Multi-Criteria Decision Making and Applications

Prof. Raghu Nandan Sengupta
Industrial Engineering and Management Department
Indian Institute of Technology, Kanpur

Week 04
Lecture 19

Welcome back my dear friends, a very good morning, good afternoon and good evening to all of you at whichever time you are hearing these lectures and whichever part of the world you are and this course title is multi criteria decision making under NPTEL MOOC series which is total of 60 lectures spread over 12 weeks and each week we have 5 lectures and each lecture being for half an hour and my good name is Raghunandan Sengupta from the IME department at IIT Kanpur in India. So this is the 19th lecture which means we are in the almost end of the fourth week and the broader set of topics which we have been continuing for quite long time is basically about the MCDM definitions, concepts of utility theory. Just to recap because we are going to end this broader set of topics, I will just mention the important points. The concept of utility theory, how you find out this expected value and there were many examples which we have discussed. Then we considered different concepts of the properties, four properties non-satiation, concept of risk, concept of absolute risk aversion, relative risk aversion. Then we considered the four different types of utility functions with their properties related to A and R separately for all these four, four were quadratic, the exponential, power and logarithmic.

Then we considered an important concept of certainty equivalent and how certainty equivalent can be used. Then we went into the ideas of trying to find, utilize certainty equivalent to map how the utility function of a person looks and then we further on went on to geometric mean methods and based on that how you can rank decisions. Then we found out the equivalence between normality of the returns and their concept of quadratic utility function. Then we left utility went into safety first principle and considered the three principles of safety first principle.

One is minimizing the probability of R_L, $R_P < R_L$, R_L is the bound, can be replaced by R_F also and then second was maximizing of R_L, maximizing of $\bar{R}_P$. So in this lecture we will consider stochastic dominance and the concept of hyperbolic absolute risk aversion property. So hopefully I can go into hyperbolic absolute risk aversion property or else it will be continued in the 20th lecture. So I will come to the examples of stochastic dominance first order, second order within few minutes, but first the concept may be it is mathematical, but let us lay stress on that. Consider two random variables x_1 and x_2, it can be A and B also and with the respective probability density function, but for our examples we consider very simply the ideas of probability mass function.

The ideas of trying to convey using examples would be from probability mass function, but it can be replicated for a continuous distribution also. With the respective PDF and the distribution function, distribution function is basically the cumulative distribution functions, they are given for the first random variable which is x_1, I will mark with the color blue. So x_1 random variable PDF $f_1(x_1)$ and CDF of $F_1(X_1)$. For the second random variable which I will mark in, see for example green, let me use the dark green which is x_2 for the second random variable, the PDF is small $f_2(x_2)$ and the CDF is capital $F_2(X_2)$. Then by definition we will consider the following properties.

The zeroth order dominance of x_1 over x_2 will be applicable if and only if $x_1 \leq x_2$ for all realization of the random variables and $x_1 < x_2$ for at least one realization. So if I am trying to plot, I will come to the examples as I mentioned for the first order and second order very soon. With concrete examples, if I have the real line and if I find out the values of x_1, I will denote by blue, so these are the random variables. As you pick up they are on the real line and I am choosing the corresponding values of x_2 which are in green. So if they are more strictly greater, some cases they are equal to, so that means both the values should coincide which I am marking here.

So in that case you have the zeroth order dominance of x_1 over x_2, x_1 is the blue one remember. So let me denote x_1 and x_2 was the green one, sorry the color did not change, this is x_2. Now as I said we will come to the first order and second order dominance, stochastic dominance using two random variables x_1 and x_2 and consider for the example. Now the first order dominance is true if and only if of x_1 over x_2, if the probability of $x_1 > y$, y is some value < Pr (x_2) being > y for all y's on the real line such that there exist at least one y such that rather than less than equal to the less than in the strict sense prevails. Now what I want you to understand is this, if I consider the probability is less than in the sense let me write this. So if Pr $[x_1 - y] \leq$ Pr $[x_2 > y]$ this is greater than. So they are greater means on to the right hand side if I am looking at the real line from my side, so random variable increases on to the right it decrease on to the left. So probability being greater than y which means that 1 minus that would be on to the left hand side. So I basically put 1 minus first if I when I put the minus sign this inequality would change. So the inequality for a minus sign, so minus or minus it becomes greater than equal to then I add 1 plus on to the left hand side plus 1 on to the right hand side also.

So based on that when I find out it will be the probability here now it is a greater than sign. So it means the Pr $[x_1 < y] \geq$ Pr $[x_2 < y]$. Which means if I draw this diagram or let me make a blank slide and then do it accordingly. So this part I will again do it in the blank side I will discard it and do it, so it is easier for us to understand. By the way the blank slide which I am going to create would not be there in the slides which are uploaded they would be there in the video, so you can understand.

Because the lectures slides are written systematically any extra portion I am just writing. The blank slide why I am creating because the space is limited as you remember in the earlier class which I did. So here I will write the probability I will again write the small state of definitions and derivations, so it is easy. The probability of x_1, so this was the case for this first order, so I make it minus sign reverses minus sign I add 1 greater than sign remains. So 1 minus basically now becomes probability what does this mean the last statement which I am putting a red arrow.
Pr $[x_1 < y] \geq$ Pr $[x_2 < y]$

So if I consider the distribution, consider the real line there are two distributions I denote one by green and which is say for example, x_1 and the other one as blue which is the y_1 which is x_2 and this is the mean value, this is the mean value. So what I want to ensure is there is some value of y which I will denote by red and consider I am taking y as say for example, fixed can change is the overall Pr [$x_1 < y$] on to the left side and which would always be more than Pr [$x_2 < y$], the probability of sense it you are trying to cover more. The second order dominance would mean when I find, so this first order dominance basically does with the distribution function. The second order dominance means for $x_1 > x_2$ if only if the integration of the whole area of the cumulative functions of $x_2 > y$ – Pr [$x_1 > y$] is always positive for all y and > 0 for some y that is second order dominance would hold true. So at least one in both the first cases should be greater and the second should be greater not greater than equal to in both the cases would give you the actual concept of first order second order correspondingly.

Now few conditions I will state without proof, the first order dominance stochastic dominance of x_1 over x_2 is a sufficient condition for the second order of x_1 over x_2. So obviously if I have the first order, second order, third order so and so forth this concept of sufficiency conditions for first to second, second to third and higher orders can be made accordingly. The second point is x_2 is a mean preserving spread of x_1 then if that is true then x_1 will be second order dominated over x_2 and what we mean by mean preserving spread is basically a change from one probability distribution to the other. Consider from A to B where B is formed by spreading out because the probability total sum is 1. So what we do is that by spreading out B by forming out one or more portions of A's probability density function such that leaving the mean value unchanged we just spread it out more.

So to make things easier let us consider two different examples diagrammatically I am not going to use any equations. The red one which you see is the uniform continuous case distribution between -2 and $+2$ and obviously if we know the mean value is given $\frac{A+B}{2}$ and if I consider these values for the red the expected value is 0 as shown here by the black vertical line where I will put a red tick mark. So this is for the red one which is $x_1 \sim U(-2, +2)$. Now consider the distribution x_2 with the same expected value which is also the vertical black line and I will use the dark blue to denote x_2 and it is spread between -3 and $+3$. So $x_2 \sim U(-3, +3)$ again the average value would be 0.

So 0 is fixed mean value is fixed for this is much more spread out if you can see spread out more while the first one. So this horizontal lines are 0 actually the distribution is this. So this would be the concept of means preserving spreads and if x_2 is a means preserving spread of x_1 as in the diagram where you have the uniform distribution of -3 to $+3$ with respect to that x_1 which is the first distribution then $x_1 > x_2$ in the second order sense. Let us consider a second diagrammatic representation the first one is basically in the blue color is for standard normal deviate x_1 which is distributed if I will use the blue color $x_1 \sim N(0,1)$ and this is the distribution which you see here. And x_2 if I consider such that both the mean values of x_1 and x_2 are same because they are given as 0.

So $x_2 \sim N(0,2)$ and the distribution is I will just mark it once this is the distribution. So if

x_2 is the mean preserving of x_1 then x_1 strictly in the second order sense would dominate x_2. Now the question is that just for your information you saw the examples where the mean values was 0. So is it necessary for the mean values to be 0? No, I could have basically taken any other distribution for the case of discrete and the normal case with different means. If I consider this example may be I could have considered x_1 ~ N(3,2), or variance of 2^2 and x_2 ~ N(3,4) or 16 is the variance.

So this can be also done. Similarly for the first example I need not choose the mean value as 0 it could have been, say for example, x_1 ~U(-10, -5). So obviously the average value would be what $\frac{-10-5}{2}$ = -7.5. Similarly, I could have x_2 done accordingly which is uniform and in that case I want to ensure -7.5 is the value. So I could have had, say for example, -12 and the values would be, say for example, if it is -12 and -3. So the value is $\frac{-12-3}{2} = -7.5$, but if in the second case which is the violet one the spread is much more in the red case it is much more concentrated. The values would give you the idea of mean preserving spread such that $x_1 > x_2$ in the case where the example was the discrete distribution and for the case when the distribution is considered as normal. I just thought these two examples would be a good one to illustrate.

Now till now I have said mentioned about stochastic dominance, but I have not discussed the examples. So these are the examples one by one for the stochastic dominance for the first order and the second order. So let us consider the example as such. So there are states of nature which we denote by 1, 2, 3 the outcomes what is the probabilities and there are two different decisions. Decisions I will denote by A and B.

Now the amounts of the outcome for the states I have taken arbitrary values they can be changed and you can formulate different problems to illustrate that. Another idea is that question would be that states of nature is only 1, 2, 3 can they be more? Yes they can be more they can be less also they can be two outcomes or they can be 4, 5, 6 outcomes also and decisions I have taken as A, B they can be more than two also like A, B, C, D accordingly, but I have made it simple. The amounts are given for state of nature as 10, 30, 70 and as per the decision A and decision B the corresponding probabilities for decision A for the states of nature of 1, 2, 3 are I will mark with the different color see for example red. The probabilities are 0.4, 0.6, 0 and you see the probability sums up to 1 obviously it has to be decision A is out of the state of natures only they cannot be anything outside. Similarly, if I consider decision B, I will use a green color the corresponding probabilities are 0.3, 0.3, 0.4 again the sum is 1. I need to find out the expected value first I am just giving.
So, the expected value for decision A would be what $10 \times 0.4 + 30 \times 0.6 + 70 \times 0$ because the last probability 0 the value comes out to be using the same coloring scheme comes out to be 22. And when I use the expected value formula for decision B the corresponding values are which I sum them up is $10 \times 0.3 + 30 \times 0.3 + 70 \times 0.4$ and the value comes out to be 40. Now let us draw the step functions of the decisions accordingly along with the expected value for decision A and decision B corresponding to the states of nature 1, 2, 3 separately and then I merge them I have used different colors so that you can understand them. Now consider $F_1(x_1)$ which is blue in color and

$F_2(x_2)$ is red in color. So, if I considered the corresponding probabilities so this consider only on the blue line. So, if I consider the blue line the jumps are accordingly jumps are how if I consider the decision A it is 0.4 first and then the jump happens 0.6. So, the cumulative values are 0.4 and 1 where is 0.4 is happening 0.4 is happening as the state of nature amount is 1, amount is 10 state of nature is 1. And the second jump is happening which becomes the cumulative distribution function becomes 1 is happening and when the state of nature is 2 where the amount is 30.

So, here it is 10, 30, 0.4, 0.6 let us see for the blue one it is 0 till 10 and there is a jump. So, the jump happens of 0.4 if you remember. So, the jump happens here I should use so this is the I should use the red color black one to denote. So, this is the first jump of 0.4 then it is constant, constant then when the value 30 occurs it jumps by 0.6 and it becomes 1. So, the blue one is basically for x_1 which is decision A. Now, consider let us consider decision B what are the jumps? Jumps are 0.3, 0.3, 0.4 at 10, 30, 70. So, at 10 it becomes 0.3 which is the red one continuous till 30 again jumps 0.3 because 0.6 continuous till 0.7 and then jumps 0.4. So, 0.3, 0.3, 0.4. Now, the interesting part is two things number one we see the $E(x_2)$ which is 40 which we calculated and the $E(x_1)$ decision A is 22, but interestingly throughout the distribution x_1 which is the blue one always dominates and how I am saying it dominates because wherever you see in the graph x_1 the blue line for x_1 is always above the x_2.

So, which you see here I will just mark at this portion obviously it is more again this portion is more. So, there is no portion of x_1 which is blue one is below the red one it can be equal in many of these cases we can formulate the problem, but it would never be fall below the x_1 if $x_1 > x_2$ in the first order sense. And obviously if I am going to consider in the strict sense not in the greater than equal to sense in the strict sense for $x_1 > x_2$ I will draw the graph I did not draw the theoretical one because for the practical one it will be easy for you to understand. If I draw the graph consider the axis is given with the axis R along the X you can I am plotting the amounts as usual or the decisions based on which I am trying to find. And if I plot x_1 like this because if you remember the probability is always 1 it cannot exceed and x_2 as the blue one ultimately the probability will be 1.

In this case the red one which we will denote by x_1 with respect to the blue one which will denote by x_2, $x_1 > x_2$ throughout because there is no portion it is equal, equal to sign if you remember would be for the case when this not strictly dominating. So, with this I will end this 19th lecture and continue discussing about the concept of stochastic dominance further on. Thank you very much and have a nice day.