Multi-Criteria Decision Making and Applications
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Week 04
Lecture 17

A very good morning, good afternoon, good evening to all the participants and the students for this multi criteria decision making course which is under the NPTEL MOOC and my good name is Raghunandan Sengupta from the IME department at IIT Kanpur and as you know this course is for 12 weeks which is spread over 60 lectures and each week we have 5 lectures each lecture being for half an hour and this is the 17th lecture out of the 60 that means we are in the fourth week. So the overall so called super set or umbrella under which we are discussing the topics we have been discussing under these topics many different sub topics. So, topic was basically the simple definitions of MCDM concepts of utility, safety first principle, stochastic dominance. So obviously, definitions concepts in details for utility theory then the concept of expected value, certainty equivalent, then geometric methods, mean methods, then different type of utility functions, their absolute risk aversion, relative risk aversion using diagram as well as equation we have discussed with different examples. For the 17th lecture and obviously it in go into the 18th also, so we will discuss safety first principle, the concept of stochastic dominance and also discuss the hyperbolic absolute risk aversion function and how hyperbolic absolute risk aversion function will make sense for all the 5 or 4 different simple examples of utility function we have taken which was quadratic, exponential, then power and logarithmic. So what is the concept of safety first principle? Under this basic tenet the decision maker is unable or unwilling to consider the utility theory and the concepts for making her decision.

So he or she always wants to play safe and wants to minimize the risk or the loss risk what I am using very generally, but the loss is essentially not based on the concept of trying to make the use of utility theory. In utility theory what we did? We always saw that we wanted to maximize the expected value or given the fact of certainty values were known we need to basically rank the certainty values. So under this safety first principle people make the decision placing more importance to bad outcomes. So people are risk averse, so they want to avoid and they want to basically give more importance to bad outcomes and based on that their main decision set of information would be.

So yes this placing more importance to bad outcomes and we will see that in few minutes. Now there are different concepts of safety first principle. So we will consider three of them and as given here the equations are given I will draw them each one at a time in order to emphasize the fact. So I will be coming back to this slide for two reasons because the three main concept of safety principle is there and there is a quite a lot of blank space where I can draw the diagrams and explain. So consider for the time being the distribution is anything why I am talking a distribution because the decisions are random and there is a probability and if you remember in utility theory the concept of probability is used extensively considering I want to find out the relative frequency of the occurrences of any decision which is being taken based on the wealth and its subsequent utility function.

So consider I will draw the x-axis and y-axis in black color, so this is the x axis which is the random variable and random variable what it is I will come to that this is the y-axis and the distribution for the random variable is this. It looks like normal but it is not normal I have drawn it very generally. Now here what we mean by the random variable is basically the return, so return can be total return can be rate of return, total return mean R, rate of return mean r and we have discussed what is R and r. For just convenience I will write it down and again erase it, so I will use a different color red. So if there are two values of investment initial I_1 final I_2 that means I invest I_1 amount of money and end of the time period I get I_2.

So I_2 can be < I_1 can be = I_1 can be > I_1. So $R = I_2/I_1$ and $r = \frac{I_2 - I_1}{I_1}$. So the decisions which I am mentioning here safety first principle they do consider R but they can be replaced with r also. And if you remember even though as immediately not important for the discussion of safety of principle we also mentioned that if the distribution of the returns are normal then the quadratic utility function would be utilized or vice versa that means it is both way implication if and only if. Now coming back to the graph which was drawn, so here along the x-axis we would have the random variable which is the return and for simplicity as the equations are given in R, I will denote R_p which is the portfolio of the conglomeration of assets which you have along the x-axis and along the y-axis we have the PDF of R_p probability density function.

And this value which I am marking with a star the vertical line consider for simplicity that it is the average value $\bar{R}_P$ it could be on the left it could be on the right also but for simplicity I am taking it. Say for example, $\bar{R}_{P,}$ I will erase this line could have been here average or could have been here but the analysis of R_p where it is placed the diagram and its figure would make things much clear. So, erase oh sorry the x-axis was deleted I will again make it. Now what we want to do and consider there is a lower return R_L, so which is I am marking with a blue colour this is R_L. So, R_L consider is here if it is on the right hand side of $\bar{R}_{P,}$ I will denote that how it can be done, so this is R_L.

So, what you would want to do is in the case that I want to minimize the probability that $R_p < R_L$ R_p not $\bar{R}_P$ which means I want to find out the overall area on to the left I am marking with green. So, if $R_p > R_L$ which is R_L is basically the minimum requirement of threshold, so of $R_p > R_L$ it is better if $R_p < R_L$ it is definitely not desired. So, I want to minimize the probability in case R_L was on to the right then this is also will be the same consider R_L is here. So, in this case I want to minimize the area on to the left, so always concentrating on the left. So, in case if I denote that area with the hashed line here, so this would be violet area on hashed which I am denoting would be on to the left, so I want to minimize this area.

Consider R_L is here on to the higher side of $\bar{R}_P$ and in this case I want to minimize for the green area provided R_L is on to the left of $\bar{R}_P$. So, considering that I want to minimize the probability which is the first bullet point probability of $R_p < R_L$ and this would basically mean that I want to minimize as I have told either the green area or the violet area. Now, what advantage do we have with the normal distribution let us first discuss, so I am only concentrating on the first part first bullet point. So, if it is a normal distribution I will draw it here, so this is the normal distribution this is the average which is $\bar{R}_P$ and if R_L is on to the left or to the right. So, this blue line is for R_L I am not marking the points R_L on the left or the right because it will become very cluttered.

So, I want to minimize this area on to the left or I want to minimize this area which is also on to the left, but in this two cases the difference being R_L is on to the left and other case R_L being on to the right. Now how do I utilize the normal distribution and the standard normal table. So, I will erase this, so diagram which was drawn with for any random distribution let us consider for the normal case and I am going to basically erase this diagram in order to do the simple not derivation simple methodology of how you calculate. So, this normal distribution is now true and I will write the equation I will use the green color for this. So, what we have if minimization of probability R_p minus is less than R_l which would mean P[$\frac{R_p - \overline{R_p}}{\sigma_p} < \frac{R_L - \overline{R_p}}{\sigma_p}$] σ_p is the standard deviation and $\bar{R}_P$ is the average value, => $P[Z \leq z]$ and I need to find out.

So, here from the data everything is known R_p is the normal distributed random variable expected value is known which I will put a tick mark is valid. So, $\bar{R}_P$ is known, σ_p is known I also put a tick mark and R_L is also known. So, based on that I find out the value of the probability and using the standard normal table I can find out this value is α. And once I have different type of decisions with different α's, I will rank them accordingly. So, I want to minimize that means I will basically minimize all the values of α, consider α_j consider there are j number of investments or decisions j is 1 to n, I will take the minimum one and take the best decision because it is minimizing my area on to the left.

Now, coming to the second bullet point which is maximization of R_L. So, I will continue the same diagram considering the normal distribution even if it is the other the general distribution which I have drawn the analysis remains same. So, I will basically erase this part. So, this was basically for the first bullet point. So, what we are doing is as follows diagrammatically. So, this diagram which is there I will just try to expand it. So, it is not cluttered. So, this is the normal distribution which we have this is the R_p value $\bar{R}_P$ and this what we are measuring is R_p and along y-axis is the pdf of R_p. So, what we want to ensure by the second bullet point is this is R_L. Now, considering there are different distribution, I will try to push this R_L on to the right hand side which means that if I consider the concept of the first one which was probability minimization on to the left.

So, for all the distribution which were already given considering their respective mean values their respective standard deviation if I push R_L. So, many of these distributions may slowly have their $\bar{R}_P < R_L$ provided R_L is on to the left hand side. So, say for example, R_L^1first instant I will put the suffix 1 this is not to the power it is just a suffix to denote then it moves to the right. So, this is R_L^2suffix 2 this is again not square a suffix then it moves here this is R_L^3 suffix 3. So, more I move to the left right I will be choosing that set of distribution where the probability of $R_L > R_p > R_L$ remains which is

in a sense the probability on to the left hand side which is for the case where I want to find out
$R_L < R_p < R_L$ continues to be minimum.

So, because the probability is 1 so if I consider the minimization on the left hand side which means what I am trying to consider the maximization on the right hand side and if I push R_L. So, obviously the concept of minimization on to the left maximization on to the right both the ideas remain. So, I will maximize R_L based on the fact I can increase to such level where I choose the best decision. So, the moment why moment $\bar{R}_P$ becomes less on to the left hand R_L the value of R_L which is on the right hand side many of these decisions would basically not be worth investing because they would give a loss. And if you remember in the safety first principle we did discuss that the decision maker is more importance to bad outcomes.

The last part is maximization of $\bar{R}_P$ and I will remove some of these here. So, R_L is already here R_L and what I am doing is this I am trying to move the average value on to the right which means lock stock barrel the distribution changes. So, if $\bar{R}_P$ now moves to the right hand side say for example here. So, I will denote by R_p^2 suffix again not square just a suffix and this is consider R_p^1 not a suffix is not a power just a suffix. So, in that case the distribution has moved to the right which means it is like this now.

So, technically if I consider the area on to the left of because the chances of R_p being on to the right as $\bar{R}_P$ increases also increases which technically means it fulfills the first norm. So, first norm and third norm and the second norm are interrelated. So, in this case I will maximize $\bar{R}_P$ and then do the corresponding calculation. Now the question may come up so can we do it for any other distribution yes we can provided we can do the calculations to find out its mean $\bar{R}_{P,}$ σ_p and do the calculations either mathematically or simulation wise. Normal distribution has a table and is easy to do hence I discuss those two examples.

But remember that for the three cases which I had in front of me for the discussion the first moment which is $\bar{R}_P$ and σ_p should exist. So, if there is some distribution which does not have the variance you cannot solve the problem provided you are considering that distribution as being the case for the example to consider the safety first principle. So, the rule and discussion which I have written down in details which has diagrammatically explained is exactly this. So, I want to find out $R_i > R_f$, R_f is basically the risk free interest rate which I was talking about R_l the minimum some value. So, I want to basically if I go here it is exactly the same what we did in the last slide I will write it down it was$P[\frac{R_p - \overline{R_p}}{\sigma_p} \geq \frac{R_f - \overline{R_p}}{\sigma_p}]$ being greater than equal to in the first case it was minimization of less than equal to here you will try to maximize the greater than.

So, $\frac{R_f - \overline{R_p}}{\sigma_p}$. So, this one and the first of the safety first principle are the same. In the safety first principle we needed to minimize because it was less than in this case which I have drawn diagrammatically we need to maximize because either on to the left or to the right the ideas would be same, but how you approach would be different minimization maximization. Here for the calculation which is shown here the two important points which I have already mentioned the second one whichever highlight is the returns are all normal and I did answer that if the returns are non normal you can use it, but there were few important points which I mentioned and R_L is the risk free interest rate which is the minimum case of R, R_f for the risk free interest rate we took R_L in the previous slide. Now continuing the discussion of normality normally distributed returns then in that case the optimum portfolio would be the one where R_L was the maximum number of standard deviations away from the mean further it is better because in that case the area on to the left is minimized. Let us consider an example I have drawn and explained using the example for the normal case and all the three safety principles were considered, but still let us continue further and this is of that.

Let us consider an example for minimization of $R_p < R_L$ which is the first remember we consider the returns are normal and the suffix p denotes the portfolio while R_L means a fixed level of return which considered for our case as 5 %, but I am using only the values. Now consider the R_p value $\bar{R}_P$ value is given and you have for decision A as which I am putting a tick mark in red the values of R_p are 10 for B which I am putting a tick mark in green it is 14 and for C which tick mark in violet is 17. The corresponding values of σ_p are, if I read the second row, it is 5, 4, 8 and if I consider the value of R_L as given as 5 then the corresponding difference from 5% which is the return R correspondingly for A, B and C are given as -1 $\times$ σ, standard deviation 2.25 standard deviation and 1.5 standard deviation which means that if I consider the distribution.

So, let me make a blank slide and explain it even though that is not part of the set of slides which will be uploaded, but I still want to discuss that. So, I will use the diagram data here and explain I will create a blank one yes. So, this is the slide just after this. So, I will keep switching between the blank slide where I will draw and this slide. So, consider $\bar{R}_P$ σ_p for A, B, C are given. So, let me draw it 10 and 5. So, values are so let me write down the first the mean 10, 14, 17. So, this is 10, 11, 12, 13, 14, 15, 16, 17 and if I consider it is 5, 4. So, consider for the first one consider 5 I am trying to draw as for the second one it where the average is 14 is 4 it is 4. So, it will be a little bit more narrow and for the average value being 17 it spread is 8.

So, I will use a different color. So, this is the average value I have tried to draw it hopes clear to all of you average value average value and what was R_L? R_L was here was 5. So, it is 9, 8, 7, 6, 5. So, I denote the basis of R_L here. So, if I consider the graph going down say for example, green is here I should have extended it. So, what I want to do is this for the red portion if you see I will use the red now pen because I do not want to make it for the red portion I will minimize the whole area which is on to the left for the whole red.

Now, when I come to the blue one which is for the second yes I will try to minimize this small area sorry if it is too minute, but this is the small area which I will have and for the green portion this will be that area corresponding under the green for the blue it was under the blue for the red it was under the red. So, if I zoom in this portion if I zoom in I would have this. So, this is R_L I should denote by I will take few more minutes this is R_L.

So, this I am considering zoom in part here this is the left portion the red distribution is coming like this the green distribution is coming like this blue distribution is coming like this somewhere. So, in the red case I will minimize this on the left for the green one I will minimize this for the blue one I will minimize this there are three ways idea remains in minimizing this.

So, with this I will end this class and we will continue discussing further on. Thank you very much. .