Multi-Criteria Decision Making and Applications
Prof. Raghu Nandan Sengupta
Industrial Engineering and Management Department
Indian Institute of Technology, Kanpur
Week 03
Lecture 14

A very good morning, good afternoon and good evening to all the students at whichever time you are hearing this lecture. This is the multi-criteria decision making MOOC NPTEL series course and my good name is Raghunandan Sengupta from the IME department at IIT Kanpur. So as you know this is a 60 lecture series which is spread over 12 weeks and each week we have 5 lectures, each lecture being for half an hour. So we are in the third week doing the course. So the broad outline we are covering which is a broad umbrella which is about the MCDM definitions, concepts, utility functions, safety first principles, stochastic dominance and this is the 14th lecture which we will do. So the coverage would be even though planned accordingly as the slides are uploaded you will understand what is the exact coverage. So we will plan to cover the concepts of the last example of utility function. Then what is the concept of certainty equivalent, geometric mean methods and why geometric mean methods are used, safety first principle, what are the three rules. There are many rules we will only discuss three. The concept of stochastic dominance, first order, second order and the general hyperbolic absolute disc aversion function and how it leads to all the initial set of our examples of utility functions we have discussed. Three we have discussed in the last class and one we will discuss today. So today's portion would be the last utility function, the simple ones quadratic, exponential and power we have already discussed.

So the last one is logarithmic utility function and the logarithmic utility function is given by ln(W), so an Napierian log and let us check the, this is my mistake this should be R(W), R'(W). So this we will consider A(W), A'(W), R(W), R'(W) and see the graphs accordingly as we have done for the other set of utility functions. So, U'(W) which is based on the first property would be 1/W, this is d(ln(W))/dW and U''(W) = $-1/W^2$.

So, if I want to find out A(W), I will use a different color, so in order to make it much more readable and differentiable. So A(W) is given by – U''(W)/ U'(W) = –1/W, – –becomes +, so divided by $1/W^2$, so this becomes W^2/W and this would become let me check whether 1/W this is $-1/ W^2$, so U'(W) this would be W^2 and this is 1/W. So, when I consider, so this is W^2 and there in the numerator you have W this is with a + sign, so the final value becomes 1/W. So, when I differentiate and find out A'(W) it is $-1/ W^2$ as given here. So, $W > 0$ obviously, $W^2 > 0$, so it has got decreasing absolute risk aversion factor considering the negative sign.

So, now let us come to R(W) which is relative risk aversion property, I will remove the portion which is marked in green which was the calculation done for related to A(W) and A'(W). So, you use the blue color, so as usual we need to find out R(W),

$R(W) = -W \times U''(W)/U'(W) = -W$, $U'(W)$ was already $1/W^2$, so with the + – – becomes + and this is $U'(W)$ is $1/W$. So, this becomes W from the denominator goes up and in the numerator of W it is 1, so $R'(W)$, differentiation of constant is 0, so it is 0 here and is a constant relative risk aversion property. So, just so we have derived it, understood it, I will erase this blue portion and also the red portion which was related to $U'(W)$ and $U''(W)$. So, only thing which I want to make you know this second bullet point is basically $R'(W)$, so based on that I need to draw.

So, the logarithmic utility function first case is basically the fact that this is ln of W logarithmic utility function given as here and we are plotting the following, you are plotting W along the X-axis and U(W), $A'(W)$, $R'(W)$, along the Y-axis and the corresponding $A'(W)$ and $R'(W)$ are even though drawn, but it will become much more clear when we go to the next set of diagrams. So, $A'(W)$ looks linear, but it is not $A'(W)$ is marked here similarly $R'(W)$ which we have derived is also marked here. So, now we will go to the separate figures for U(W), $A'(W)$, $R'(W)$. So, this is also the first figure where we plot W along the X-axis only U(W) along the Y-axis. This is the logarithmic utility function, the shape is exactly the same what we have seen in the last diagram logarithmic utility function versus W. And the corresponding $A'(W)$ is this even though it was a straight line, but actually it is this because it is if you remember it was given by $-1/W$ and the negative signs are here.

So, this is basically dA(W) d(W) which is $A'(W)$ for the logarithmic utility function versus W which is being plotted just for information even though it is a repetition please bear with me we are plotting W along the X-axis and $A'(W)$ along the Y-axis. The last figure is basically given as d(R(W)) dW which is basically $R'(W)$ we are plotting it for the logarithmic utility function versus W which is the wealth. And as usual we plot W along the X-axis $R'(W)$ along the Y-axis and if you remember in this in the derivation we did derive that the fact that $R'(W)$, R(W) was a constant was 1. So, hence $R'(W)$ was basically 0 and if you see this graph it is a straight line with the value of 0, 0 being here. So, we have considered in the last class 3 different utility functions quadratic, exponential and power and this class which is the 14th lecture we have considered the last example of logarithmic utility functions.

There are other utility functions also we will cover one more which is the general one hyperbolic absolute risk aversion factor HARA. Now all these things we are discussing about utility function actually in practical sense would not make much sense. So, that was the background, but what we need in order to understand the concept of utility function is known as the concept of certainty value which I am marking or certainty equivalent. So, the actual value of expected utility is of no use except when we are comparing with other alternatives. So, there are many alternatives and you want to compare them.

So, expected value is basically rank from the highest to the lowest and we make the decisions accordingly. So, another important concept which will make much better sense when we want to analyze utility function is the certainty equivalent and we will use the symbol C to denote the certainty equivalent. So, this is the amount of certain amount of wealth which is under considered on the disc free concept wide is free I will come to that that has the utility level exactly equal to this expected value which we want to find out for some U(W). So, I denote the U(W) as in red color. So, you want to find out some equivalent C such that the corresponding expected value of C would be equal to the expected value of U(W) based on the fact that the certainty equivalent is being considered another risk free concept.

So, what does it mean? Actually it means the following and which I will mark in yellow which means the expected value of the certainty equivalent and it is under risk free. So, expected value of the certainty equivalent which is marked here this one this portion would be exactly equal to the expected value of the utility which has non-deterministic outcomes. So, let me draw the diagram it will be there also later on, but let me make it clear. So, I will use blue color for the certainty value and red color for the utility. So, certainty does not have any risk which means the probability is 1.

So, you will basically have C which when utilized will give you the concept of U(C) which is the utility based on c and is expected value of E[U(C)] .
C is a fixed value = U(C) × Pr[U(C)] which as we know is 1, because it is another risk free case. So, the value would be U(C) × 1 because this probability which you have this portion I am marking in green has a value of 1. Like you are tossing a coin both sides are heads. So, the probability of getting a head is 1.

Now, this value of U(C) which you find out should be exactly equal to the expected value of the utility which I said I will mark in red. So, the utility is given by U(W). So, its corresponding expected value I will denote by E[U(W)]= U(W) × Pr[U(W)]. So, both are being balanced. So, on the left hand side you have the certainty value whose utility is U(C) × 1 because it is a sure case.

On the right hand side you have the utility and its expected value is found out by multiplying the corresponding utility with its corresponding probability. We will see that in a problem. So, if I draw the diagram I will continue using this slide erase the writings which I have done and draw the diagram. The blue one for the certainty part and the red one for the utilities. Certainty part means utility function in both the cases are the same, but certainty means with the probability of 1.

So, let me erase it. So, the certain value I used blue. So, I will continue using blue. So, you have C which gives you U(C) probability is 1. So, the expected value comes out as we know is E[U(C)] which you have already done that would be equivalent. I will use the equivalent sign with black and on the right hand side I will use the red colour in order to denote the utility which has different outcomes.

So, consider there are three outcomes. Outcome 1, outcome 2, outcome 3. So, here the wealth is W_1 corresponding utility is $U(W_1)$. For the next arm I will write the probabilities. Consider wealth is W_2 corresponding utility is $U(W_2)$ and third arm is W_3 and the corresponding utility is $U(W_3)$.

Now the utility function in both the cases are same. Now how they are equivalent? Because if I have probabilities P_1 for the first arm, P_2 for the second arm, P_3 for the third arm, then the corresponding expected value of this utility on the right hand side which is red in colour would be given. I will write down it down in the top would be $U(W_1) \times$. This star is the multiplication sign $P_1 + U(W_2)$, then $\times P_2 + U(W_3) \times P_3$. So, what is this value? This value as we know is expected value of utility W and equivalent signs has been placed in the sense that this expected value and this expected value are same.

So, that is why the equivalent sign. So, given utility function you need to find out the value of C. So, how is the value of C useful? So, suppose that we have a decision process with a set of outcomes as I showed you three outcomes. The probabilities are given which is $P_{1,}$ P_2, P_3 for the example and the corresponding utility values are also given $U(W_1)$, $U(W_2)$, $U(W_3)$. So, in case we want to compare this decision process we first find its certainty value and if you want to compare with other utility decisions we will compare just the certainty values for two different decisions.

As it will be much easier for us to compare because we will have a value and we can find them. To find the exact form the utility function for a person the certainty value is also useful and the person may not be clear about the form of utility function, but you can basically ask him or her some questions based on the certainty concept and then at least understand what his or her utility function would be. So, there are two important facts ranking, comparing and also to find out what type of utility function a person has. So, let us do a simple problem. So, those were the equations now let us do a very simple problem.

Suppose you face two options. So, option one which I will mark in green is this under option I you toss a coin and if a head comes you win 10, 10 rupees while if a tail appears you win 0. So, 1/2 of probabilities outcomes and under option II you get an amount of M fixed there is no change. That means you are considered tossing a coin and where both are heads and head comes probability 1, you get m. So, I will mark this case of option II with blue colour. So, green for the tossing the coin where head and tail are different 1/2 and 1/2 and for option II it is blue and consider for this case the utility function is quadratic in nature as given which I marked $W-0.04\ W^2$. So, the quadratic utility function which you have already considered. Now for the first option the corresponding expected value of the utility would be given by this. So, when I put the value of 10 for the utility function it is $(10 - 0.04 \times 10^2)\times 1/2$ because that is the probability that is the first arm. So, this is the first arm, this is the second arm.

Here you have 10, you have 0 probabilities are 1/2 and 1/2. In the second arm the value is 0. So, it is $(0 – 0.04 \times 0^2) \times 1/2$ and the value comes out to be 3, expected value for the utility with two arms. Now for the second option the expected utility is given by because the probability is 1. So, this is 1 and the utility is given because the value is M. So, $[M – 0.04 \times M^2]$ when you put it in the equation this becomes a quadratic equation. So, you equate this utility function which is $[M – 0.04 \times M^2] = 3$, equate them and based on the simple quadratic equation you find it and the value of C comes out to be C is 3.49 which is the certainty value for the case based on which you had option I. Which means technically that if the person has a quadratic utility function of this form $W-0.04\ W^2$ and on one table you keep a coin unbiased if a head comes it is 10 rupees given to him or her and if a tail comes he or she does not get any amount of money and on the right on the table there is 3.49 rupees given. So, if the person wants to decide he will try to basically decide the expected value of the utility based on the fact that the value is 3.49.

So, in case if he chooses that then in some way you can say the person is indifferent we will discuss that later on and if the person wants to take the gamble which means he is willing to take the risk why the risk concept is coming I will come to that and if he wants to take the certainty value which means he does not want to take the risk he is a risk hater or risk avoider. Why in the case if he takes the gamble he is willing to take the risk is because there are 2 arms one is 10 and one is 0. So, he or she thinks that with the chance of 1/2 he or she may get 10 which is more than 3.49 and if you remember the example which you have done for the fair gamble. So, we are increasing the stake and for the example which was 2000, 200000, 200000 and so on and so forth and other arm being 0 and for the fixed value in the sense probability is 1 the value was either 1 or 1000 corresponding to 2000 then 1 lakh corresponding to 2 lakh and 0.

So, there are 2 arms another was 10 lakhs for the for probability 1 and the other side it was 2000000 and 0. So, there it was basically being balanced and if you remember we use the utility function as W. So, whatever the value was we multiplied it by its corresponding value. So, certainty value will give you in a sense that how you will balance your decision. Now let us go back to the venture capital example or the consider a venture capitalist is considering of investing in an investment this is the second example where the falling amounts are there which is 10 lakhs, 5 lakhs, 1 lakhs with probabilities of 0.2, 0.4, 0.4 which means the sum of the probabilities is 1. So, there are 3 arms. So, let me use the red colour first arm its 10 lakhs I am using the word L for lakhs with a probability of 0.2, the second arm 5 lakhs with probability of 0.4, third arm 1 lakh with probability of 0.4. So, the sum is 1 you can find it out. Now it says that the you are using the power utility. So, power utility if is the case then the corresponding expected value would be calculated by this. So, $(10\ \text{lakhs})^{1/2}$ that is the utility $\times$ 0.2 is the probability. So, this is the first arm. The second arm is value is 5 lakhs, $(5\ \text{lakhs})^{1/2} \times 0.4$ which is basically the second arm. And the third arm was basically $(1\text{lakh})^{1/2} \times 0.4$ which is the third arm. If I find out the expected value the value comes up to be 609 which is just circle.

Now consider certainty value C the symbol which I said. So, obviously the utility function is also same. So, the corresponding utility for the certainty value which should match 609 is like this. So, I will use a different colour. So, the certainty value C, $(C^{1/2} \times 1)$ is basically the value of expected value of the certainty value that should be equal to what? That should be equal to 609. So, when I find out the value of C, the C comes out to be 609^2 which is 3 lakhs 70881 which means for the utility function 1/2 the certainty value for this decision where there are three outcomes 10 lakhs, 5 lakhs, 1 lakhs with probabilities of 0.2, 0.4, 0.4 and the utility function being $W^{1/2}$ which is the power utility function. Here power utility function means that $\gamma = 1/2$, C value is 1 and C and γ if you remember, were ≤ 1.

So, this value would be 3 lakhs 70881 we just balance it. Now, considering the discussion of certainty equivalent and the gamble, I will discuss few important points. A risk averse person will select an equivalent certainty event rather than the gamble. So, if he wants to avoid risk he will go for the sure event not the gamble. A risk neutral person will be indifferent to both of them and for the case when person is seeking the risk he or she will select the gamble rather than the certainty event and then highlight mention this point which I have already mentioned few minutes back that for the risk averse person he is thinking on the downside risk. So, if you consider the last example of the last to last example where there was an outcome of 10 and 0 the person he or she if he or she is a risk averse person is only concentrating on the value of 0 even though in actuality the expected value should be used to balance.

While the risk neutral person is thinking of so called the expected value for both the cases certainty values, sure event and the gamble. And the risk seeking person is concentrating thinking that what if the chance of the head comes he or she gets 10. It is not that 0 would not come with the probability 1/2 it will come both for the risk averse person and risk seeking, but the risk seeking person is concentrating on the value of 10. So, with this I will end this 14th lecture and continue discussing about utility functions in the later class. Thank you very much and have a nice day.