering decision A and B but they are given along the columns not along the rows. The amounts of the states of nature are ordered from the minimum to the maximum with amounts being 1, 4, 5, 6, 8, 12 it can be other thing also but I have taken simple values and the probabilities are given based on the fact that for A and B I am trying to find out where decisions can be taken.

Now why I have made this it will become clear, if you see decision A and then mark with the green one, so for decision A for amounts 0 there is no investment hence the probability is 0. For an amount 4 for state of nature is 4 the probability is 15% which I am putting a tick mark then for an amount 5, state of nature 5, probability is 0.4 then for nature condition being 6, or amount being 6, probability is 0.25, for 8 outcome in nature, there is no investment so it is 0 and finally, for 12 it is 2. If you add up the probabilities 15 + 40 is 0.55, 0.55 and 0.25 would be basically 0.8, 0.8 + 0.2 is 1 so the probability is 1. For decision B amounts I have kept the same but obviously there would be changes in the probabilities some may be 0 which were not earlier because they can be different decisions. The probabilities I will read it out for amounts state of nature being 1, 4, 5, 6, 8, 12 are given as 0.1, 0.1, 0.1, 0.3, 0.2, 0.2 so the probability sum is 0.1 + 0.1, 0.2, 0.2 + 0.1, 0.3, 0.3 + 0.3 is 0.6, 0.6 + 0.2 is 0.8, 0.8 + 0.2 is 1 again it is 1. Now, let us find out the expected value and there is a reason why I am trying the expected value and as shown in the first order graph also and as will be shown in the second order graph also stochastic dominance. The expected values are corresponding state of nature amounts multiplied by probabilities for A it is $1 \times 0 + 4 \times 0.15 + 5 \times 0.4 + 6 \times 0.25 + 8 \times 0.0 + 12 \times 0.2$. The value comes out to be I will use a different color say for example red for decision A it is 6.5. Similarly for decision B where I should use a different color violet so it is the corresponding expected value can be found out by $1 \times 0.1 + 4 \times 0.1 + 5 \times 0.1 + 6 \times 0.3 + 8 \times 0.2 + 12 \times 0.2$ the value comes out to be 6.8. Now, let us see the diagram and again the same concept it will be a cumulative distribution drawn and the jumps will occur at the places where the probability is non zero. So, I will take decision A and decision B with different colors and explain that illustrate that.

So, let us consider the decision A where the amounts are if you remember in an order for better explanation ease of explanation I have taken the amount same which is in column 1 and column 3. Now, let us consider the probabilities jumps which are happening for A it is 0 at 1, 0.15 and 0.140 and so on and so forth. So, let us consider so point 4 if I consider the x_1 as in blue the corresponding jumps are happening so for 1 it is 0 and then at 4 it jumps at 0.15. So, this is the one the blue one is happening jumps at 0.15 then again the jump happens at 5 which is 0.4, 0.4 is this height. So, it becomes now 0.15 + 0.4 is 0.55 then the jump happens at 6 which is here. So, 6 is again 0.25 which becomes 0.8 total cumulative probability and finally, the jump happens as at 12 which is 0.2 makes it 1. So, the blue line I will draw the blue line then separated the red line and then mark the concept of second order dominance. So, the blue line for x_1 jumps. Now, let us come to the second random variable x_2 the jumps or the probability jumps I am using for the probabilities are starting from 1 to 12 are given in the fourth column 0.1, 0.1 and so on and so forth. So, 0.1 happens for 1 again 0.1 accordingly. So, if I consider 0.4 again 4 it again jumps 0.1. So, if I draw it using the red line this is the case for x_2. Now, what is important is now here if you see the red and the violet one they sometimes red is over the violet and sometimes violet is below the red and that is the concept of stochastic dominance of the second order in the sense that there would be some portions of x_1

which I am marking as blue where it is above this is above and there are some portions which I will mark in green where the red portion graph is below. So, this is below. So, this would be the concept of utilizing second order stochastic dominance for the case of x_1 and x_2 where x_1 stochastic dominance x_2. The mean values are given here the red vertical line and the violet or the blue vertical line and if you recollect the values were given for decision A and decision B as 6.5 and 6.8. So, this was the simple example of stochastic dominance. We will try to wrap up this fourth week and maybe for utility concept using the concept of hyperbolic absolute risk aversion function. Now, this is a little bit theoretical concept applications can be there in the theoretical sense quite a lot of application, but that would not come under the purview of this course I will mention them. They would not be directly related when you solve the problems for this course, but for people who are curious if they want to pursue and understand I will mention that accordingly. So, HARA function which is hyperbolic absolute risk aversion function is used in decision science which portrays a type of risk aversion which specifically refers to von Neumann's Morgenstern utility functions.

And this is a very general function which corresponds to the final wealth or some related variable it can be returns also, returns means R or r and it describes the decision makers degree of satisfaction with the outcome of the wealth. So, in this hyperbolic absolute risk aversion function there are three parameters a, b, γ and W is the wealth and if you see the parenthesis terms which is given here $b > 0$ and the parameter terms which for a, b, γ, I will put the green tick mark where it is mentioned parameters. They will be decided accordingly such that you can find out few things and what are the important things I will just mention. If somebody wants to check they can check the slides for the derivations, but I am not going to come into the derivations from the point of view of detailed solving.

So I will mention it you can refer. So consider the linear or risk neutral utility function which you remember we have used to find out the concept of certainty value. The people were risk averse, people were risk lover and people were neutral. So the utility function is basically given by $U(W) = W$ which is I am circling and for the case when $\gamma = 1$ which is the important factor assumption you can consider the limiting case when $\gamma = 1$and use the idea of absolute risk aversion this hyperbolic risk aversion function which is HARA. So I will mention HARA accordingly and for the case in the limiting case when $\gamma = 1$ and considering a= 1 and mark this two important a = 1 and the idea of $\gamma = 1$ you can have the utility function in the simple sense as the derivation is given as converted to linear or risk neutral which is the simple case of utility function. For the second case which is the examples which you have considered, was the quadratic utility function.

What quadratic utility function is given if you remember by $W - 1/2\, cW^2$ and $c > 0$. So for considering as $\gamma = 2$ and the assumptions and using the concepts of simple expansion we can assume with few assumptions like $c = a^2$, a^2 a, b and γ are the parameters in the HARA utility function. You can convert and have the quadratic utility function from the HARA utility function. The third part is or third example is basically one of them which you have done is the exponential utility function. And when I consider the HARA we will have two important assumptions here to convert the HARA into the exponential one and if you see remember the exponential one a was the parameter.

So, the two important facts for the HARA to be considered and is one when consider bis as 1 and $\gamma > 0$ in ∞. In one case it was taken in the first case it was taken as + 1. So considering these two assumptions you can convert the gamma into the exponential utility function. I am just giving the highlighted points you can check the slides or you can check the book, but again they are not directly related to the problem solving part, but they give you a lot of ideas. The next fourth example is again the power utility function we have considered. So the power utility function is given by cW^{γ} and if you remember c and $\gamma \leq 1$ as per the assumption. If I consider HARA utility function with the assumptions of $b = 0$ we can utilize the concepts and basically have derived the idea. So I should mark it as do as I was doing. If you consider $b = 0$ you can have the case of the power utility function from the HARA. The other case is basically where you consider the logarithmic utility function we have already considered if I consider HARA function.

HARA function is the one which you have considered in the first definitions. With the following case of the assumptions as $a = 1$ and $b = 0$ the parameters in the HARA we will get the utility function as the logarithmic one and derive it accordingly. And this in this case of trying to derive the logarithmic one we have to use the concept of L'Hôspital's rule in order to derive that. With this I will end the 20th lecture which is the last lecture for week four. I know it may be little bit short but I do not want to start any new concepts we will start in the 5th week accordingly.

Have a nice day and thank you very much for your attention. .