Multi-Criteria Decision Making and Applications

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Week 04
Lecture 18

Good morning, good afternoon, good evening to all the participants and the students for this course titled multi-criteria decision making, which is under the NPTEL MOOC series and my good name is Raghunandan Sengupta from the IME department at IIT Kanpur. So, as you know we will be starting and finishing today the 18th lecture out of this 60 lecture series, which is spread over 12 weeks with each week being 5 lectures, each lecture being for half an hour. And the broader topic which we are still under with is basically definitions, concepts of utility theory, safety first principle and stochastic dominance. In the last class we did start with the safety first principle and I did explain the concepts using the diagrams, we will continue that the example which we left going to stochastic dominance as time permits and then also the hyperbolic absolute risk aversion. Again I am mentioning whatever the coverage is being mentioned in the slides they are updated accordingly and you can refer them. So, considering the example which I drew in details for the normal distribution the same one, you returns are normally distributed and you want to find out the minimization of the return or the decision being less than equal to R_L, which R_L is the minimum point or risk free interest rate.

And for three decisions which are given as A, B and C the corresponding values of $\bar{R}_P$, σ_P and difference being 5% away are given in the second column, which I highlight first the third column and the fourth column on the diagram I have drawn in detail. So, if you see here this is what I was hinting at. So, sorry for the repetition, but it will make things much easier for all of us to understand. So, we have B the distribution which is marked in blue, the $\bar{R}_B$ is given on the x-axis similarly, green is for decision A, $\bar{R}_A$ is marked, bar values being for the mean for both the cases and σ values with suffix B and A are denoted to signify the standard deviation and $\pm$ 2σ , +σ on to the right +σ on to the left and if I convert into the concept of standard normal $\pm$ σ +2σ and -2σ in the sense I mean the total width is 2. So, +1 to the right +1 to the left. So, if I consider the standard normal I would be covering about 67% and these things are known to all of us. So, if it was 4σ, total width which is +2σ on to the right -2σ on to the left. So, the total coverage would be about 97 and if I go into the concept of 6σ, the total width being 6 standard deviation which is +3σ on to the right, -3σ on to the left the total coverage area would be 99.97 percentage and given R_L which is on to the left I want to find out as based on the fact the minimum area.

So, if I use the green one so I want to minimize the area here which I am again drawing based on the first principle which was minimization $R_P \leq R_L$. Now, if I consider the case the green one so this would be this area again the second concept the first principle being applied to the second distribution which is A. Now, if I consider the other two principles I will use a different color for the safety first principles to be written down. First one was minimization max of R_L and the third one was max of $\bar{R}_P$. So, $\bar{R}_P$ for both the cases if I denote and Max[R_L] if I denote for the max case what will happen is this R_L will be shifted on to the right distribution being same which means the areas would be calculated accordingly.

If I go to the max of Max[$\bar{R}_P$]for these two distributions for A it will be shifted similarly for B it will be shifted R_L remains there. So, there is no change in the R_L for the third principle. In order to determine how many standard divisions R_L lies below the mean we calculated R we calculate R_L minus the means it are divided by the standard division we are normalizing it. Normalizing it in the sense that mean can be different standard division can be different. So, you want to find out the excess of the average return divided by the standard division or the difference in the negative side of R_L and $\bar{R}_P$ divided by standard division that means we are trying to find out so called efficiency.

Because if standard deviation is very high or low so it should also come into the picture. Thus we have to minimize the value as given here min $\frac{R_L - \overline{R_p}}{\sigma_p}$. This is the same thing which we have considered when we are trying to find out the minimization of $R_P \leq R_L$ the standard deviation concepts. If I minimize a function and or try to maximize the negative of the function idea is same. So, we basically Max[$\frac{\overline{R_p} - R_L}{\sigma_p} > 0$], it will continue. This is exactly equal to the concept of Max[$\frac{R_p - R_F}{\sigma_p}$], because R_L is now being replaced by R_F by standard division and we solve the problems accordingly. Even though for our example we have simplified our assumption by considering only the normal distribution which I did mention but this whole would hold true for any distributions having the first and the second moments because first moment being mean second moment being standard division and if the second moment does not exist you cannot utilize this principle accordingly.

Further extending it according to Chebyshev's inequality for any random variable such that the expected value and variance exist first moment and second moment exist the following are the same. So, in the concept of Chebyshev's inequality diagrammatically what it is. So, if I consider the line random variables and this is basically expected value which is E(X) and random variable x is any side onto the left or the right. So, what I want to find out is the deviation being bound between a limit. So, for x being less it is onto the left hand side for x being more it is onto the right hand side.

So, I want to find out a bound for fluctuations and divide by standard division because that will normalize. So, that being greater than some constant t according to Chebyshev's inequality the P[$t \leq 1/t^2$]. Now, in case if I replace x by R_p random variable expected value E[x/$\bar{R}$P] which is obvious and square root of variance which is standard division for random variable x would be replaced by σ_p. So, x is replaced and the random variable is now R_p and its corresponding mean and standard division have come. So, if this is greater than some k it will always be less than equal to $1/k^2$ according to Chebyshev's inequality.

Now, consider what how can k be calculated as we are interested in the lower limit hence we simplify it as follows. So, if we are interested in the lower limit. So, what if I draw the distribution again the normal case this is the distribution normal and this is the average value which I denote by $\bar{R}_P$, R_p is the random variable and consider R_L is given. So, I am trying to find out the probability out to the right hand side. So, if I denote it green color this is the area I want to find out.

In the previous case it was trying to find out the left hand side. So, I wanted to minimize here I want to basically do it and then try to maximize. Minimizing means it will push on to the left the R_L line in the same way if I want to maximize the right hand side it is same thing if I move R_L to the left minimizing on the left is exactly equal to the maximizing on the right. So, the right part is the green one and for convenience I am writing the minimization using the see for example, violet color because the area is one. So, minimizing one will maximize or vice versa.

So, continuing the discussion and if I want $R_L < \alpha$, $P[R_L > R_p]$. So, in this case I just use the standard normal deviate concept $\frac{R_p - \overline{R_p}}{\sigma_p}$, I convert into $\frac{R_L - \overline{R_p}}{\sigma_p}$, which is in the third bullet point which I am putting a tick mark the parts in the bracket inside the bracket the left hand term which is on the left hand side of the greater than sign is Z and the part which is on the right hand side of the greater than side is z, standard normal deviate. So, if this is equal to α then according to Chebyshev's inequality if I want to draw a simile this value can be considered as k and if I want to basically find out the one to one correspondence the k^2 value becomes k^2 > some constant k or t. So, given standard deviation known given R_L known given $\bar{R}_P$ known I can find out the bound or the maximum value or the minimum value in whichever way you are looking because the overall sum is one for the probability. So, less than equal to this bound means that it cannot cross of the probability.

So, even though I have been switching between and discussing all the three norms of the safety first principle. So, yet I would like to take your time in trying to discuss the second criteria and then the third. The second criteria was maximization of R_L maximization of R_L means that if I only consider the distribution this is $\bar{R}_P$ this was R_L and I want to find out R_L and push it on to the left hand side on to the right hand side and consider I am putting a constraint now. The constraint I will put on different colour is probability of $R_p \leq R_L$. So, I will try pushing R_L on to the right hand side, but at the same time try to basically bound the probability that $R_L > R_p$ or $R_p < R_L$ with the value of α.

And let us consider we have a value of α as 0.05. Then we should have if I consider the such that conditions again it can be converted into standard normal.
So, is $\frac{R_p - \overline{R_p}}{\sigma_p} = Z < \frac{R_L - \overline{R_p}}{\sigma_p} = z < \alpha$ and α is 0.05. So, how would we solve it? So, this is the case we have $\bar{R}_P$ given the green vertical line distribution is given for R_p for the decision B. So, this is $R_{P,B}$ for decision B is the average value and considering this what we want to assume if you superimpose the condition of trying to maximize subject to conditions. We are trying to basically consider R_L and push it on to the right. This is what we are trying to do. Now this implies is what if I consider the this equation which I will put a violet mark.

So this would mean that if I convert that value of α which is 0.05 into how many standard deviation it is on to away. So, standard deviations would give me how many what quantum of distance it is based on the standard deviation multiple. Consider that I write the equation as $R_p > R_L + k\sigma_p$. This k is the constant value which I can change.

σ_p is given from the data fixed R_L given from the information fixed $\bar{R}_P$ given from the data fixed. So, I want to find out depending on the α value what is k and I want to find out that depending on the k value whether this equation is met. Now if you see this equation it is a straight line equation like $y = mx + c$, but here inequality sign being there greater than. You change this fixed value of return R_L so keeping R what is the fact that either you can keep changing α which will have an effect on k or you can change keep changing R_L. Now the problem was problem in the sense as the formulation was given you want to maximize R_L subject to the conditions that is $\leq \alpha$, α is fixed.

So, you cannot do anything with α you need to concentrate on R_L only. So, that is what is written in the second bullet point change the fixed rate of R_L which you want to have and keep finding the straight lines accordingly because if you keep changing R_L, $\bar{R}_P$ is given from the data, k is given from the data, σ is given from the data and why I mentioned a straight line this will become clear to you. If I consider the straight line we all know equation is $y = mx + c$ what is y the independent variable what is x the dependent variable based on which you want to find out y. What is m? m is the slope per rate increase of y as x changes or decreases depending on $m > 0$ or $m < 0$ and c is the intercept where the line cuts the y axis. So, if I draw it this is y this is x the line is this.

So, $y = mx + c$. So, this is c. Now if I consider similarity and in the same way this equation $\bar{R}_P > R_L + k\,\sigma_p$. So, this is $\bar{R}_P$ I am writing the equation again in red color for better differentiation. Now it becomes very obvious the value of c is akin into this equation is R_L the value of k is almost equal to the concept of m and R_p and σ_p are the concepts which we can utilize considering $\bar{R}_P$ is y σ_p is the standard deviation. You can consider in the other way round also like σ_p as y and $\bar{R}_P$ as x.

So, we can change the equation. If I consider this equation as given the line looks like this. So, if you remember in the equation I said $\bar{R}_P$ was y σ_p was x I have drawn it exactly as this for simplicity for understanding and the equation was I will write in black it was $\bar{R}_P \geq k\,\sigma_p$, and I would not write R_L now. Why I have not written R_L because I want to use a different color because it is changing it can be changed so it is R_L. So, consider no let me use the blue color because as it is in the graph.

So, I keep changing R_L. So, the first case one I take it as R_L^1 and based on this the equation is given when I circulate R_L^1. When I change to R_L^2 the equation is the second one change it to R_L^3. This equation is the third one change it to R_L^4 the fourth one and if you see only the intercept is changing the m value which is the tangent here in this case is k which is given as 0.5 is fixed. So, all these lines are parallel to each other. So, R_L^1 where it intercepts R_L^2 where it intercepts R_L^3 where it intercepts R_L^4 where it intercepts all these lines which I am denoting by 1, 2, 3, 4 are all parallel and which point do you consider.

The curve which is the best decision which you have if you consider the concept of risk and return for a portfolio you consider the marginal utility case to three cases increasing and increasing rate, increasing at a constant rate, increasing and decreasing rate. So, as I mentioned human beings are always discovers the moment one of these lines is just tangent that is the best point which I have denoted by P* and this red arrow means it is moving in the top left corner and goes increasing the moment it touches its tangent to that curve because the green curve is fixed only the blues are changing and you are able to find out. So, conceptually the diagram is like this. Now coming back to the criteria is maximization of $\bar{R}_P$ which was the third safety first principle and again the same concepts of the constraints, constraints which was there in the second safety first principle is being replicated here also, but in the second criteria it was maximization of R_L and the third criteria is maximization of $\bar{R}_P$. So, if I consider the constraint I will just mark where α is predetermined depending on the investors decision makers own constraint sets of safety concepts and the idea when converted is like this which is maximizing R_P and obviously this is probability, but again it can be converted into the concept if I remember the straight line $y = m x + c$.

So, this is $\bar{R}_P$ is greater than $R_L + z\,\sigma_p$, but now here very interestingly in the second principle it was maximizing of R_L. So, only the intercepts were changing. So, it was parallel lines now it is a different idea. Now what we need to do is R_L is fixed. So, this is if I again consider the same diagram type of σ_p on the x-axis $\bar{R}_P$ on the y-axis R_L is fixed which is this point which I will denote by blue and the green curves remains which is the best frontier.

Now what we do is that I want to maximize R_L. So, if I consider I want to maximize the maximize R_P. So, I counter clockwise turn the graph which is shown by the red line and this value of θ increases. So, I will use θ_1 then it increases to θ_2 is increasing θ_3. So, these are the angles which I have considered this is θ_2 this portion is θ_2 and so on and so forth.

The moment that line is tangent which is P* here. So, that will give me the best value of trying to basically maximize $\bar{R}_P$ set to the conditions that you have again I am writing the condition in violet color which is $P[\, R_P \geq R_L \,]$ which when with the some α value which when converted becomes $R_P \geq R_L$ + some k σ_k and z are the same depending on standard normal deviate which you have. So, in the first principle what we considered is we had two ideas of trying to maximize or minimize the probability maximize on the right hand side. Second principle was maximizing the concept of R_L third principle was maximizing the concept of $\bar{R}_P$ and here remember the decisions $R_{A\,B\,C\,D}$ accordingly and we will basically take $\bar{R}_P\,\sigma_p$ for those corresponding decisions only. With this I will end this lecture and continue discussing more further more about stochastic dominance and the concepts are coming. Thank you very much have a nice day. .