Multi-Criteria Decision Making and Applications
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Lecture 15

Welcome back my dear friends and participants for this course multi criteria decision making another MOOC and NPTEL series and as you know my good name is Raghunandan Sengupta from the IME department at IIT Kanpur. So we are in the phase of discussing the concepts of multi criteria decision making and the broad umbrella which we will continue for few more lectures is basically concept of utility theory, concept of safety first principles, stochastic dominance and this is the 15th lecture where the coverage is in general we have considered certain equivalent and we have considered two different examples. We will further on go into the utilization of certain equivalent why, what is the use and if you remember there were two points one was basically to understand how utility functions or utility concepts can be used for ranking and also to find out the so called functional form of the utility function considering that the person can take a decision but you as an academician want to find out what type of utility function and which class of utility function the person has or if the parameters are to be found out. Then we will go into the concept of geometric mean methods and how geometric mean methods can be utilized for different sort of ranking. And later on obviously we will cover safety first, stochastic dominance, hyperbolic absolute risk aversion properties but these are as we proceed. So hopefully we will be able to cover the certain equivalent and geometric mean models in this 15th lecture out of the 60.

So if you see this graph and this is where certain equivalent concept and gamble will be utilized to understand what type of person he or she is. So whether the person wants to take risk, one is indifferent of taking the risk and one is basically trying to avoid risk. So first let us specify what we are drawing and why this diagram is here. I will use different colors to denote.

The first is that we will denote the values of the wealth which will denote by A, B, A_1, B_1, A_2, B_2 along the X-axis as noted and we will denote the certainty value which is given by C. If you remember we have used that symbol C being denoted on the Y-axis. So the values again on the X-axis as I am marking putting a single tick mark and certainty value along the y axis where I am putting a double tick mark. Now what we do and how we proceed. So we consider a 45° line as given.

So why this 45° line? So this we should at least mention this angle is 45° and why you will understand. Now consider on one table, table 1 which I mark by 1, we place wealth of amount A and B which are different amounts as given here along the X - axis and we consider intrinsically from our side as we are doing the experiment we considered the utility function is linear which means the utility function which we are going to consider for our comparison is U(W) = W and the wealth we place A and B the outcomes are based on the fact when you toss a coin you get probability 1/2 and 1/2 which is an unbiased coin. So if I consider that overall expected value for A and B so I will denote by E only. So consider E's case basically or I will put the E(A,B) will come out to be, the U(A) would be A because the utility function is linear so it will be A this star sign is multiplication sign, A × 1/2 because 1/2 is the probability of getting ahead considering A comes out for a head and B comes out for a tail so it would be B × 1/2.

So the value would be (A + B)/2. Now what we do is that we I am using the black color such that the consistency of the notation and there is no confusion. So (A + B)/2 would be a middle point we draw a straight line. So that so and if you remember I have mentioned that we place (A + B) on table 1 and table 2 there are no values and the person comes and we ask him or her that this is a gamble with A coming out with probability 1/2 B coming out with probability 1/2 and what will make Q indifferent or say for example we place C on table 2 to make things much more explicit and simple C value is placed. So the person comes and we ask that what will he take or she take the gamble with A B or the certainty value with the probability 1 which is C.

If the person says the value of C which has been placed on the table 2 if he or she agrees which means the point based on which we have found out this would be C because we have formulate the C value such that (A + B)/2 = C is the midpoint. So the point would definitely be the point which is marked by the this black dot would definitely be on the 45° straight line. Now in case the person says no he or she will not be able to take the gamble but will go for a certainty value higher. So obviously in that case the C value which will be told by him or her with respect to the C value which is there on the table the person says the C value is higher. So obviously the X value which is (A + B)/2 remains same the C value is higher.

So consider the C value according to him or her is here. So we note it down and we note down this point the second point here. So let me mark it with the red so this is the point this one and in case so I should mark with green because the green line is there just wait one minute. So the person says green so this is the point. If the person says no he or she is would be sure to take that gamble with respect to a certainty value which is lower than C.

C is already placed then the persons corresponding C based on which he is proceeding would be below. So we will denote the higher so consider C is which is kept on the table again I am mentioning and the person 1 who comes he or she says that the C value is above which is on the green point and we will denote it by C with a green color. This is C so which means this is C and actually value which have placed on the table was C the black color and for the second person the C value would be here which is red color. So I am not using any suffix I am just differentiating with the color so that it is easy for us to communicate. Now what we do is that we change once we know whether the person value C green is above or C red is below or C black one is on the line we change the values of A and B.

So consider A and B are changed so here is see for example new value is A_1 new value of B is B_1 probability remains 1/2 of the midpoint is now here. Again the values are as I mentioned A_1 B_1 and a value of black mark is C for example C_1 where $C_1 = (A_1 + B_1)/2$ and this C was if you remember we have mentioned is (A + B)/2. So again the person is asked he or she says no the C_1 value which is given does not satisfy him or her. So C_1 value for him or her which they say is higher so this is C_1 so this would be C_1 or else it can be below this is C_1 which means this is C_1 or this is C_1 and the third case can be where the person agrees with the C_1 and is indifferent so this would be C_1. So C_1 but with different colors denoting different values but C_1 is the symbol which I am using for the second case.

Now I keep repeating this question so I change A_1 B_1 to A_2 B_2 same probabilities I put a C_2, C_2 is basically $(A_2 + B_2)/2$ the person can give the answer as the value is exactly matches so which is indifferent along the straight 45° line so it will be C_2 color is different again I am mentioning variable concept is C_2 but color different means different values or else it can be lower which is red C_2 and if it is higher it will be green C_2. Similarly for the case again I change A and B this is A_3 B_3 there is an C_3 which is half of $(A_3 + B_3)/2$ and correspondingly the person gives the answer accordingly which is it can be C_3 above the 45° line C_3 along the 45° line and C_3 below. So what do you see as we keep asking the question change values of A and B and also corresponding value of C you can have three different types of profile. Profile 1 would be along the 45° line which we have drawn which means the person is based on A and B and C value A B and C changing with suffixes 1 then 2 then 3 which means the person is indifferent based on the gamble and the certainty value which is being placed because the rate of increase is increasing but at a constant rate if you remember the three graphs. For the case when the person is risk averse or risk lover the profile for the person who is risk averse would be increasing but at decreasing rate which is the green curve and for the person who wants to take the risk is increasing and in increased rate which is the red curve.

So these are the curves which we had considered the marginal utilities. So in this way certainty value can be utilized considering that we very simply do the experiment. Now what I said is exactly mentioned here in this slide. So A and B are wealths as we have discussed A and B are wealths and we consider the utility function to be W hence the straight line 45° line. You form a lottery that has an outcome of A with probability 1/2 as already mentioned B with the probability 1/2 then obviously the utility which I will only denote by U would be (A + B)/2.

Now we place C exactly equal to the 1/2 of (A + B)/2 as we have mentioned. You change keep changing the values on A B and ask the investor how much certainty value will make him or her take that gamble. So again over the 45° line below the 45° line or on along the 45° line. So the expected value of the lottery is which I have mentioned time again is (A + B)/2 = C. The risk averse risk neutral and risk seeking person would prefer C would be based on the concept would basically take the decisions accordingly.

If we are C risk averse person would prefer a higher value of C would be indifferent between C and (A + B)/2 the value which you are given and would prefer a value which is lower depending on that the person profile would be a straight line which is increasing under constant rate curve the green one which is increasing and decreasing rate and the red one is increasing at an increase and increasing rate. Plot the values of C as you vary A and B and this variation of C for the answer which he or she is given that means the red colored C or the green color C or the blue color C the value of C which you are putting is fixed but his answer or her answer is basically what you are trying to plot. A risk neutral seeking an averse person would have as I mentioned the part if it is a neutral straight line again I am mentioning the person is risk averse the curve which is green above and slowly sloping down and for a risk seeking person would be below and sloping up. Now how can we use the concept of C which is certainty value further on to find out an explicit form of the utility function. Suppose you know that the actual form of the utility function form you know but exact parameters you do not know the form which you know I will just highlight is an exponential part with a values parameters not known.

So you will ask the person who has which you think or you are doing an experiment with the utility function should be exponential and you ask that person give a lottery with a 50-50 chance probability 1/2 and 1/2 an unbiased coin being tossed and the values of winning 50 with 50% probability is 10 lakhs and another 50% winning is 1 lakh. Now you place that and ask that person what is the amount he or she is going to pay to buy that lottery. So obviously somebody has to buy and consider the person says okay I am willing to pay 4 lakhs. So 4 lakhs means that he or she is trying to balance the expected value of the lottery with the value of 4 lakhs because in that case the person would exactly understand the amount of 4 lakhs I am paying to buy in the long run if he wins any two amounts either 10 lakhs and 1 lakhs the overall expected return would be 4 lakhs. So in that case for the certainty value the overall expected value of the utility would be this which is because I put W which is 4 lakhs into the equation A value is not known and the probability is 1 because he or she has already purchased and the corresponding side for the lottery would be the utility for 10 lakhs is what I am underlying with the little bit darkish yellow color and obviously multiplied by 1/2 and for the value which the outcome is 1 lakh this is the expected value where A value is again unknown and this is the probability.

So you put this equation do some iterative methods and try to solve and find out the values of A. So this concept which I used can be extrapolated for any other utility functions and the parameters can be estimated not found out exactly. Now few concepts for every gamble there is a certainty equivalent such that a person would be indifferent. So obviously we saw from the experiment that if C is the value or the certainty value then a gamble whatever the value is A, B, X, Y they would be a certainty value for which the person would definitely be indifferent between the gamble and the certainty value which means that if I draw the diagram and I denote the certainty value with blue color which I have drawn but I am still mentioning it again this is the blue color probability is 1 I am marking it here and if I consider the certainty value of C then the corresponding utility would be U(C) utility function is given whatever form is and that would be equivalent. Consider the gamble has two outcomes it can be 3, 4 so and so forth but consider very simply a two outcome and I will denote the gamble using the green color.

One arm, two arm, probabilities being 1/2, 1/2, the wealth being W_1, W_2 then the corresponding expected value which should match U(C) which is in the blue color and using the green color for the gamble. So $U(W) = 1/2U(W_1) + 1/2\ U(W_2)$. So whatever the values of the wealth is given the utility function these two values would exactly match I will use the red color to symbol where this one U(C) and U(W) would exactly match they has to match and they would be some value of C which will make them equal. Now extending that idea to the capital asset pricing model which very used different concept of optimization so few important points investors have quadratic utility function and this is important which I will mention and share prices are log normally distributed. Here one important fact is which I am marking in red is quadratic utility function would mean that if the utility function is I am only using U is $W - 1/2\ aW^2$ or to extend it that if you have investment and you consider the return and the return concept I am trying to utilize the green color so there are two returns one is total return R, one is rate of return r.

So if $r \sim N(\mu_r, \sigma_r)$ then the wealth or the investment based on which the small r or the rate of return is being realized would be quadratic that means if r is normal the utility function is quadratic if the utility function is quadratic r is normal this would be true. Share prices are log normally distributed and based on that if you remember one of the examples where we plotted and did the optimization when we are discussing the introduction part optimization of the either the return or minimization of the risk maximizing return and minimization risk for a portfolio considering of NSE stocks. So there we consider the r as log normally return and we used if you remember again a different color being used in order to make things. So r was calculated by Napierian ln (P_2/P_1) this P_2 and $P_1$are the prices which are basically wealth if you can understand price P_2 is the closing price for day 2 and P_1 is the closing price for day 1. Now comparing the point which I just mentioned normality and quadratic utility function, utility function being quadratic and rate of return being normal.

So comparison between mean variance which is considering normal distribution utility function where we consider the utility function as given here which I am underlining is basically quadratic. So this is the same form $W - 1/2\ aW^2$. So here a/2 is considered as b. So this is just a constant and we are trying to utilize that. Considering there are three assets and the assets are A B C the prices are given second column, third column price for B, fourth column price for C and I use the total return.

So I could have also used r there is no only the few formulas would change the concept remains the same. Just for information I will write down here if my wealth or investment the word wealth investment for this example I will be using interchangeably. Consider R which is total return would be investment at time period 2 divided by investment at time period 1 that means if I invest 100 today and I get back 110 then the capital value of R = 110/100. If I invest 100 and if I get the value of 90 then R = 90/100. Similarly r would be the difference of $(I_2 - I_1)/ I_1$.

So if I have 100 investment I_1 I get 110 so in the numerator it would be

(110 – 100)/100 = r. So this is the r return distribution I am talking about normality. So R let us consider based on the fact that R variation would also be normal. So consider we find out R. So R would be what? For A = (110/100) which is 1.1, (115/110) = 1.05, (120 – 115) and so on and so forth. So the values are found out. Similarly for R(B) which is for asset 3, asset 2, it is 115/105 for B and then it will be for 120/115, 125/120 and 130/125. So this is for B. And for C I have the values as 90/80, 95/90, 105/95, 130/105 I get the second last column of R(C) and the probabilities I consider as $1/5^{th}$ for each of these outcomes.

So based on that I find out the average return. So average return for I am marking with orange in this a little bit darkish yellow color $\bar{R}_{A,}$ $\bar{R}_{B,}$ $\bar{R}_{C}$ what is that? This returns R which I have calculated for A add them up divide by the total number of such instances. There are four instances because prices are 5 but when I calculate R it will be 4 in number. Similarly for , $\bar{R}_{B}$ again adding up the four returns for B/four, for C it is adding up four returns for C/4. So the corresponding values are given here 1.06, 1.05, 1.14. Similarly, I find out the standard deviation for A, B, C from the sample this is the sample. So given that standard deviation formula for the sample given the mean value for the population is unknown mean values are given from the sample as here 1.06, 1.05, 1.14 I find out the standard deviation. And just for information the value of the variance given by population mean is known. We know that so this was done in DADM I the course which I have taught or in any statistics which you can find. If the mean value for expected value of the population is known which is μ the best estimate for the variance is given as $1/n \sum_{i=1}^{n}(Xi - \mu)^2$ where n is the sample size. If I want to find out the this is a different estimate and why because the population mean is unknown I replace the population mean with the sample mean. This is for the normal distribution remember because x bar n is the best estimate for the population mean $\bar{X}_n$ is the sample estimate.

So if we use any of these formulas obviously the second one will be used which I am just marking in a square box because the population mean is unknown. So based on that I find out then I find out the weights with the weights are just simply the wealth my approach is with wealth I find out just the sum of the wealth for each investment divided by number of instance because there are 5. So they with these values come out to be 114, 119, 110. So considering the time limitation for this lecture I will continue this discussion further on in the next lecture. Sorry for that but because we have been discussing many things so I will continue from the same slide and consider in the 16th lecture the ending of this problem.

Thank you very much and have a nice day. Thank you. .