Multi-Criteria Decision Making and Applications
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Lecture 16

Now, welcome back my dear friends and participants for this multi criteria decision making course under MOOC and NPTEL and my good name is Raghunandan Sengupta from the IME department at IIT Kanpur. So, this is the 16th lecture which is the starting of the fourth week first lecture and as you know it is a 60 lecture series spread over 12 weeks, each week we have 5 lectures of half an hour each. The broader outline we are still continuing of the discussion is about MCM definitions concept of utility theory, safety first principle, stochastic dominance and if you remember we have spent a lot of time in discussing definitions concept of utility theory. We are in the fag end of trying to finish of utility theory and then start with safety first principle and stochastic dominance concept. The coverage obviously as you know that the plan based on what we are covering and if you again to mention to all of you, all the slides would be available in with NPTEL and with exact starting and ending with all the slides being included for each lecture and the coverage will be exactly mentioned as we finish each set of slides which is basically the lecture number on the day for that week. So, we will cover geometric mean which we would do and safety first principle stochastic dominance and hyperbolic absolute risk aversion function.

So, this is the example where we left in the 15th lecture which is the last lecture of the third week and this slide was discussed how we find out R what is the difference of small r R what is the relationship of quadratic utility function with normality of returns of r and R we have discussed that I mentioned that. Then we showed this slide and we did mention how we find out R values they were marked with different colors and how we find out the standard deviation for each and every assets then the wealth average. So, all these are average values $\bar{R}_A$, $\bar{R}_B$, $\bar{R}_C$, $\hat{\sigma}_A$, $\hat{\sigma}_B$, $\hat{\sigma}_C$, A, B, C are the assets and then wealth bar A, B, C. So, all these values are calculated based on the table.

Now I use if you remember normality quadratic concepts which was the highlight of this example. So, we find out the return now consider the risk free interest rate is given as 1.05. Risk free interest rate is basically the interest rate which has no variance but constant which is of expected value. So, like the treasury bills of the government even though treasury bills change values change but we are considering theoretically the treasury bills 91 day treasury bills of the government. So, based on that 1.05, I find out the excess return. So, what we are doing this is normality if you remember the graph would look like this. This is the values of probability this is the values of the random variable and this is the distribution of normality. So, here we have X we have f(X), which is the probability density function. Now consider the values of 1.05 are marked here consider you want to basically have this and 1.05 say for example, is marked here. So, for the first one which is $\bar{R}_A$ 1.06 it is on the right hand side. So, I will use a different colour consider green.

So, I want to find out the green one the probability that 1.06 would be more than 1.05 what is the probability. So, I cannot use standard normal deviation concept which is we all know P(X). So, this is the general formula $P[X \geq x]$ would imply that $P[\frac{X-\mu}{\sigma} \geq \frac{X-\mu}{\sigma}]$

this can be converted into $P[Z \geq z]$.

So, this is the conversion for the standard normal deviate. So, we use this formula and we when we use this formula the first case which we have is this I am using the green colour for A. So, $(\overline{R}_A - R_F)$ which is the excess which we have. So, mean value concept if I consider. So, I am only considering the case where we have the concept of the excess return.

So, excess return $(1.06 - 1.05)/\sigma$gives me the probability value is such that the standard value is 0.40 this is not the probability this is the Z value. Similarly when I use the red colour to denote no not red the blue colour to denote for B.

So, B is exactly on 1.05 line. So, the value comes out to be probability of 0 and when I use C again a different colour the value is $(114 - 1.05)$/its corrsponding σ comes out to be 1.36. So, from the standard normal table, I need to find out the probabilities. So, if I plot it. So, I will erase this and so erase this one I will erase this one also and now the plot which we have done is now actually converted to standard normal deviate. So, this is capital Z this is f(Z, z) and the value of probability 0.4. So, this value is 0. So, 0.4 is here on the right hand side. So, this is the green one I should use the green one this is 0.4 the probability of 0 which I have used is blue colour this is along the central line and 1.36 which is on the right hand side more this is here. So, I need to find out the excess probability excess probability means higher the value which means on to the left if you see this. So, I am going to consider for the green one this portion these values for the second case which is B green was A asset A second case which is the blue one I will be utilizing this. If I come to C, I will be utilizing this portion higher the probability better. So, if I use that concept then the ranking would be you have the values my here it should be just the other way round.

So, this I will use so the probability being highest being for probability 1.36 which is

C > A > B. So, now corresponding to the fact that normality concepts we have rank them then I come to the concepts of. So, to double check even though the values are different. So, now I consider the quadratic utility function and the values are as given.

So, the value of wealth is a quadratic utility function. So, the equation is $W - bW^2$ with a b value of 0.002 and I plot the values or in the table I note down W(A), W(B), W(C), these are the values based on W(C). And considering the utility function I find out or note down the W(A) which I will denote by green which is column 2, I am marking in the green. I am using this utility function $W - bW^2$ where
$b = -0.002$ and these values are given here the wealth based on second column I find out the utility. For the case of B the wealth as given third column I find out the utility values

utility again equation is quadratic. Finally, for C which I use the blue color the wealths are given in the fifth column which are W(C) headed by W(C) and U(W(C)) are the corresponding utility for C wealths C assets and again utility function is same. The corresponding probabilities are given 0.2 accordingly which is 1/5 and for A, B, C I find out the corresponding values and the expected value.

So, I have to double check this the values of A comes out to be 28.02 and for B 29.49 and for C comes out to 24.11 and there the ranking as given is B > A > C. This I will clarify because the idea being that they can be many examples where the corresponding values of utility based functions with the quadratic one actually do not turn out to be true.

I will clarify this in the definitely in the next class so as an editor. So, we will consider this example and then upload the slides accordingly. Now the corresponding concept of the deterministic versus probabilistic idea. So, if you see this diagram this is what the concept of so called certainty concept expected values all these things we have considered. So, the diagram is like this there is I am only considering on one arm at a time consider the value which you have here first arm.

So, that is being equivalently broken down into a gamble how and there would be other replications also which we will discuss. So, if I have the corresponding value of W_1, W_1 so called the corresponding utility of W_1, $U(W_1) = P(h_1)\, U(b_1) + (1- h_1)\, U(0)$, which means that I am replacing this gamble which is with outcome $(b_1, 0)$ probability h and $(1- h_1)$ by a so called sure outcome which is W_1. Now can that be extended yes for each arm if I consider the coloring. So, W_2 can be equated with they would be which I have not drawn a value of h_2, $(1- h_2)$, value of b_2 and 0. So, the expected value again would be equal.

Similarly for W_3 I am using a different color it would be h_3, $(1- h_3)$, the probabilities the and value is being B_3 and 0. And finally, for the fourth one I am using the color dark red which would be I am writing it here it would be h_4, $(1 - h_4)$ and values of $(b_4, 0)$. Now can that be extended, this is equivalence what I am saying. If I draw the equivalence on to left hand side which is not there I can replace the second case I denote it by II and this by III by a certainty value such that the certainty value consider the wealth is W, I am using the W only such that the U(W) multiplied by the probability we know would be equivalent to the second case similar it with the third case. So, in the second case what I would have I will use a different color to find out the expected value blue color.

So, that would be $P_1 \times U(W_1) + P_2 \times U(W_2) + P_3 \times U(W_3) + P_4 \times U(W_4)$. So, this value if I find out should be exactly equal to the E(U(W)). So, these two are equal which are marking in yellow here and the value which I have here which I marked in. So, this marked in the blue box. So, U(W) would be equal to this in order to make things visible these two values are equivalent.

Other MCDM selection techniques under portfolio management would be general investors have the following characteristics. So, that is expected utility maximization would always be used on minimization of the risk of variance people are generally risk averse they want to avoid even if we have other forms of utility function. Yet the quadratic approximation that is the use of mean value theorem do give us good results which are much more practical in sense because the concept of returns normality and the quadratic utility function turn out to be true in many of the practical examples. So, number 1 is always maximize the utility expected value minimize the risk which is the second moment and normality and quadratic utility function normality for the returns and utility function in quadratic concept will be utilized. So, there are other selection models also will consider geometric mean safety first criteria stochastic dominance and analysis in terms of characteristics of the returns distribution.

So, we will consider the concept of geometric mean first. So, as you know so there are for any value there are three different average values arithmetic mean geometric mean harmonic mean. So, generally we use the arithmetic mean we denote the arithmetic mean by AM, geometric mean by GM, and harmonic mean by HM. Arithmetic mean is very popular I will circulate we will use geometric mean which I am putting a tick mark. Geometric mean would be used generally for financial cases arithmetic mean for general cases harmonic mean see for example, you want to find out the average speed if you are travelling from city 1 to city 2 and find it out.

So, under the concept of geometric mean what is the overall idea for the selection process we consider maximizing the geometric mean concept and we consider the maximum geometric mean because it will have the following two properties the highest probability of reaching or exceeding any given wealth level in the shortest possible time that would be true for the geometric mean and the second point would be the highest probability of exceeding any given wealth level over any given period of time would also be true for the geometric mean. And we will very simply see the utilization of simple example of the geometric mean utilization. So, consider the formula first then the example. So, we will denote the return for the geometric mean return will be capital R_{Gj} as I am highlighting we denote the geometric mean utilization formula and j is basically the portfolio or the combination we are considering. So, j would basically be 1, 2, 3, 4, so and so forth depending on how many such combinations you can find.

In the formula which you see $R_{G,j}$ would be equal to the multiplicative factor of the following the total return for the first when I am utilizing the first concept would be for the case where I am using the idea what are the constituents inside that portfolio on the basket. So, considering if there are 3 baskets and each basket had 10 same type of items 10 in number but combined differently. So, in that case would be 1, 2, 3 and the basket has 10, so i would be 1 to 10. So, we will find out here accordingly the total return to the power the probability of how much percentage it will be there the proportions or the first one in the jth one and then obviously the values can change I will come to that later on we will see that value concept later on multiplied by the term for the last set of constituent in that portfolio which you have is $(1+R_{n,j})^{p}{}_{n,j}$ to the power is to its corresponding. So, this would be n because there are and market here because there are n such constituent things in the basket for each portfolio and this −1 comes because you are using the capital R, this −1 would not come if you are using the small r.

So, R_{ij} is the ith possible return for the jth portfolio and p_{ij}'s are the probabilities of the ith outcome for the jth portfolio and do remember the sum of the probabilities on investing would always be 1 which I am highlighting now. And n is the number of investment for each portfolio as denoted. So, we will choose maximum value of GM and rank them accordingly. So, this is the example consider we have the following combinations of asset A, B, C. So, this would basically be the constituent in each and every portfolio and the portfolios are marked I will use a different colour 1, 2, 3 and the corresponding returns are given for A, B, C obviously once returns are therefore, A it will be fixed for any investment you are doing for either 1 or 2 or 3 portfolio.

So, they are given as here 10, 20, 30 percentage wise and the proportions and investment of A, B, C in 1 then 2 then 3 are given like this and use the same colour green to mark them 0.2 0.2 0.6 for A, B, C in 1 such that the sum adds up to 1. Similarly I have used I consider A, B, C for the second portfolio which I am again using the same colour to mark 1/3, 1/3, 1/3, sum equals 1.

For the third case for the third portfolio again A, B, C it is 25%, 25%, 50%, which is 0.25 0.25 0.5 the sum comes out to be 1. Now, I need to find out the geometric mean returns for combination 1, which is 1 combination 2, 2 combination 3, 3 which is portfolio 1, 2, 3 and these are denoted by $R_{p,1}$, $R_{p,2,}$ $R_{p,3,}$ and the values when I put in that formula would come out to be 0.237, 0.197, 0.222. See if I rank them obviously for A portfolio or 1 it would be highest then you will have 0.222, which is the second which is the third is coming to the second position and second is going to the third. That means if I use 1, 2, 3 as portfolio I am using the word portfolio. So, portfolio number 1 > portfolio number 3 > portfolio number 2.

So, this is what is mentioned here. So, I have ranked them accordingly. So continuing geometric mean return few concepts maximizing geometric mean return is equivalent to maximizing the expected value of the log utility function. Now remember for the case of quadratic utility function on the left hand side is the utility on the right hand side is the distribution. Similarly, the first point if you consider log utility functions logarithmic if you have remember the logarithmic one utility would be equivalent to the geometric mean returns being true. The finally, the portfolio that maximize geometric return are also mean variance efficient if returns are log normally distributed if you remember the concept of log normality we have mentioned.

So, we will be discussing all these things later on the concept will be utilized and with this I will end this lecture. Have a nice day and thank you very much. .